On lumped ILU preconditioners

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1. Introduction

Partial differential equations in 2D or 3D

unstructured grid, FEM or FVM discretization

$$Ax = b$$

Properties of $A \in \mathbb{R}^{n \times n}$

- Matrix A is large and sparse.
- SPD symmetric positive definite
- A is non singular

Solved by iterative methods (black-box, robust, efficient):

Krylov subspace methods combined with incomplete LU decomposition

A is SPD

The method of choice is CG combined with a preconditioner M (which should be SPD)

Preconditioned CG

$k = 0,\ r_0 = b - Ax_0$

while $\|r_k\|_2 > \varepsilon$ **do**

Solve $M z_k = r_k$

$k = k + 1$

if $k = 1$

$p_1 = z_0$

else

$\beta_k = \frac{(r_{k-1}, z_{k-1})}{(r_{k-2}, z_{k-2})}$

$p_k = z_{k-1} + \beta_k p_{k-1}$

end

$\alpha_k = \frac{(r_{k-1}, z_{k-1})}{(p_k, Ap_k)}$

$x_k = x_{k-1} + \alpha_k p_k$

$r_k = r_{k-1} - \alpha_k A p_k$

end while

2. IC preconditioners

If A is SPD the Choleski decomposition LL^T exists. Only the lower triangular part of A is stored in memory.

Incomplete Choleski (IC) decomposition

Compute L which satisfies:

- $l_{ij} = 0$ for $a_{ij} = 0$ and $i > j$
- $(LL^T)_{ij} = a_{ij}$ for $a_{ij} \neq 0$ and $i \geq j$

This preconditioner exists if A is an M-matrix.

$$a_{ij} \leq 0, \quad j \neq i \text{ and } A^{-1} \geq 0.$$

MIC preconditioners

A is SPD. Only the lower triangular part of A is stored in memory.

Modified IC (MIC) decomposition

Compute L which satisfies:

- $l_{ij} = 0$ for $a_{ij} = 0$ and $i > j$
- $(LL^T)_{ij} = a_{ij}$ for $a_{ij} \neq 0$ and $i > j$
- rowsum(LL^T) = rowsum (A)

This preconditioner exists if A is *weakly* diagonal dominant: $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$.

Break down IC preconditioner

Suppose A is not an M-matrix.

$$l_{ii}^2 = a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2$$

Break down when right-hand side is negative.

Typical examples

- large convection and central differences
- linear elements with bad triangles
- quadratic elements
- coupled problems: mechanical problems, Navier-Stokes

Solution strategies

1. Use LDL^T and MINRES
2. Replace right-hand side by a small positive number $(a_{ii}/100)$
3. Use IC of $A+\gamma I$ or $A+\gamma \text{diag}(A)$ with a suitable chosen $\gamma > 0$.
4. Construct $\hat{A}$ such that $\hat{A} \approx A$ and $\hat{A}$ is an M-matrix. Use IC of $\hat{A}$.

3. Lumped IC preconditioners

Suppose A is SPD but not an M-matrix. Construct $\hat{A}$ in the following way:

for $i = 1, n$
  $\hat{a}_{ii} = a_{ii}$
  for $j = 1, i-1$
    if $a_{ij} < 0$ **then**
      $\hat{a}_{ij} = a_{ij}$
    else
      $\hat{a}_{ij} = 0$
      $\hat{a}_{ii} = \hat{a}_{ii} + a_{ij}$
      $\hat{a}_{jj} = \hat{a}_{jj} + a_{ij}$
    end if
  end for
end for

$\hat{A}$ is an M-matrix. Its IC decomposition exists and is used as a preconditioner for $Ax = b$.

Lumped MIC preconditioner

Note that rowsum($\hat{A}$) = rowsum(A). (*)

Define the vector $\mathbf{e}$ as: $\mathbf{e} = (1, \cdots, 1)^T$.

Lemma

If $A \cdot \mathbf{e} \geq (0, \cdots, 0)^T$ then $\hat{A}$ is weakly diagonal dominant.

Proof

Using (*) gives: $\displaystyle\sum_{j=1}^{n} \hat{a}_{ij} = \sum_{j=1}^{n} a_{ij} \geq 0.$

Since $\hat{a}_{ij} \leq 0, \; j \neq i$

$$\hat{a}_{ii} \geq \sum_{j \neq i} -\hat{a}_{ij} = \sum_{j \neq i} |\hat{a}_{ij}|$$

Corollary: MIC of $\hat{A}$ exists.

4. Lumped ILU preconditioner

Krylov subspace methods: Bi-CGSTAB, GMRES, GCR,...

Preconditioner necessary

ILU decomposition exists if A is an M-matrix

MILU decomposition exists if A is a weakly diagonal dominant matrix

Problem $u_{ii} \approx 0$ or $u_{ii} \rightarrow \infty$.

- division by a small number
- slow convergence

Lumped ILU preconditioner

It is possible that $a_{ii} \cdot a_{jj} \leq 0$

Construct $\hat{A}$ in the following way:

for $i = 1, n$
    $\hat{a}_{ii} = a_{ii}$
    for $j = 1, i-1$ and $j = i+1, n$
        if $a_{ij} \cdot a_{ii} > 0$ **then**
            $\hat{a}_{ij} = 0$
            $\hat{a}_{ii} = \hat{a}_{ii} + a_{ij}$
        else
            $\hat{a}_{ij} = a_{ij}$
        end if
    end for
end for

The (M)ILU decomposition of $\hat{A}$ is used as a preconditioner for $Ax = b$.

