An unstructured staggered scheme with quadrilaterals for the Navier-Stokes equations

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abstract

Problems in fluid dynamics are often characterized by the Navier-Stokes equations, which express the conservation of mass, momentum and energy. In the first part of the report these equations are explained. In physics such equations are usually made dimensionless. This is done by using a Mach-formulation for compressible fluids, which is not suitable for incompressible fluids. In reality it is hardly possible to solve the non-linear Navier-Stokes equations analytically, therefore they will be treated numerically. In order to do this we need a discretization. In the second part of the report the dimensionless equations are discretized by the finite volume method. We are making use of a staggered placement of the variables on an unstructured grid with quadrilaterals. Finally the implementation has to be made. This has not been done yet, so in the concluding part the steps that need to be done in order to make an implementation are briefly discussed.

Contents

1 Introduction **5**
1.1 Literature 7

2 The Basic Equations of Fluid Dynamics **9**
2.1 Conservation of mass 9
2.2 Conservation of momentum 10
2.3 Conservation of energy 12
2.4 Equations of state 13
2.5 Euler equations 14
2.6 Condition for incompressibility 16
2.7 Dimensionless form 16
2.8 Boundary conditions 19

3 Discretization **21**
3.1 Finite volume method 21
3.2 Notation 23
3.3 Discretization of the equations of motion 24
3.4 Discretization of the momentum equation 26
3.4.1 The inertia term 27
3.4.2 The pressure term 27
3.4.3 The convection term 28
3.4.4 The stress tensor 30
3.5 Discretization of mass conservation 34
3.6 Discretization of the energy equation 35
3.6.1 The first time derivative 36
3.6.2 Second time derivative 36
3.6.3 Kinetic energy convection 36
3.6.4 Enthalpy convection 36
3.6.5 The viscous term 37
3.6.6 The diffusive term 37
3.7 The upwind scheme 39
3.8 Treatment of the boundary conditions 41

4 Conclusions **47**
4.1 Time schedule 48

Chapter 1

Introduction

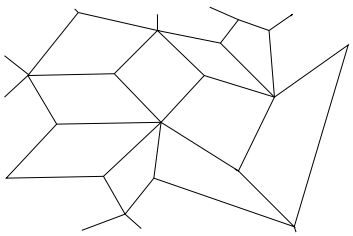


Figure 1.1: An unstructured grid.

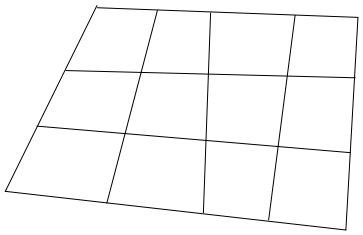


Figure 1.2: A structured grid.

Problems in fluid dynamics, for example the modeling of a flow in a pipe or the airflow around a wing, are usually described by the Navier-Stokes equations. These equations describe the conservation of mass, momentum and energy. The Navier-Stokes equations form a set of partial differential equations.

In practice it is impossible to solve the Navier-Stokes equations analytically except for some very simple cases. For that reason one applies a discretization method such as finite differences, finite elements or finite volumes. Over the last two decades the finite volume method has become very popular to solve these equations because it is based on conservation laws, thus on physics.

For discretizations we need a grid. In general we distinguish between two types of grids, namely structured and unstructured grids. A structured grid usually consists of blocks of cells. Each block can be mapped onto a rectangular grid with a constant number of cells in each of the coordinate directions. The number of cells that meet each other in an internal vertex is always 4 in a 2-dimensional space for quadrilaterals. See Figure 1.2. In an unstructured grid the arrangement of the cells is more or less arbitrary, see Figure 1.1.

In practice it is much easier to generate an unstructured grid for a com-

plex domain than for a structured grid. However, the price to pay is that datastructures become more complicated and that it is a harder task to develop a discretization. In the numerical analysis group of Delft University a start has been made to create a finite volume Navier-Stokes solver for an unstructured grid consisting of triangles. The literature however, shows that when the aspect ratio of the cells (the ratio between the length and width) becomes large, it may be better to use quadrilaterals. Such cells are especially needed in the neighbourhood of a boundary layers. The ultimate goal is of course to use a mixture of quadrilaterals and triangles. However, in this report we shall limit ourselves to use quadrilaterals only.

Another important issue is the position of the unknowns. In a collocated grid all variables are stored in the same position, usually either the centre of the cells or the vertices. For an incompressible flow however, it is known that a staggered grid arrangement, where scalars and vector components are positioned at different places, have advantages with respect to accuracy. Since we want to develop a code that is suited both for compressible and incompressible flows we have chosen to apply this staggering. However, in first instance we shall limit ourselves to compressible flows only.

In this report the discretization of the Navier-Stokes equations on an unstructured and staggered grid of quadrilaterals will be discussed.

In Chapter 2 the Navier-Stokes equations and the equations of state are discussed. In order to use these equations in (numerical) models these equations are made dimensionless. A nondimensionalization method for compressible fluids is used that is not suitable for incompressible fluids. In the last Section boundary conditions in a channel are discussed for inviscid flows (Euler) and viscous flows (Navier-Stokes).

Chapter 3 deals with the discretization of the dimensionless Navier-Stokes equations by the finite volume method. In the first section this method is explained. Notations are introduced. The idea of a staggered grid is explained and the discretization of the equations is briefly discussed. The next sections are devoted to the discretization of the dimensionless equations. In Section 3.7 upwind discretization will be discussed for the convection terms in the equation of motion. The boundary conditions, introduced in Chapter 2, have to be treated numerically. This point is discussed in the last section.

The last chapter concludes the report and gives a brief description of the things that need to be done in order to make an implementation.

1.1 Literature

In Chapters 1 of Wesseling's manuscript "Principles of computational fluid dynamics" [5] and Bachelor's "An introduction to Fluid Dynamics", [1] the principles of fluid mechanics are explained. It is interesting to know what happens inside the flow, but we need also some information on the boundaries. Section 16.4 of the book "Numerical Computation of Internal and External flows", written by Hirsch [2] deals with the boundary conditions for Euler. Finally Wenneker's report "An unstructured staggered scheme for the Navier-Stokes equations" [4] gives some ideas about the discretization of the Navier-Stokes equations with triangles. In this report quadrilaterals are taken instead.

First, in the next chapter, a brief overview of fluid mechanics will be given. The equations of motion and the equations of state are introduced. Next these equations are made dimensionless and finally boundary conditions are discussed.

Chapter 2

The Basic Equations of Fluid Dynamics

In the first three sections of this chapter we will consider the basic equations of fluid dynamics. These equations, the Navier-Stokes equations, describe: the conservation of mass, the conservation of momentum and the conservation of energy. In order to have as many equations as unknowns one or more additional relations are required. These so-called equations of state will be discussed in Section 2.4. The Euler equations, obtained by inserting zero viscosity and zero thermal conductivity in the Navier-Stokes equations, are discussed in Section 2.5. In the Section 2.6 the conditions under which a flow can be considered as incompressible will be given. The equations are made dimensionless in Section 2.7. In order to get a unique solution we also need to define boundary conditions. These are discussed in Section 2.8.

2.1 Conservation of mass

Consider a fixed volume V in the fluid enclosed by a surface S. Let $\mathbf{n}$ be the outward unit normal on the surface. Conservation of mass tells us that the change of mass in the volume V equals the netto flux through the surface S (we take for granted that there are no sources inside volume V). In mathematical form we get the following:

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int\int\int_V \rho dV = -\int\int_S \rho \mathbf{u} \cdot \mathbf{n} dS. \tag{2.1}$$

Here M is the total mass in volume V. Application of the Gauss-theorem to the right-hand side gives:

$$\frac{\partial}{\partial t} \int\int\int_V \frac{\partial \rho}{\partial t} dV = -\int\int\int_V \text{div}\,(\rho \mathbf{u}) dV. \tag{2.2}$$

Since this has to be true for all V it is clear that:

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \mathbf{u}) = 0, \tag{2.3}$$

For incompressible flows the continuity law reduces to the following expression:

$$\text{div}\,\mathbf{u} = 0. \tag{2.4}$$

The conditions for incompressibility will be discussed in Section (2.6).

2.2 Conservation of momentum

The law of conservation of momentum says that the change in momentum in a volume equals the total force in that volume. There are two kinds of forces: *body forces* ($\mathbf{F}_b$) and *surface forces* ($\mathbf{F}_s$). A body force is a force proportional to its mass. An example is gravitation force. A surface force is proportional to the area of the volume element, for example friction. The body force and surface force can be defined in the following way:

$$\mathbf{F}_b = \mathbf{f}^b \rho\, dV(t), \tag{2.5}$$

$$\mathbf{F}_s = \mathbf{f}^s dS(t). \tag{2.6}$$

Here $\mathbf{f}^b$ is the force per mass unit and $\mathbf{f}^s$ is the force per area unit. Applying the conservation of momentum to a material volume, where u_i are the velocities in each direction ($i = 1 \ldots 3$) yields:

$$\frac{d}{dt}\int_{V(t)} \rho u_i dV = \int_{V(t)} f_i^b \rho dV + \int_{S(t)} f_i^s dS, \tag{2.7}$$

We take $f_i^b = 0$. Now we use the transport theorem which says that a material volume $V(t)$ permanently consists of the same material particles:

$$\frac{d}{dt}\int_{V(t)} \phi dV = \int_{V(t)} (\frac{\partial \phi}{\partial t} + (\phi u_j)_{,j}) dV. \tag{2.8}$$

Taking $\phi = \rho u_i$ and substituting the right-hand side of (2.8) into the left-hand side of (2.7) gives:

$$\int_{V(t)} \{\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j}\} dV = \int_{S(t)} f_i^s dS. \tag{2.9}$$

The notation $_{,j}$ means differentiation with respect to x_j. The summation convention says that if an index appears twice, a summation over that index is intended. Hence:

$$u_i u_i = \mathbf{u} \cdot \mathbf{u}. \tag{2.10}$$

It can be shown that there are nine quantities σ_{ij} (the so called stress tensor) such that:

$$f_i^s = \sigma_{ij} n_j. \tag{2.11}$$

The vector $\mathbf{n}$ is the outward unit normal on the surface dS. Substituting of (2.11) into (2.9) yields:

$$\int_{V(t)} \{\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j}\} dV = \int_{S(t)} \sigma_{ij} n_j dS. \tag{2.12}$$

From the Gauss-theorem follows:

$$\int_{V(t)} \{\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j}\} dV = \int_{V(t)} \sigma_{ij,j} dV. \tag{2.13}$$

This equation is true for every volume $V(t)$ and therefore the integrals can be dropped:

$$\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j} = \sigma_{ij,j}. \tag{2.14}$$

Tensor σ_{ij} has to be related to the movement of the fluid. We can use the constitutive relations to that purpose, namely:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(\epsilon_{ij} - \frac{1}{3}\Delta\delta_{ij}). \tag{2.15}$$

Here p is the pressure, δ_{ij} is the Kronecker delta, μ is the dynamic viscosity and ϵ_{ij} the *rate of strain tensor*:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.16}$$

and:

$$\Delta = u_{i,i}. \tag{2.17}$$

The last expression represents the change in volume of V. In case of incompressible fluids this term is equal to zero, see equation (2.4). If we substitute (2.15) into (2.14) we get:

$$\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j} = -p_{,i} + \{2\mu(\epsilon_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j}. \tag{2.18}$$

These equations are the so-called Navier-Stokes equations.

2.3 Conservation of energy

Again we consider a volume element V with a surface S. The element moves along with the fluid, and therefore there is no transport of mass through the walls of the volume element. The first law of thermodynamics says the following:

$$\frac{d}{dt}\int_{V(t)} \rho E dV = Q + W. \tag{2.19}$$

In words: the change in total energy equals the heat Q that is added and the work W exposed to that volume element. The total energy is equal to the internal energy and the kinetic energy:

$$E = e + \frac{1}{2}u_i u_i. \tag{2.20}$$

The body force does work at rate $\rho u_i f_i^b dV(t)$ and the surface force does work at rate $u_i f_i^s dS(t)$. This means for the total work:

$$W = \int_{V(t)} \rho u_i f_i^b dV + \int_{S(t)} u_i f_i^s dS. \tag{2.21}$$

Again we set $\mathbf{f}^b$ to zero. With the aid of equation (2.11) and the divergence theorem of Gauss this can be written as:

$$W = \int_{V(t)} (u_i \sigma_{ij})_{,j} dV. \tag{2.22}$$

Supposing that there is no heat production inside the volume, Q equals:

$$Q = \int_{S(t)} \tau dS. \tag{2.23}$$

Flux τ follows Fourier's law, i.e.:

$$\tau = kT_{,i}n_i, \tag{2.24}$$

where k is the thermal conductivity and T is the temperature. Substituting (2.24) into (2.23) and using the divergence theorem we find:

$$Q = \int_{V(t)} (kT_{,i})_{,i} dV. \tag{2.25}$$

Applying the transport theorem to (2.19) and substituting (2.22) and (2.25) into the right-hand side of (2.19) yields:

$$\int_{V(t)} \{\frac{\partial \rho E}{\partial t} + (\rho u_i E)_{,i}\} dV = \int_{V(t)} \{(u_i \sigma_{ij})_{,j} + (kT_{,i})_{,i}\} dV. \tag{2.26}$$

This is the energy equation in integral form. Since this equation is applicable for all volumes $V(t)$ we can deduce the differential form of the heat equation:

$$\frac{\partial \rho E}{\partial t} + (\rho u_i E)_{,i} = (u_i \sigma_{ij})_{,j} + (kT_{,i})_{,i}. \tag{2.27}$$

Inserting the constitutive relations (2.15) we get the following relation for the energy:

$$\frac{\partial \rho E}{\partial t} + (\rho u_i E)_{,i} = -(u_i p)_{,i} + \{2\mu u_i (e_{ij} - \frac{1}{3}\Delta \delta_{ij})\}_{,j} + (kT_{,i})_{,i}, \tag{2.28}$$

or:

$$\frac{\partial \rho E}{\partial t} + (\rho u_i H)_{,i} = \{2\mu u_i (e_{ij} - \frac{1}{3}\Delta \delta_{ij})\}_{,j} + (kT_{,i})_{,i}, \tag{2.29}$$

where E is given by (2.20) and H is the total enthalpy, i.e.:

$$H = h + \frac{1}{2} u_i u_i. \tag{2.30}$$

Here h is the enthalpy.
The first term in the right-hand side of (2.29), the viscous term, is very small.

We now have 5 equations, ie. the continuity equation (2.3), the momentum equations, given by (2.18) and the energy equation, given by (2.29). However, we have 7 unknowns, namely: u_i, ρ, p, T, h. So we need 2 extra equations. These equations are called the equations of state. In the next section these will be introduced in order to complete the system.

2.4 Equations of state

A perfect gas is a gas for which the following statement holds:

$$p = \rho R T \tag{2.31}$$

This is the first equation of state, where R is a constant that is characteristic for the gas. Furthermore, in case of a calorically perfect gas, e (the internal energy per mass unit) only depends on the temperature T. This yields our second equation of state:

$$e = c_v T, \tag{2.32}$$

where c_v is called the specific heat at constant volume. The enthalpy is also only a function of the temperature, namely:

$$h = c_p T, \tag{2.33}$$

where c_p is the specific heat at a constant pressure.
Furthermore:

$$h = e + p/\rho. \tag{2.34}$$

We then get the following relationship for h and e:

$$\frac{h}{e} = \gamma, \tag{2.35}$$

where:

$$\gamma = \frac{c_p}{c_v}. \tag{2.36}$$

For a perfect gas $\gamma = \frac{7}{5}$.

It follows immediately from (2.31), (2.32), (2.33) and (2.34) that:

$$R = c_p - c_v. \tag{2.37}$$

We have here a so-called simple system, i.e. there are at most two independent variables of state, on which the other variables of state depend. Our independent variables of state are ρ and h. Since we will use density and enthalpy as primary variables (and not the temperature), equation (2.31) is rewritten as:

$$p = \frac{\gamma - 1}{\gamma}\rho h, \tag{2.38}$$

where (2.33), (2.35), (2.36) and (2.37) are used.

2.5 Euler equations

A fluid is called ideal if there is no friction nor diffusion/conduction, thus $\mu = k = 0$.

With $\mu = 0$ in (2.18) and $\mu = k = 0$ in (2.29), together with (2.3) and (2.38) using the definition of the total energy (2.20) and the total enthalpy (2.30) we get:

$$\frac{\partial \rho}{\partial t} + (\rho u_i)_{,i} = 0, \tag{2.39}$$

$$\frac{\partial \rho u_i}{\partial t} + (\rho u_i u_j)_{,j} + p_{,i} = 0, \tag{2.40}$$

$$\frac{\partial \rho E}{\partial t} + (\rho u_i H)_{,i} = 0, \tag{2.41}$$

$$p = \frac{\gamma - 1}{\gamma}\rho h, \tag{2.42}$$

where:

$$E = e + \frac{1}{2}u_i u_i, \quad H = h + \frac{1}{2}u_i u_i, \tag{2.43}$$

and:

$$h = e + p/\rho, \tag{2.44}$$

The speed of sound

Sound is a small disturbance in a fluid in which the entropy is constant. The speed of this wave is the speed of sound. If the entropy change of a process is zero then the process is called reversible (this means that the process can be reversed in time). On the other hand, if the entropy change of a process is not zero, this entropy change is always larger than zero and the process is irreversible (this means that the process cannot be reversed in time). Examples of reversible processes are flows in which diffusion and shocks are absent. A flow in which the entropy is constant is called an homentropic flow. In order to define the speed of sound we need to define the entropy first.

The entropy per mass unit s is defined by:

$$T\delta s = \delta e + p\delta(1/\rho), \tag{2.45}$$

where δ indicates small changes in the state variables. For homentropic flows, where $\delta s = 0$, we find the following relation for the pressure for ideal fluids (Wesseling [5]):

$$p = C\rho^{\gamma}. \tag{2.46}$$

Here C is a constant. The speed of sound is:

$$c^2 = (\frac{\partial p}{\partial \rho})_s, \tag{2.47}$$

The subscript $_s$ refers to the fact that $\partial p/\partial\rho$ must be taken at constant entropy.
Using (2.31) and (2.46) in (2.47) we see:

$$c^2 = \gamma RT = (\gamma - 1)h. \tag{2.48}$$

Later on we need the speed of sound in order to define the Mach number, which will be used in making the equations dimensionless.

2.6 Condition for incompressibility

The condition for incompressibility is (see [5]):

$$M \ll 1. \tag{2.49}$$

Here $M = U/c$ is the Mach number (the ratio between speed of the flow U and the speed of sound c). In practice one assumes a fluid to be incompressible if $M \leq 0.2$.

2.7 Dimensionless form

In this section a dimensionless form of the equations will be discussed. We can build a model with a small scale to simulate a model with a larger scale. By using the fact that laws of nature do not depend on physical units defined by convention, we can say something abouut the similarity between the two cases. We choose the following primitive variables: ρ (density), $\mathbf{m}$ (momentum), h (enthalpy). The velocity $\mathbf{u}$ equals: $\mathbf{m}/\rho$.

The governing equations can be made dimensionless by using units that are not conventional (such as the mks system) but are characteristic for the problem at hand. We will need four units for the length, density, enthalpy and momentum from which all other reference values will follow. If we have a flow in a pipe then the characteristic unit for length would for example be the radius of the pipe. By changing these characteristic units one can scale the model to one specific problem. The character of the flow depends on these characteristic units. The four independent characteristic units are chosen as follows: L_r, ρ_r, h_r, m_r. This gives the following dimensionless quantities, indicated by a tilde:

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{L_r}, \quad \tilde{\rho} = \frac{\rho}{\rho_r}, \quad \tilde{h} = \frac{h}{h_r}, \quad \tilde{\mathbf{m}} = \frac{\mathbf{m}}{m_r}. \tag{2.50}$$

Velocity and time can be scaled as:

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{u_r}, \quad \tilde{t} = \frac{t}{t_r}, \tag{2.51}$$

with:

$$u_r = \frac{m_r}{\rho_r}, \quad t_r = \frac{L_r}{u_r} = \frac{L_r \rho_r}{m_r}. \tag{2.52}$$

Mass conservation

For the mass conservation equation we have equation (2.3).
Using:

$$\frac{\partial}{\partial t} = \frac{m_r}{L_r \rho_r} \frac{\partial}{\partial \tilde{t}}, \quad \frac{\partial}{\partial x_i} = \frac{1}{L_r} \frac{\partial}{\partial \tilde{x}_i}, \tag{2.53}$$

we see that the scaled mass conservation equation is invariant with respect to the scaling.

The equation of state

Using (2.38) we can deduce the following expression for the reference pressure:

$$p_r = \frac{\gamma - 1}{\gamma} \rho_r h_r, \tag{2.54}$$

The dimensionless pressure is:

$$\tilde{p} = \frac{p}{p_r}. \tag{2.55}$$

The dimensionless equation of state becomes (tildes have been omitted):

$$p = \rho h. \tag{2.56}$$

Momentum equation

The momentum equation is given by (2.18). Inserting (2.38) in (2.54) leads to:

$$p_r = \gamma^{-1} \rho_r c_r^2, \tag{2.57}$$

where c_r represents the reference speed of sound. Using the scaling for the viscosity $\mu = \tilde{\mu}\mu_r$ we get (omitting the tildes):

$$\frac{\partial m_i}{\partial t} + (\rho^{-1} m_i m_j)_{,j} - \frac{\mu_r}{L_r m_r} \{2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j} = -\frac{\rho_r^2 c_r^2}{\gamma m_r^2} p_{,i}. \tag{2.58}$$

Using the definition of the *Mach number* and the *Reynolds number*:

$$M_r = \frac{m_r}{\rho_r c_r}, \quad Re_r = \frac{L_r m_r}{\mu_r}, \tag{2.59}$$

we get:

$$\frac{\partial m_i}{\partial t} + (\rho^{-1} m_i m_j)_{,j} - \frac{1}{Re_r} \{2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j} = -\frac{1}{M_r^2 \gamma} p_{,i}. \tag{2.60}$$

Since both sides of (2.60) are finite, we see that a problem arises when $M_r \downarrow 0$. In order to keep the right-hand side of (2.60) of order unity, the pressure becomes independent of position $\mathbf{x}$ in the incompressible limit. This is not very physical. However, since we restrict ourselves to compressible flows with $M_r \geq 0.2$, see Section 2.6, the singularity in $M_r \downarrow 0$ is avoided.

There is another dimensionless parameter that characterizes the flow, namely the Reynolds number. A very high Reynolds number indicates that convection dominates the flow, on the other hand if the Reynolds number is small we have a strongly viscous flow.

Energy equation

Expanding the terms of (2.29), using (2.33) and omitting the tildes yields:

$$\frac{m_r h_r}{L_r}\left[\gamma^{-1}\frac{\partial(\rho h)}{\partial t} + (m_i h)_{,i} + \frac{1}{2}\frac{m_r^2}{h_r\rho_r^2}\left\{\frac{\partial(\rho m_i m_i)}{\partial t} + (m_i u_j u_j)_{,i}\right\}\right]$$
$$-\frac{m_r h_r}{L_r}\left[\frac{k_r}{m_r L_r c_p}(kh_{,i})_{,i} + \frac{m_r\mu_r}{\rho_r^2 L_r h_r}\{2\mu u_i(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j}\right] = 0. \quad (2.61)$$

With the aid of (2.48) we see that:

$$h_r = \frac{c_r^2}{\gamma - 1}. \quad (2.62)$$

It is common practice to introduce the dimensionless *Prandtl number*:

$$Pr = \frac{c_p\mu_r}{k_r}. \quad (2.63)$$

The Prandtl number is the ratio between viscosity and heat conduction. Its value depends on the medium. For air it is approximately constant (≈ 0.72). Substituting (2.62) and (2.63) into relation (2.61) and using the definition of M_r leads to:

$$\gamma^{-1}\frac{\partial(\rho h)}{\partial t} + \frac{1}{2}M_r^2(\gamma - 1)\left\{\frac{\partial(\rho^{-1} m_i m_i)}{\partial t} + (m_i m_j u_j)_{,i}\right\} + \quad (2.64)$$
$$+(u_i h)_{,i} - (\gamma - 1)\frac{M_r^2}{Re_r}\{2\mu u_i(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j} - \frac{1}{Re_r Pr}(kh_{,i})_{,i} = 0.$$

System of equations

The resulting system of dimensionless equations in primary variables is:

$$\frac{\partial\rho}{\partial t} + m_{i,i} = 0, \quad (2.65)$$

$$\frac{\partial m_i}{\partial t} + (u_i m_j)_{,i} + \frac{1}{Re_r}\{2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j} = -\frac{1}{\gamma M_r^2}p_{,i}, \quad (2.66)$$

$$\gamma^{-1}\frac{\partial(\rho h)}{\partial t} + \frac{1}{2}M_r^2(\gamma - 1)\left\{\frac{\partial(\rho^{-1} m_i m_i)}{\partial t} + (\rho^{-2} m_i m_j m_j)_{,i}\right\} \quad (2.67)$$
$$+(m_i h)_{,i} - (\gamma - 1)\frac{M_r^2}{Re_r}\{2\mu u_i(e_{ij} - \frac{1}{3}\Delta\delta_{ij})\}_{,j} - \frac{1}{Re_r Pr}(kh_{,i})_{,i} = 0.$$

$$p = \rho h. \quad (2.68)$$

In order to have a unique solution of these equations it is necessary to have sufficient initial and boundary conditions. In the next section we shall consider the types of boundary conditions that are commonly used.

2.8 Boundary conditions

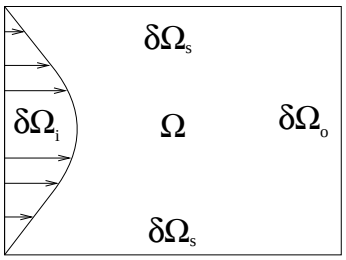


Figure 2.1: Boundaries for a channel

The Euler equations are given by (2.39) to (2.42). We have the independent variables $\mathbf{m}$, ρ and h. For a flow in a channel the situation will be the following: we have an inflow boundary, written as: $\partial\Omega_i$, two solid walls $\partial\Omega_s$ and the outflow boundary $\partial\Omega_o$. In order to get a unique solution we need boundary conditions. The Euler equations are hyperbolic equations. Therefore the number of boundary conditions depends on the number of incoming characteristics. For supersonic flows other boundary conditions are needed than for subsonic flows, because the number of incoming characteristics is different. One can prove (see [2]) that the number of boundary conditions for a subsonic flow in two dimensions is given by: three conditions on the inlet and one on the outlet and for a supersonic flow by: four conditions on the inlet and none on the outlet. For subsonic flows in a channel the following choice is possible:

$$\begin{aligned} h_i(t) &= \textbf{given}, \quad \mathbf{m}_i(t) = \textbf{given}, \\ p_o(t) &= \textbf{given}, \\ u_s^n &= \mathbf{u}_s \cdot \mathbf{n}_s = 0. \end{aligned} \tag{2.69}$$

We see clearly three boundary conditions on the inlet and one on the outlet.

The fourth condition implies that there is no flow through a solid wall. If we solve the Navier-Stokes equations, theoretically the situation is much more simple. The Navier-Stokes equations are parabolic or elliptic (in the stationary case). Hence on each boundary exactly two boundary conditions for the velocity and one for the enthalpy is required.

However: the flows we consider are convection dominant. As a consequence large parts of the flow behave more like Euler than like Navier-Stokes. The numerical scheme does not feel much influence of the relatively small viscosity and behaves as if the equations were hyperbolic. Especially for the outflow conditions this means that formally we have to prescribe 2 velocities, but in the case of Euler only one boundary for the momentum equations is

required (the pressure). As a consequence prescribing the velocity of the outflow might result in unrealistic wiggles since a kind of artificial obstacle is introduced unless we know the exact velocity corresponding to the numerical scheme. For that reason one always tries to use boundary conditions that disturb the flow as little as possible. For example at the outflow one sets:

$$\sigma^{nt} = 0, \quad \sigma^{nn} = -p, \quad \left.\frac{\partial h}{\partial \mathbf{n}}\right| = 0, \tag{2.70}$$

where σ^{nt} and σ^{nn} are the components of the normal stress. Note that this is strictly speaking not correct for Euler: if we have a differential equation of the first order we can only give boundary conditions of the order zero.

For a flow with viscosity we also have the no-slip condition at the solid walls, i.e. the velocity vector equals zero:

$$\mathbf{u}_s = \mathbf{0}, \tag{2.71}$$

At the solid wall we need one more boundary condition. At the solid wall we give a homogeneous Neumann condition for the enthalpy in normal direction. This gives us the following boundary conditions for Navier-Stokes:

$$\begin{aligned} h_i(t) &= \textbf{given}, \quad \mathbf{m}_i(t) = \textbf{given}, \\ \left.\frac{\partial h}{\partial \mathbf{n}}\right|_o &= 0, \quad \sigma_o^{nt} = 0, \quad \sigma_o^{nn} = -p, \qquad (2.72) \\ \mathbf{u}_s &= \mathbf{0}, \quad \left.\frac{\partial h}{\partial \mathbf{n}}\right|_s = 0. \qquad (2.73) \end{aligned}$$

The compressible Navier-Stokes are in general much too complicated to be solved analytically. For that reason it is necessary to use a numerical method. In our case we have decided to discretize the equations by a finite volume method (FVM).

Chapter 3

Discretization

In order to discretize the differential equations the finite volume method (FVM) will be used. This will be described in Section 3.1. In Section 3.2 some notation is introduced. The FVM will be applied to the equations of motion. In Section 3.3 a short introduction is given. Sections 3.4, 3.5 and 3.6 deal with the discretization of respectively the momentum, continuity and enthalp equation. The upwind scheme will be applied in Section 3.7 and in the last section the boundary conditions, described in Section 2.8, will be treated numerically.

3.1 Finite volume method

The finite volume method is a general discretization method for partial differential equations that can be applied if the equations are written in conservative form; i.e. as a divergence of a vector. An example is the Poisson equation, written as:

$$\nabla \cdot \nabla u = f, \tag{3.1}$$

where u is the velocity in one dimension and f a function. In the finite volume method we start by subdividing the region into subregions, for example in $\Re^2$: triangles or quadrilaterals. In this report we will limit ourselves to quadrilaterals only.

Next we choose special points per subregion where we want to put the unknown quantities. The points are for example the vertices of the subregion or the centroids of the quadrilaterals.

Next we have to define the equations in order to be able to solve the unknowns.

An important step in the FVM is the choice of a control volume. This

volume is chosen such that for each unknown we have a control volume.

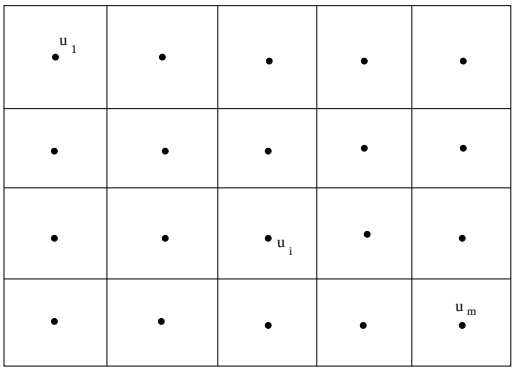


Figure 3.1: A structured grid with quadrilaterals

The unknowns are positioned in the centres of the control volumes. Therefore we take the control volume the same as the grid cell. After choosing the control volume the left- and right-hand side of equation (3.1) are integrated over the volumes Ω_i, with $i = 1 \ldots m$ where m is the number of control volumes:

$$\int_{\Omega_i} \nabla \cdot \nabla u d\Omega = \int_{\Omega_i} f d\Omega. \tag{3.2}$$

Let the boundary of control volume Ω_i be Γ_i. Application of the Gauss theorem to the left-hand side gives:

$$\int_{\Gamma_i} \mathbf{n} \cdot \nabla u d\Gamma = \int_{\Omega_i} f d\Omega. \tag{3.3}$$

Here $\mathbf{n}$ is the outward unit normal on Γ. Let the edges Γ_i consist of straight parts $\Gamma_{i_1}, \Gamma_{i_2} \ldots \Gamma_{i_4}$ with lengths: $h_{i_1} h_{i_2} \ldots h_{i_4}$. The integral in the left-hand side can be approximated by using the midpoint rule:

$$\int_{\Gamma_i} \mathbf{n} \cdot \nabla u \, d\Gamma \approx \sum_{k=1}^{4} \mathbf{n}_{i_k} \cdot \nabla u(\mathbf{x}_{i_k}) h_{i_k}, \tag{3.4}$$

where $\mathbf{x}_{i_k}$ is the midpoint of the edge Γ_{i_k}. The right-hand side is approximated by a midpoint rule:

$$\int_{\Omega_i} f d\Omega \approx f(\mathbf{x}_i)\Omega_i, \tag{3.5}$$

where $\mathbf{x}_i$ is the centre of cell i. This leads to the following equations for the FVM for all i:

$$\sum_{k=1}^{n} \mathbf{n}_{i_k} \cdot \nabla u(\mathbf{x}_{i_k}) h_{i_k} = f(\mathbf{x}_i)\Omega_i. \tag{3.6}$$

For two volumes lying next to each other we see that the outward normals $\mathbf{n}$ on the common edge point in opposite directions. The contributions at the edges of the cells cancel out each other. The absolute contribution is the normal component of the total flux through the edge.

$\nabla u \cdot \mathbf{n}$ is the normal derivative $\partial u/\partial \mathbf{n}$. This can be approximated by a numerical scheme, for example central differentiation.

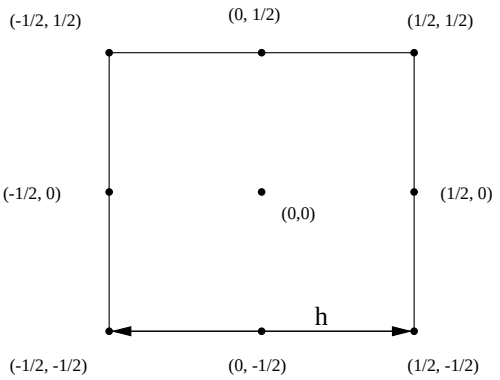


Figure 3.2: A primary cell

Application of this procedure to equation (3.1) to the case of square control volumes with $h_{i_k} = h$, see Figure 3.1, leads to the following discretization for internal cells:

$$h\left[\left(\frac{\partial u}{\partial \mathbf{n}}\right)_{0,1/2} + \left(\frac{\partial u}{\partial \mathbf{n}}\right)_{-1/2,0} + \left(\frac{\partial u}{\partial \mathbf{n}}\right)_{0,-1/2} + \left(\frac{\partial u}{\partial \mathbf{n}}\right)_{1/2,0}\right] = h^2 f_{0,0}. \quad (3.7)$$

In Cartesian coordinates we get:

$$h\left[\left(\frac{\partial u}{\partial y}\right)_{0,1/2} - \left(\frac{\partial u}{\partial x}\right)_{-1/2,0} - \left(\frac{\partial u}{\partial y}\right)_{0,-1/2} + \left(\frac{\partial u}{\partial x}\right)_{1/2,0}\right] = h^2 f_{0,0}. \quad (3.8)$$

Note that the unknowns are positioned at the coordinates (mh, nh) with m and n integers. Central differentation leads to:

$$u_{0,1} + u_{-1,0} + u_{0,-1} + u_{1,0} - 4u_{0,0} = h^2 f_{0,0}. \quad (3.9)$$

Here we recognize the standard finite difference formula in Poisson's equation. Before we apply the finite volume method to the Navier-Stokes, a couple of notations will be introduced in the next section.

3.2 Notation

The cells, i.e. quadrilaterals, will be ordered in a grid.
The midpoint of the face that connects the vertices i and j is given by:

$$\mathbf{e}_{ij} = (\mathbf{v}_i + \mathbf{v}_j)/2. \quad (3.10)$$

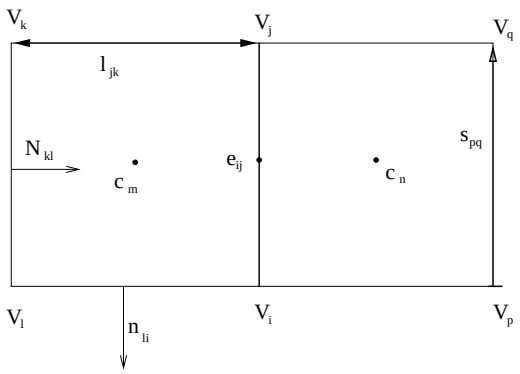


Figure 3.3: Notations

The position of the centre of cell m with vertices i, j, k, l can be calculated by averaging the the positions of the vertices:

$$\mathbf{c}_m = (\mathbf{v}_i + \mathbf{v}_j + \mathbf{v}_k + \mathbf{v}_l)/4. \tag{3.11}$$

The tangential vector of the face from vertex i to j is defined by:

$$\mathbf{s}_{ij} = \mathbf{v}_j - \mathbf{v}_i. \tag{3.12}$$

The length of the edge is given by:

$$l_{ij} = |\mathbf{s}_{ij}|. \tag{3.13}$$

It is important to note that if $\mathbf{s}_{ij}$ is part of a counter clockwise directed contour of cell m then $\mathbf{s}_{ji}$ will be a part of the counter clockwise contour of its neighbour n. For the discretization of the momentum equation we need a uniquely defined normal on the edges of each cell. The choice of the unique normals can be done in the following way: $\mathbf{N}_{ij}$ points from cell m to cell n if $m < n$, and points from cell n to cell m if $m > n$. This means that $\mathbf{N}_{ij}$ always points to the cell with the higher number. The advantage of this choice is that it is can be trivially extended to three dimensions.

With these definitions we now will consider the discretization of the equations of motion: the momentum equation, the energy equation and the conservation of mass.

3.3 Discretization of the equations of motion

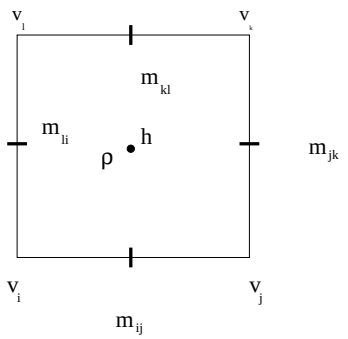


Figure 3.4: A cell of a staggered grid

For the discretization of the Euler and Navier-Stokes equations we shall use a so-called staggered grid arrangement, see Figure 3.4. That means that the scalar unknowns such as enthalpy, pressure and density will be positioned in the centres of the cells. The vector quantities, i.e. velocity $\mathbf{u}$ and $\mathbf{m}$ will be stored in a special way. On the midpoint of each edge the normal component of this vector quantity is stored in the midpoint. Our primary variables are: $\mathbf{m}$, ρ, h. For time discretization we will use Euler-backward. If we have the equation:

$$\frac{\partial u}{\partial t} = f(t, u(t)), \tag{3.14}$$

then Euler-backward says:

$$\frac{u^{n+1} - u^n}{\Delta t} = f(t^{n+1}, u^{n+1}). \tag{3.15}$$

Hence we have an implicit scheme. The advantage of implicit schemes is that they are more stable than explicit schemes, thus we can take larger time steps. Suppose that the function in the right-hand side is linear, then we have an equation that can be solved directly. In the case of a non-linear right-hand side we must linearize the equation and possibly iterate. The Navier-Stokes equations are non-linear so they have to be linearized first. Per time step we solve the equations of motion independently. The discretization of the equations is done in the following order:

1. The momentum equation will be discretized with the aid of m^n, u^n, ρ^n and p^n. The new momentum m^{n+1} will be solved from the resulting system (Section 3.4),

2. The continuity equation is used to obtain ρ^{n+1} from m^{n+1} and ρ^n (Section 3.5),

3. The velocity u^{n+1} is calculated by $\frac{m^{n+1}}{\rho^{n+1}}$,

4. the new enthalpy h^{n+1} is solved from the energy equation (in Section 3.6), using h^n and p^n and the computed m^{n+1}, u^{n+1} and ρ^{n+1},

5. the new pressure follows directly from the new density and the enthalpy with aid of the equation of state, ie. $p^{n+1} = \rho^{n+1} h^{n+1}$.

3.4 Discretization of the momentum equation

In this section is dealt with the discretization of the momentum equations. After a short introduction the discretization of the terms will be given.

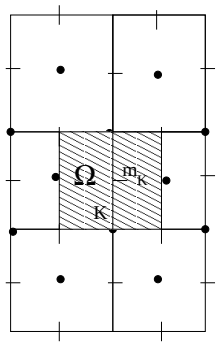


Figure 3.5: A control volume for the momentum

To derive an equation for the normal momentum m_K on the face $\mathbf{e}_K$ we project equation (2.66) on the uniquely defined normal $\mathbf{N}_K$ (defined in Section 3.2) on that face, denoting $m_K = \mathbf{m}_K \cdot \mathbf{N}_K$. For the momentum m_K we take the control volume Ω_K, as is sketched in Figure 3.5, with boundary $\partial\Omega_K$. Integrating (2.66) over Ω_k gives:

$$\int_{\Omega_k} \frac{\partial m^\alpha}{\partial t} d\Omega + \int_{\Omega_k} (u^\alpha m^\beta)_{,\alpha} d\Omega - \frac{1}{Re_r} \int_{\Omega_k} t^{\alpha\beta}_{,\beta} d\Omega = -\frac{1}{M_r^2 \gamma} \int_{\Omega_k} p_{,\alpha} d\Omega, \tag{3.16}$$

with $t^{\alpha\beta} = 2\mu(e^{\alpha\beta} - \frac{1}{3}\Delta\delta^{\alpha\beta})$. Here we use Greek superscripts for the components and Greek subscripts for the partial derivatives. For example:

$$t^{\alpha\beta}_{,\beta} = t_{ij,j}. \tag{3.17}$$

Application of the Gauss theorem gives:

$$\int_{\Omega_k} \frac{\partial m^\alpha}{\partial t} d\Omega + \int_{\partial\Omega_k} u^\alpha m^\beta n^\alpha d\Gamma - \frac{1}{Re_r} \int_{\partial\Omega_k} t^{\alpha\beta} n^\beta d\Gamma = -\frac{1}{M_r^2 \gamma} \int_{\Omega_k} p_{,\alpha} d\Omega, \tag{3.18}$$

where n_K^β is the outward unit normal of Ω_K. Taking the inner product with the uniquely defined normal N_K^α gives:

$$\int_{\Omega_k} \frac{\partial m^\alpha}{\partial t} N_K^\alpha d\Omega + \int_{\partial\Omega_k} u^\alpha m^\beta n^\alpha N_K^\beta d\Gamma - \frac{1}{Re_r} \int_{\partial\Omega_k} t^{\alpha\beta} n^\beta N_K^\alpha d\Gamma = -\frac{1}{M_r^2 \gamma} \int_{\Omega_k} p_{,\alpha} N_K^\alpha d\Omega. \quad (3.19)$$

The contour integrals are taken counterclockwise. The system consists of four terms, which will be discussed in the following subsections.

3.4.1 The inertia term

The first term on the left-hand side is the inertia term and is approximated as:

$$\frac{d}{dt} \int_{\Omega_K} m^\alpha N_K^\alpha d\Omega \approx \Omega_K \frac{dm_K}{dt}. \quad (3.20)$$

The time derivative is approximated in the following way:

$$\frac{dm_K}{dt} \approx \frac{m_K^{n+1} - m_K^n}{\Delta t}, \quad (3.21)$$

where Δt is the time step.

3.4.2 The pressure term

The pressure term is given by:

$$-\frac{1}{\gamma M_r^2} \int_{\Omega_K} N_K^\alpha p_{,\alpha} d\Omega = -\frac{1}{\gamma M_r^2} \int_{\Omega_K} \nabla p \cdot \mathbf{N}_K d\Omega. \quad (3.22)$$

This term can be evaluated by the so-called path-integral formulation. The basic idea is that first an approximation for the gradient ∇p is made. Then the inner product with respect to $\mathbf{N}_K$ is taken. The path-integral method is based on the following identity:

$$\int_{\mathbf{x}_n}^{\mathbf{x}_m} \nabla p \cdot d\mathbf{s} = p(\mathbf{x_m}) - p(\mathbf{x_n}). \quad (3.23)$$

With the aid of Taylor series approximation we see:

$$p(\mathbf{x}_m) - p(\mathbf{x}_n) = p_m - p_n \approx \nabla p_K \cdot (\mathbf{x}_m - \mathbf{x}_n). \quad (3.24)$$

Here ∇p_K is an approximation of the pressure gradient at $\mathbf{e}_K$. The unknowns are the elements for the gradient of p_K. In two dimensions we need one more equation to solve ∇p_K.

We can also take the path from p_o to p_s or from p_q to p_r, provided the path crosses the edge by $\mathbf{e}_K$ yielding:

$$p_o - p_s \approx \nabla p_K \cdot (\mathbf{x}_o - \mathbf{x}_s). \tag{3.25}$$

and:

$$p_q - p_r \approx \nabla p_K \cdot (\mathbf{x}_q - \mathbf{x}_r). \tag{3.26}$$

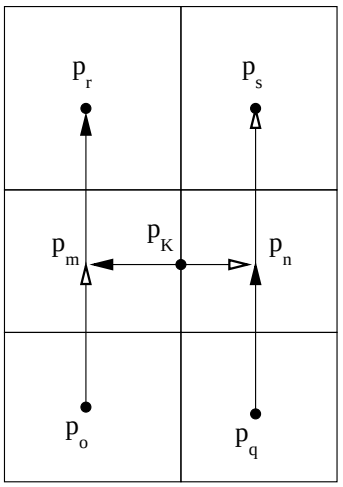


Figure 3.6: The paths from p_o to p_s and from p_q to p_r

From Figure 3.6 it is clear that there is no reason to prefer one of them, hence we average the value of both. It is clear that the common contribution of the path from p_m to p_n cancels. In conclusion: the gradient ∇p_K is solved from the following system:

$$\begin{aligned} p_m - p_n &\approx \nabla p_K \cdot (\mathbf{x}_m - \mathbf{x}_n), \\ p_o - p_r + p_q - p_s &\approx \nabla p_K \cdot (\mathbf{x}_o - \mathbf{x}_r + \mathbf{x}_q - \mathbf{x}_s). \end{aligned} \tag{3.27}$$

By using the midpoint rule we find for the second term in the right-hand side:

$$-\frac{1}{\gamma M_r^2} \int_{\Omega_K} \nabla p \cdot \mathbf{N}_K d\Omega \approx -\frac{1}{\gamma M_r^2} \Omega_K \nabla p_K \cdot \mathbf{N}_K. \tag{3.28}$$

3.4.3 The convection term

For the convection term the following discretization will be used:

$$\int_{\partial\Omega_K} (m^\alpha N_K^\alpha)(u^\beta n^\beta) d\Gamma \approx \sum_e l_e u_e (\mathbf{m}_e \cdot \mathbf{N}_K)(\mathbf{N}_e \cdot \mathbf{n}_e),$$

where e refers to the boundary edges of the contour $\partial\Omega_K$. $\mathbf{m}_e$ is the momentum at the centre of e, l_e is the length of the edge e. Since both $\mathbf{n}_e$ and $\mathbf{N}_e$ are unit normals pointing in either the same or opposite direction it follows that $\mathbf{n}_e \cdot \mathbf{N}_e = \pm 1$.

Since it is not desirable to solve non-linear systems directly, it is necessary to linearize these equations. One of the simplest methods is to take one of the values $\mathbf{u}_e$ or $\mathbf{m}_e$ at the old time level. From numerical experiments it is known that with respect to stability one should choose $\mathbf{u}_e$ at the old and $\mathbf{m}_e$ at the new time level.

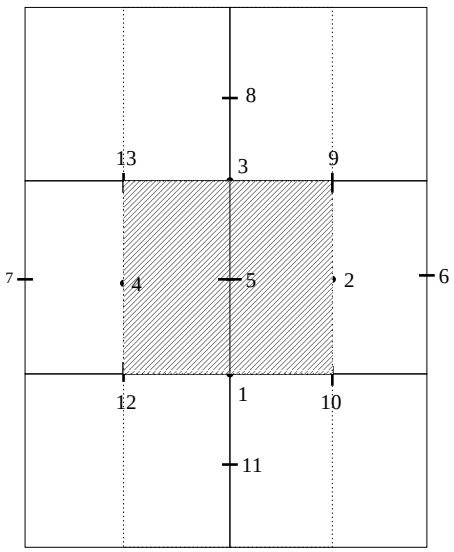


Figure 3.7: The neighbours of m_5

One of the problems we are faced with is that the momentum and the velocity are not available in the integration points. Hence we must use an interpolation. An extra complicating factor is that both quantities are vectors and we have only one of the components available per point. For the momentum vector $\mathbf{m}$ we write $\mathbf{m} = (m, \tilde{m})$. Here m represents the normal momentum component and $\tilde{m}$ the tangential component. Note that the values for m inside the control volumes are the unknowns. m and $\tilde{m}$ on the edges follow from a reconstruction procedure, in which we want to express $\mathbf{m}$ in neighbouring unknowns.

A simple approximation is to assume that the sides of the quadrilaterals are more or less parallel. This leads for example to the following approximation:

$$m_2 = \frac{m_6 + m_5}{2}. \tag{3.29}$$

$$\tilde{m}_2 = \frac{m_9 + m_{10}}{2}. \tag{3.30}$$

The normal and tangential components of $\mathbf{m_1}$, $\mathbf{m_3}$ and $\mathbf{m_4}$ are calculated in the same way. Instead of averaging the neighbours the upwind scheme can be used. This method is presented in Section 3.7. Hence we get an expression with the original momenta, stored at the edges of the primary cells. The same is done for the momenta $\mathbf{m}_1$, $\mathbf{m}_3$, $\mathbf{m}_4$. Performing this procedure for all control volumes Ω_K, we get a system of linear equations:

$$\mathbf{A}\mathbf{m} = \mathbf{b}, \tag{3.31}$$

where $\mathbf{m}$ is a vector containing all normal components of the momenta of the control volumes. The length of $\mathbf{m}$ equals the number of control volumes. $\mathbf{A}$ is a square matrix.

3.4.4 The stress tensor

The stress tensor consists of partial derivatives of velocities, positioned at the edges of the cells. We use the control volume defined in Figure 3.5. The contour integral over the stress tensor will be approximated in the following way:

$$\begin{aligned} & -\frac{1}{Re_r}\int_{\partial\Omega_k} t^{\alpha\beta} n^\beta N_K^\alpha d\Gamma \approx \\ \approx \quad & -\frac{1}{Re_r}\sum_e \mu_e \left[\{(u^\alpha_{,\beta})_e + (u^\beta_{,\alpha})_e\} N_K^\alpha n_e^\beta - \frac{2}{3}(u^\alpha_{,\alpha})_e (N_K^\beta n_e^\beta) \right] l_e \end{aligned} \tag{3.32}$$

In formula (3.32) μ_e is the viscosity on edge e of the control volume. In

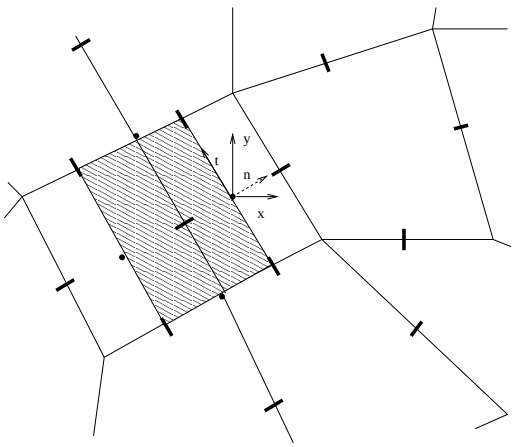


Figure 3.8: The Cartesian components of the velocity and normal components at the edges.

general it depends on the local properties of the fluid. The problem is how to evaluate the $t^{\alpha\beta}$ at the edges of the control volume. We are working with normal components of the velocities and momenta, but if one takes a gradient then one takes the partial derivatives with respect to the Cartesian system. Hence we have x and y components instead of normal and tangential components. Generally these components do not point in the same direction as the normal and tangential components. By adding the first component of the gradient of the first velocity component and the second component of the second velocity component we find the divergence $u^{\alpha}_{,\alpha}$.

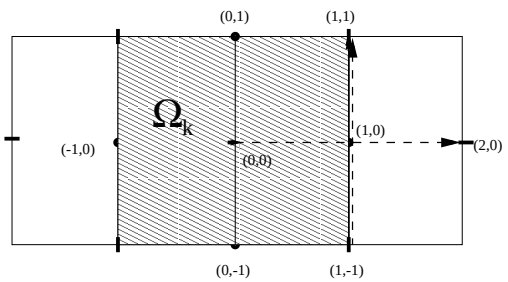


Figure 3.9: Two paths for the gradient of $\mathbf{u}$ in (1,0)

We take the edges as is illustrated in Figure 3.9. With the path integral formulation the derivatives of the vector $\mathbf{u}$ in for example (1,0) can be expressed in neighbouring u's by taking the path from (0,0) to (2,0) and from (1,-1) to (1,1), leading to respectively:

$$\mathbf{u}_{(2,0)} - \mathbf{u}_{(0,0)} = \int_{(0,0)}^{(2,0)} \nabla \mathbf{u}_{(1,0)} \cdot d\mathbf{x} \approx \nabla \mathbf{u}_{(1,0)} \cdot (\mathbf{x}_{(2,0)} - \mathbf{x}_{(0,0)}) \qquad (3.33)$$

and:

$$\mathbf{u}_{(1,1)} - \mathbf{u}_{(1,-1)} = \int_{(1,-1)}^{(1,1)} \nabla \mathbf{u}_{(1,0)} \cdot d\mathbf{x} \approx \nabla \mathbf{u}_{(1,0)} \cdot (\mathbf{x}_{(1,1)} - \mathbf{x}_{(1,-1)}) \quad (3.34)$$

Hence we have two equations and per unknown component of $\mathbf{u}$ two unknown elements of the gradient. We now have the gradient of $\mathbf{u}$ expressed in surrounding u.

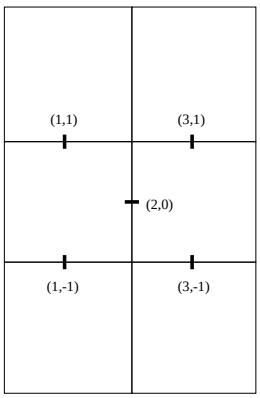


Figure 3.10: Interpolation of the velocity

The problem is that we do not have $\mathbf{u}$, only a normal component per momentum as unknown. The velocity $\mathbf{u}$ can be expressed in the normal and tangential components. The tangential components can be reconstructed from surrounding normal components with the same method as in the previous section, i.e. by averaging the neighbours. We take $\mathbf{u}_{(2,0)}$ as an example. The velocities on (2,-1) and (2,1) are not given. For that reason we average the normal components of four velocities closest to it, thus:

$$\tilde{u}_{(2,0)} \approx \frac{1}{4}(u_{(1,1)} + u_{(1,-1)} + u_{(3,1)} + u_{(3,-1)}). \quad (3.35)$$

This will only work properly for quadrilaterals with parallel edges as remarked before.

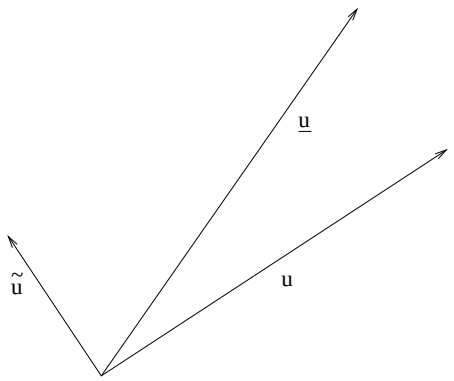


Figure 3.11: The normal and tangential component of **u**

Instead of the normal and the tangential component we want to have the x component and y component of **u**. In the situation of Figure 3.11 the x component and y component of **u**, written as u_x and u_y, are calculated from the normal and the tangential component of the momentum. Let **t** be a unity vector pointing into the tangential direction. Both the unity normal **n** and the unity tangential vector **t** consist of x and y components:

$$\mathbf{n} = (n_x, n_y), \quad \mathbf{t} = (t_x, t_y). \tag{3.36}$$

The velocity **u** is now written as:

$$\mathbf{u} = (u, \tilde{u}) = u\mathbf{n} + \tilde{u}\mathbf{t}. \tag{3.37}$$

This yields the following for the x and y components:

$$u_x = un_x + \tilde{u}t_x, \tag{3.38}$$

$$u_y = un_y + \tilde{u}t_y. \tag{3.39}$$

$$\tag{3.40}$$

The other gradients at the boundary of the control volume Ω_K are dealt with in the same way. However: we do not have the velocities as primary

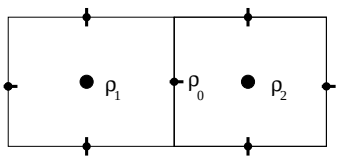


Figure 3.12: Interpolation of the density

variables, only the normal components of the momenta. The new velocity u_K can be expressed as $\frac{m_K}{\rho_K}$. We do not know the new density ρ_K, but we can linearize around the old density, ρ_K^n. The density on the edges is

calculated by averaging the pressures in the centres of the primary cells, for example as in Figure 3.12:

$$\rho_0^n \approx \frac{\rho_1^n + \rho_2^n}{2} \tag{3.41}$$

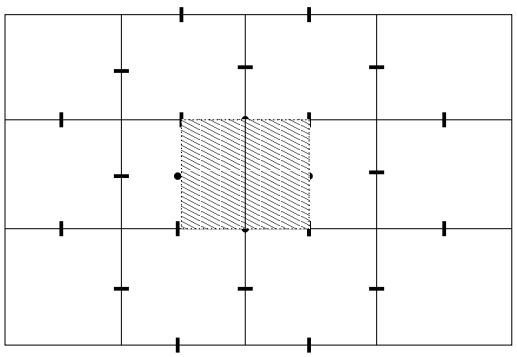


Figure 3.13: A stencil with 21 unknowns

Hence we get a stencil with 21 unknown momenta in total, marked by a stroke in Figure 3.13. This means that we have 21 non-zero elements per row, while for Euler flows, in which the viscosity is absent, there are only 9 non-zeroes. This means that one needs more memory to store the matrix.

From equation (3.19) we find the new momentum m with the aid of the old m, u, ρ, p.

3.5 Discretization of mass conservation

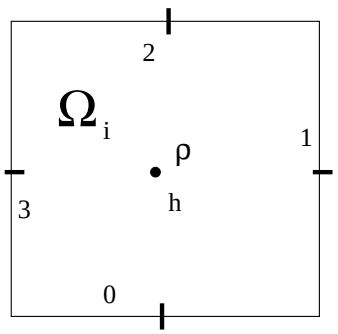


Figure 3.14: A control volume for the continuity equation

In this section we deal with the discretization of the continuity equation or mass conservation equation. The control volume Ω_i for the continuity

equation is the same as the primary cell i with ρ in the centre and the momenta m_e on the edges e. In Figure 3.14 $e \in \{0,1,2,3\}$. Integrating over the control volume Ω_i yields:

$$\int_{\Omega_i} \frac{\partial \rho}{\partial t} d\Omega + \int_{\Omega_i} m^\alpha_{,\alpha} d\Omega = 0. \tag{3.42}$$

Applying the Gauss-theorem to the second term yields:

$$\int_{\Omega_i} \frac{\partial \rho}{\partial t} d\Omega + \int_{\partial\Omega_i} m^\alpha n^\alpha d\Gamma = 0. \tag{3.43}$$

Using the midpoint-rule on both terms gives:

$$\Omega_i \frac{\rho^{n+1} - \rho^n}{\Delta t} + \sum_e m_e^{n+1} l_e N_e^\alpha n_e^\alpha = 0, \tag{3.44}$$

From this equation we find the new ρ which will be used in the enthalpy equation.

3.6 Discretization of the energy equation

In this section the discretization of the energy equation will be discussed. The control volume for the enthalpy is the same as the control volume for the density, see Figure 3.14. m^{n+1} and ρ^{n+1} are already known and will be used here. From the energy equation we will calculate the new enthalpy h. Integrating (2.67) over a control volume Ω_i gives:

$$\begin{aligned}
&\gamma^{-1} \int_{\Omega_i} \frac{\partial(\rho h)}{\partial t} d\Omega + \frac{1}{2} M_r^2 (\gamma - 1) \int_{\Omega_i} \frac{\partial(\rho^{-1} m^\alpha m^\alpha)}{\partial t} d\Omega + \\
&+ \frac{1}{2} M_r^2 (\gamma - 1) \int_{\Omega_i} (\rho^{-2} m^\alpha m^\beta m^\beta)_{,\alpha} d\Omega + \int_{\Omega_i} (m^\alpha h)_{,\alpha} d\Omega - \\
&- (\gamma - 1) \frac{M_r^2}{Re_r} \int_{\Omega_i} \{u^\alpha t^{\alpha\beta}\}_{,\beta} d\Omega + \int_{\Omega_i} \frac{1}{Re_r Pr} (k h_{,\alpha})_{,\alpha} d\Omega = 0.
\end{aligned} \tag{3.45}$$

Applying the Gauss-theorem yields:

$$\begin{aligned}
&\gamma^{-1} \int_{\Omega_i} \frac{\partial(\rho h)}{\partial t} d\Omega + \frac{1}{2} M_r^2 (\gamma - 1) \int_{\Omega_i} \frac{\partial(\rho^{-1} m^\alpha m^\alpha)}{\partial t} d\Omega + \\
&+ \frac{1}{2} M_r^2 (\gamma - 1) \int_{\partial\Omega_i} (\rho^{-2} m^\alpha m^\beta m^\beta) n^\alpha d\Gamma + \int_{\partial\Omega_i} (m^\alpha h) n^\alpha d\Gamma - \\
&- (\gamma - 1) \frac{M_r^2}{Re_r} \int_{\partial\Omega_i} \{u^\alpha t^{\alpha\beta}\} n^\beta d\Gamma + \int_{\partial\Omega_i} \frac{1}{Re_r Pr} (k h_{,\alpha}) n^\alpha d\Gamma = 0.
\end{aligned} \tag{3.46}$$

3.6.1 The first time derivative

Let Δt be the time step. Then the first time derivative is approximated by:

$$\int_{\Omega_i} \frac{\partial(\rho h)}{\partial t} d\Omega \approx \Omega_i \left(\frac{\rho_i^{n+1} h_i^{n+1} - \rho_i^n h_i^n}{\Delta t} \right) \tag{3.47}$$

3.6.2 Second time derivative

the second term in the first line can be approximated by:

$$\int_{\Omega_i} \frac{\partial(\rho^{-1} m^\alpha m^\alpha)}{\partial t} d\Omega \approx \Omega_i \frac{(\rho_i^{-1})^{n+1}(m_i^\alpha m_i^\alpha)^{n+1} - (\rho_i^{-1})^n (m_i^\alpha m_i^\alpha)^n}{\Delta t} \tag{3.48}$$

We already know $\mathbf{m}$ from the momentum equation in Section 3.4. The tangential momenta $\tilde{m}$ and normal momenta m in the centres of the cells are calculated by using a reconstruction method applied in Section 3.4.3.

3.6.3 Kinetic energy convection

The first term in the second line of (3.46) represents the so-called kinetic energy convection term. By applying the midpoint rule to this term we find the following approximation:

$$\int_{\partial\Omega_i} (\rho^{-2} m^\alpha m^\beta m^\beta) n^\alpha d\Gamma \approx \sum_e m_e \frac{(\mathbf{m} \cdot \mathbf{m})_e}{\rho_e^2} l_e \mathbf{n}_e \cdot \mathbf{N}_e. \tag{3.49}$$

Both momentum and density are taken at the new time level. To get ρ at the edges e the interpolation method in Section 3.4.4 can be used. The new momentum is found from the momentum equation.

3.6.4 Enthalpy convection

The second term in the second line of (3.46) is called the enthalpy convection term. The enthalpy convection term is discretized as follows:

$$\int_{\partial\Omega_i} (m^\alpha h) n^\alpha d\Gamma \approx \sum_e m_e h_e l_e \mathbf{N}_e \cdot \mathbf{n}_e \tag{3.50}$$

We already know the momentum, see Section 3.4. However, we only know the enthalpy in the centres of the primary cells, but we do not have it on the edges of the primary cells. For the central scheme we take the average of the neighbours. Consider for example h_3 as in Figure 3.15. This is approximated by:

$$h_3 \approx \frac{h_4 + h_5}{2} \tag{3.51}$$

The enthalpy on the edges can also be evaluated by applying upwind scheme. This is done in Section 3.7.

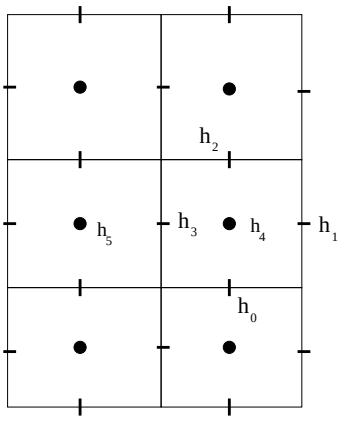


Figure 3.15: Interpolation of the enthalpy

3.6.5 The viscous term

The first term in the third row of (3.46), the so-called viscous term, is one of the most difficult terms. However, the viscous term compared to the convection and diffusion is neglected in order to save time.

3.6.6 The diffusive term

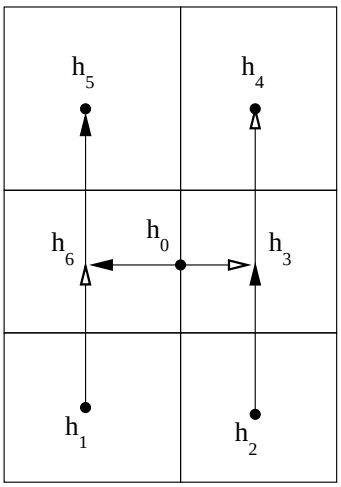


Figure 3.16: The path integral method for the enthalpy

The diffusive term can be approximated in the following way:

$$\int_{\partial\Omega_i} (kh_{,\alpha})n^{\alpha}d\Gamma \approx \sum_e k_e(h_{,\alpha})_e n_e^{\alpha} l_e. \tag{3.52}$$

The gradient $(h_{,\alpha})_e$ of the enthalpy can be approximated with the aid of the path integral method as discussed in Section 3.4.2, see Figure 3.16. If we

take the path from h_3 to h_6 the gradient of the enthalpy in 0 is evaluated by:

$$h_6 - h_3 \approx \nabla h_0 \cdot (\mathbf{x}_6 - \mathbf{x}_3). \tag{3.53}$$

We need one more equation to solve ∇h_0. To that purpose we add the path from h_2 to h_5 and from h_1 to h_4. This yields our second equation:

$$h_5 - h_2 + h_4 - h_1 \approx \nabla h_0 \cdot (\mathbf{x}_5 - \mathbf{x}_2 + \mathbf{x}_4 - \mathbf{x}_1). \tag{3.54}$$

Until now the convective terms have been treated by "central differences", since the momentum in integration points is evaluated by taking the average of the neighbouring values. However, in practice central differences may lead to numerical oscillation, the so-called wiggles. To avoid this, the upwind scheme may be applied. In the next section we deal with first order upwind.

3.7 The upwind scheme

An alternative method for central differences is the upwind scheme, which will be discussed in this section. Consider the following one dimensional differential equation:

$$u\frac{d\Phi}{dx} - \nu\frac{d^2\Phi}{dx^2} = 0, \tag{3.55}$$

with propriate boundary conditions. Here u is the velocity and ν is the viscosity. Discretization on a uniform grid with stepsizes h leads to:

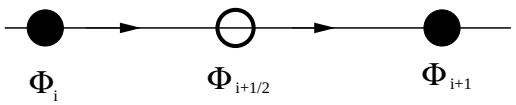
$$u\frac{\Phi_{i+1} - \Phi_{i-1}}{2h} - \nu\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{h^2} = 0, \tag{3.56}$$

where the convection term is treated with central differences. It can be shown that this discretization leads to spurious wiggles when the mesh-Peclet number $Pe = \frac{uh}{2\nu}$ is larger than one. Since we consider usually convection dominated flows, where $u \gg \nu$, the stepsize has to be unacceptably small to meet this stability criterion. It can be shown that the use of upwind differences for the convection term automatically prevents the pressure of spurious wiggles. First order upwind discretization leads to:

$$u\frac{d\Phi}{dx} \approx \begin{cases} \mathrm{u}\frac{\Phi_{i+1}-\Phi_i}{h} & \text{if } u < 0, \\ \mathrm{u}\frac{\Phi_i-\Phi_{i-1}}{h} & \text{if } u \geq 0. \end{cases} \tag{3.57}$$

Although there are much more sophisticated upwind interpolations than the first order upwind, this is the only scheme to be discussed.

A useful interpretation of (3.57) is that only information coming from the opposite flow direction (upwind) is taken into account. This idea can be applied to the momentum and enthalpy equation.



$$\Phi_{i+1/2} = \Phi_i$$

Figure 3.17: Upwind evaluation

In the first-order upwind method a value which has to be interpolated is taken equal to the last unknown that is passed as is illustrated in Figure 3.17.

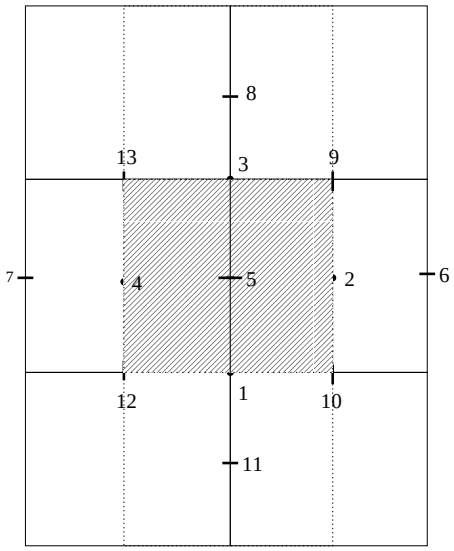


Figure 3.18: The neighbours of m_5

In the case of the convection term of the momentum equation, described in Section 3.4.3 this leads to the following evaluation for m_2 in Figure 3.18:

Let m_2^n be the momentum at the previous time level evaluated by taking $m_2^n = (m_6^n + m_5^n)/2$. If $m_2^n > 0$, we use m_5^{n+1} for m_2^{n+1}. Otherwise we use m_6^{n+1}. For $\tilde{m}_2^{n+1}$ we find:

$$\begin{cases} \tilde{m}_2^{n+1} = \tilde{m}_9^{n+1}, & \text{if } (m_9^n + m_{10}^n)/2 > 0, \\ \tilde{m}_2^{n+1} = \tilde{m}_{10}^{n+1}, & \text{if } (m_9^n + m_{10}^n)/2 < 0, \\ \tilde{m}_2^{n+1} = \frac{1}{2}(\tilde{m}_9^{n+1} + \tilde{m}_{10}^{n+1}), & \text{if } (m_9^n + m_{10}^n)/2 = 0, \end{cases} \tag{3.58}$$

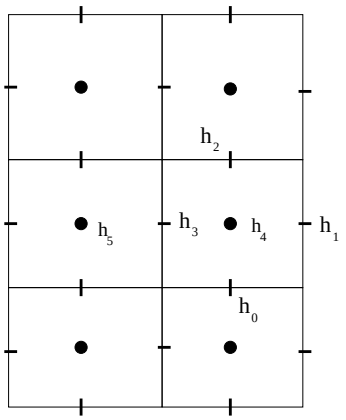


Figure 3.19: Interpolation of the enthalpy

For the discretization of the enthalpy convection in Section 3.6.4 we need to interpolate the enthalpy. The enthalpy on the edges of the cell is calculated by taking the average of the neighbouring enthalpy values. Application of upwind gives the following for h_3 in Figure 3.19:

If the direction of the flow is from h_4 to h_5 then:

$$h_3 = h_4. \tag{3.59}$$

On the other hand, if the flow is from h_5 to h_4 then:

$$h_3 = h_5. \tag{3.60}$$

3.8 Treatment of the boundary conditions

In Section 2.8 boundary conditions are given for subsonic flows. In a channel we have three types of boundaries: the inflow, the solid wall and the outflow. Furthermore we can distinguish between two types of boundary conditions: Dirichlet and Neumann. On the inflow and the solid wall Dirichlet conditions for the momentum and enthalpy are prescribed. On the outflow the normal stress is prescribed instead of the momentum and a homogeneous Neumann condition is given for the enthalpy. Difficult terms in the momentum equation and the enthalpy equation are the convection, the stress and the diffusion term. We will consider subsequently the momenta and the enthalpy in the neighbourhood of the edges.

First we consider the convection term (3.29) of the momentum equation. The control volume for the momentum m_2 is taken as in Figure 3.20. The momentum m_1 is given on the inflow and solid wall. Hence no momentum equation has to be solved, and we can put these elements directly in the

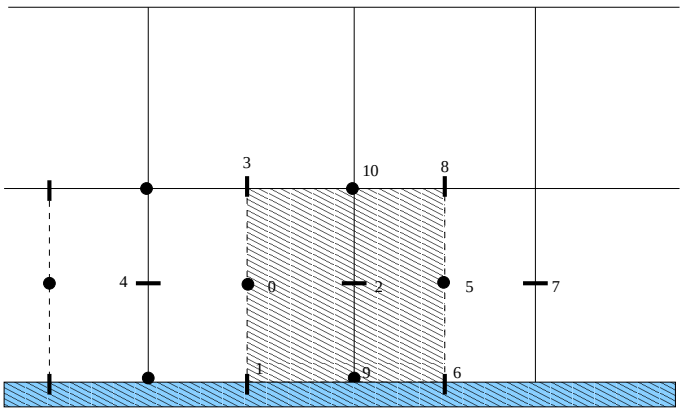


Figure 3.20: The control volume for the momenta

matrix. The contour of the patterned control volume of m_2 the contour is formed by the edges with the midpoints 9, 5, 10, 0, where we need the momentum vectors. On the boundary we have a Dirichlet boundary condition for the momentum vector. This means for example that we know $\mathbf{m}_9$. The tangential momentum $\tilde{m}_5$ is calculated by taking the average of m_6 and m_8. m_6 is situated on the boundary and is therefore given. For the normal component m_5 we take the average of m_2 and m_7, where nothing is prescribed.

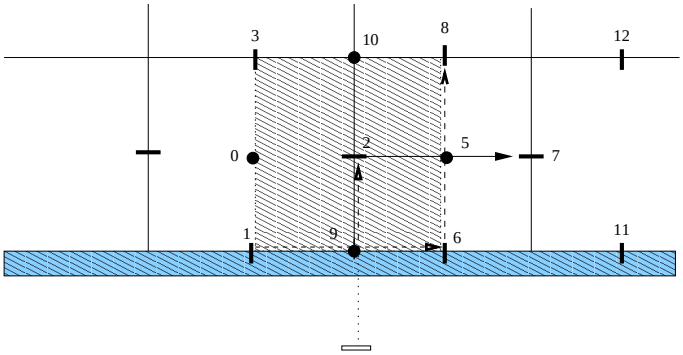


Figure 3.21: Paths on the edge

In the viscous term of the momentum equation (3.32) we need to evaluate the gradient of the two momentum components in 9, 5, 10 and 0 with the aid of the path integral method. Take for example the momentum $\mathbf{m}_9$ in Figure 3.21. The path integral method cannot be applied in the same way as in Section 3.4.4 to the boundary. The path from 2 through 9 would cross the boundary. We do not have any interpolation points outside the grid. Therefore we take one-sided difference here, in other words, we take the path from 9 to 2. The second path goes from 1 to 6 where the momenta in these points are prescribed. For the momenta in the points 5, 10 and 0 we do not meet this complication. In 5 for example we can safely take the path

from 6 to 8 and from 2 to 7, with $\mathbf{m}_6$ given. The tangential momentum $\tilde{m}_7$ is calculated by:

$$\tilde{m}_7 = \frac{m_6 + m_8 + m_{11} + m_{12}}{2}, \tag{3.61}$$

where both m_6 and m_{11} are given.

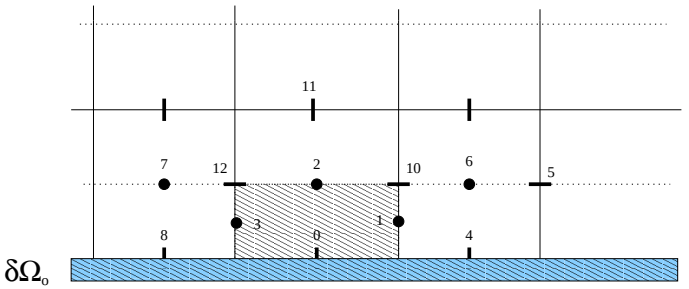


Figure 3.22: Cells at the outflow

Also some difficulties are met on the outflow. In the case of Navier-Stokes we have a prescribed σ^{nn} on the edge. We take another control volume for the momentum equations at this boundary. The control volume will be defined as in Figure 3.22.

For the convection term of the momentum equation we have a contour integral with integration points 0, 1, 2 and 3. In 0 the normal stress is given. The tangential component of m_2 is interpolated by:

$$\tilde{m}_2 = \frac{m_{10} + m_{12}}{2}. \tag{3.62}$$

The normal component is calculated by:

$$m_2 = \frac{m_0 + m_{11}}{2}, \tag{3.63}$$

The momenta $\mathbf{m}_1$ and $\mathbf{m}_3$ will be interpolated in a different way. We can move these momenta to either the points 12 and 10 or lay them on the outflow boundary. Two unknown momenta are positioned in 12 and 10, but on the outflow boundary we do not have the momenta, only the normal stress. The momenta on the edge then follow from an interpolation with the unknowns m_{12} and m_{10}. For that reason we simply take $\mathbf{m}_1 = \mathbf{m}_{10}$ and $\mathbf{m}_3 = \mathbf{m}_{12}$.

In the viscosity term we have the complications of the path integral method that we met on the other boundaries. However, on the outflow the normal stress is prescribed instead of the momentum. This can be a good motivation for working with contour integrals: For the edge 0 in Figure 3.22

equation 3.19 can be rewritten as:

$$\int_{\Omega_0} \frac{\partial m^\alpha}{\partial t} N_0^\alpha d\Omega + \int_{\partial\Omega_0} u^\alpha m^\beta n^\alpha N_0^\beta d\Gamma = \frac{1}{Re_r} \int_{\partial\Omega_0} \sigma^{\alpha\beta} n^\beta N_0^\alpha d\Gamma.$$

For the term in the right hand side we use the following discretization:

$$\int_{\partial\Omega_0} \sigma^{\alpha\beta} n^\beta N_0^\alpha d\Gamma \approx \sum_e \sigma_e^{\alpha\beta} n_e^\beta N_0^\alpha l_e, \tag{3.64}$$

where e represents the edges 0, 1, 2, 3. Since $\sigma_0^{nn} = \sigma_0^{\alpha\beta} n_0^\alpha n_0^\beta$ we see that one term in (3.64) is given, since it lays on the boundary. The σ's on the other edges (in Figure 3.22: 1, 2, 3) have to be calculated by an interpolation. We apply the same interpolation method as for the $\mathbf{m}_1$ and $\mathbf{m}_3$ in Figure 3.22, $\sigma_3 = \sigma_{12}$ and $\sigma_1 = \sigma_{10}$. For σ_2 we take the average of σ_0 and σ_{11}.

The numerical stencil of this contour integral is smaller than the stencil of the path integral method and as we have seen in the beginning of this section the numerical treatment of the boundary conditions will be more complicated if we work with path integrals.

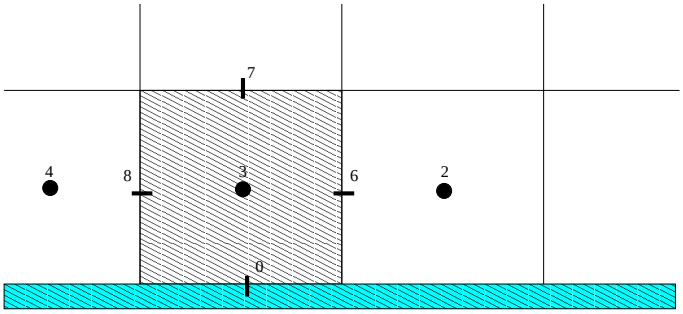


Figure 3.23: The control volume of the enthalpy at the boundary

We will now consider the boundary conditions for the enthalpy. The enthalpy is used in the energy equation. The control volume at the boundary is taken as in Figure 3.23. On the inflow boundary a Dirichlet condition for enthalpy is given, but for the outflow and solid wall we have a homogeneous Neumann boundary condition. For the energy equation the convection and diffusion are the difficult terms.

For the convection term (3.50) in the energy equation we need the enthalpy on the edges, since we are working with contour integrals. We already know the momenta from the momentum equation. In Figure 3.23 we need the enthalpy in 0, 6, 7 and 8. We have either a Dirichlet or a Neumann boundary

condition for the enthalpy. No equation has to be solved for the boundary. The enthalpy in 8 is calculated by upwind. The same method can be applied for h_6 and h_7. For h_0 we have either a Dirichlet or a Neumann condition.

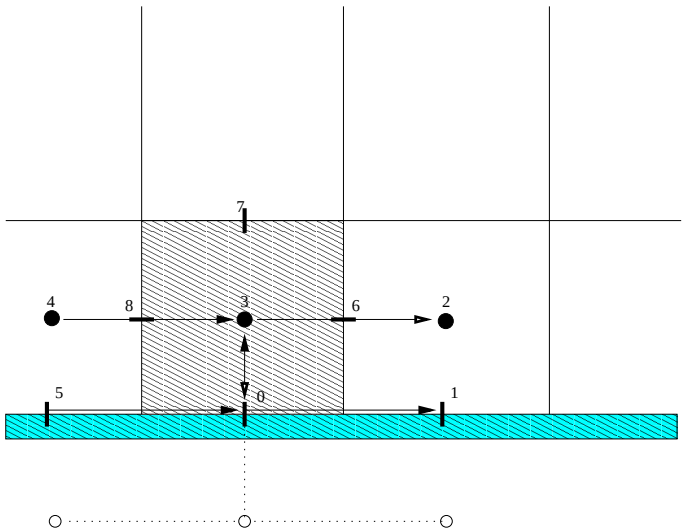


Figure 3.24: Paths on the edge

In the diffusive term (3.52) we need to have the inner product of the enthalpy gradient and the normal - the normal derivative. When we apply the path integral formulation we see that we meet the same complications on the boundary as for the viscous term in the momentum equation.

Let us consider the enthalpy in 0: we do not have the enthalpy on the other side of the boundary (marked by open dots in Figure 3.24). The paths are taken as in the figure, namely the path from 4 to 1, from 5 to 2 and from 0 to 3. The path that would have been outside the flow is moved to the boundary. On this path we know the enthalpy if we have a Dirichlet boundary condition. On the other hand: in case of a homogeneous Neumann condition, nothing has to be calculated on the boundary.

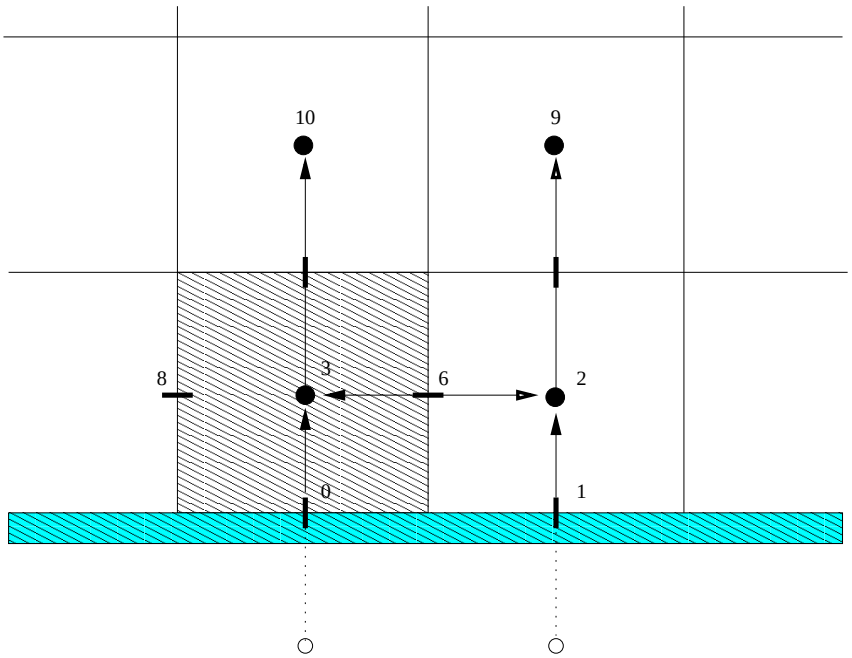


Figure 3.25: Paths on the edge

The same problem is found for the enthalpy gradient in point 6, see Figure 3.25. However, we can let the paths through 0 and 1 stop on the boundary. This yields a path from 1 to 10 and a path from 0 to 9. The same method is applied for the enthalpy gradient in 8. The gradient in 7 however does not give any problems. Here the same rules can be applied as in Section 3.6.6.

In this report only attention is paid to the theoretical part of the discretization. In order to be able to use this in an implementation datastructures are needed. The final chapter of this report gives a short description about what has to be done in order to make an implementation.

Chapter 4

Conclusions

In this report the Navier-Stokes equations and equations of statehave been discussed and discretized. Chapter 2 gives an overview of the governing equations. These equations were made dimensionless and formulated in expressions with the dimensionless Mach number and Reynolds number. The nondimensionalization method that was used is only suitable for compressible flows, where the Mach-number is larger than 0.2. The dimensionless equations were discretized in Chapter 3 with the aid of the finite volume method. An unstructured staggered grid was used to that purpose. In some terms of the momentum and enthalpy equation we needed to make interpolations, which was done by either central differences or upwind schemes. The interpolation methods discussed in this report will not work properly when the edges in the mesh are not parallel.

This report however only embodies the theoretical part. Now an implementation has to be made.

First we have to consider how we should define the datastructures. These datastructures follow from the generation of the mesh. On the datastructures the whole code of the implementation is based. We have two matrices: one for the momentum and one for the enthalpy. The variables are solved in the order which is described in Section 3.3. In the solver for the momentum and the enthalpy we also need to implement the interpolations described in 3.4 and 3.6. Our ultimate goal is to combine triangles and quadrilaterals.

4.1 Time schedule

date	subject
11 jan - 15 jan	introduction
18 jan - 22 jan	basic equations
25 jan - 29 jan	scaling and boundary conditions
01 feb - 12 feb	discretisation of momentum equations
15 feb - 26 feb	discretization of enthalpy and mass conservation equation
01 mar - 05 mar	upwind and boundary conditions
08 mar - 19 mar	finishing of the report
29 mar - 16 apr	datastructures
19 apr - 14 may	momentum solver
17 may - 28 may	density solver
31 may - 02 jul	enthalpy solver
30 aug - 17 sep	results
20 sep - 01 oct	conclusions and finishing of the report

Bibliography

[1] G.K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, UK, 1967.

[2] C.Hirsch. *Numerical Computation of Internal and External Flows. Vol.2: Computational Methods for Inviscid and Viscous Flows*. John Wiley, Chichester, 1990.

[3] Ivo Wenneker et al. *Computation of compressible and incompressible flows by a staggered finite volume scheme on unstructured grids*, Delft University, Delft, 1998.

[4] Ivo Wenneker. *An unstructured staggered scheme for the Navier-Stokes equations*. Delft University, Delft, 1999.

[5] P. Wesseling. *Principles of Computational Fluid Dynamics*. Delft University, Delft, 1998. (unpublished)