Lumped ILU preconditioner

Define diagonal matrix D: $d_{ii} = sign(a_{ii})$.

Note that $\tilde{A} = D\hat{A}$ is an element of $Z^{n\times n}$. $\tilde{A}$ is an M-matrix if it satisfies one of the conditions given in Berman and Plemmons.

If $\tilde{A}$ is an M-matrix its ILU decomposition exists: $\tilde{L}\tilde{U}$. The ILU factors of $\hat{A}$ are

$$\hat{L} = D\tilde{L}D^{-1} \text{ and } \hat{U} = D\tilde{U}.$$

If $DA \cdot \mathbf{e} \geq (0, \cdots, 0)$ then $\tilde{A}$ is weakly diagonal dominant, so its MILU decomposition exists and thus the MILU decomposition of $\hat{A}$ exists.

Implementation

Strategy

Start with (M)ILU decomposition of A

if $|u_{ii}| < \epsilon$ or $|u_{ii}| > u_{max}$ **then**

restart with (M)ILU decomposition of $\hat{A}$

if $|u_{ii}| < \epsilon$ **then**

$u_{ii} = a_{ii}/100$

end if

end if

5. Lumped ILUD preconditioner

Example 5 point FD for Poisson in 2D

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} & & \\ a_{21} & a_{22} & a_{23} & & a_{2n} & \\ & \ddots & \ddots & \ddots & & \ddots \\ a_{n1} & & a_{nn-1} & a_{nn} & a_{nn+1} & \\ & \ddots & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Incomplete Cholesky (IC = ICD) decompostion

$LD^{-1}L^T$

- $l_{ij} = a_{ij}, \; i > j$
- $l_{ii} = d_{ii}$
- $(LD^{-1}L^T)_{ii} = a_{ii}$

Lumped ICD preconditioner

Efficiency

- Only one extra vector in memory for the preconditioner
- Save one matrix vector product per iteration

Eisenstat implementation

$$\tilde{A} = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}, \;\; \tilde{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

$$\tilde{L}^{-1}\tilde{A}\tilde{L}^{-T}v = \tilde{L}^{-1}v + \tilde{L}^{-T}(v + (diag(\tilde{A}) - 2I)\tilde{L}^{-T}v$$

Problem: $d_{ii} < 0$

MILUD preconditioner

MILUD decompostion

$LD^{-1}U$

- $l_{ij} = a_{ij},\ u_{ji} = a_{ji},\ i > j$
- $l_{ii} = u_{ii} = d_{ii}$
- rowsum$(LD^{-1}U)$ = rowsum(A)

Exists if A is weakly diagonal dominant

Eisenstat implementation

$$\tilde{A} = D^{-1}A,\ \ \tilde{L} = D^{-1}L,\ \ \tilde{U} = D^{-1}U$$

$$\tilde{L}^{-1}\tilde{A}\tilde{U}^{-1}v = \tilde{U}^{-1}v + \tilde{L}^{-1}(v + (diag(\tilde{A}) - 2I)\tilde{U}^{-1}v$$

Problem: $d_{ii} \approx 0$, bad convergence

Lumped (M)ILUD preconditioner

$\hat{A}$ is the lumped version of A

Determine $\hat{L}\hat{D}^{-1}\hat{U}$ from a (M)ILUD decomposition of $\hat{A}$.

Lumped (M)ILUD decompostion

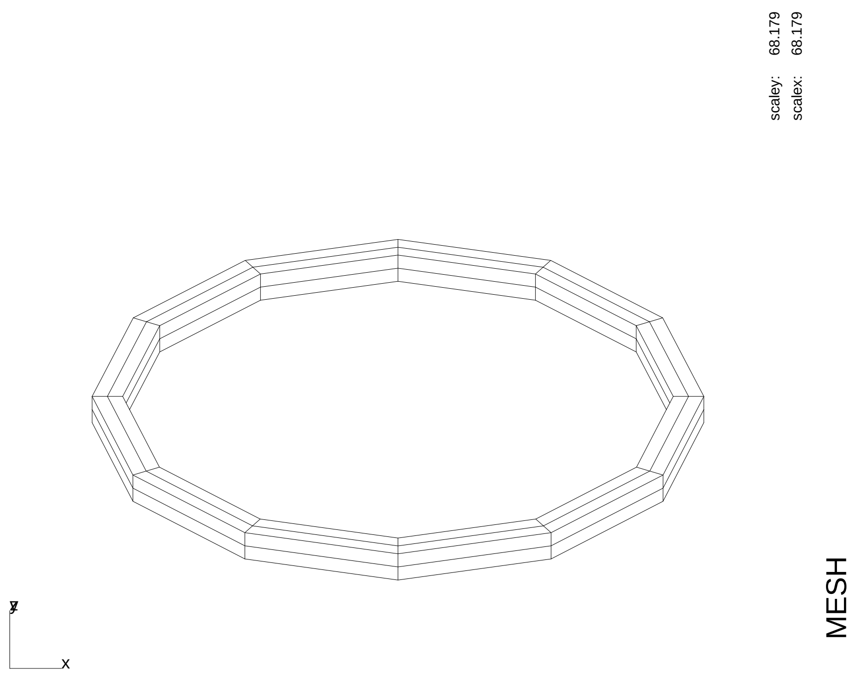
$LD^{-1}U$

- $l_{ij} = a_{ij}, \; u_{ji} = a_{ji}, \; i > j$
- $l_{ii} = u_{ii} = d_{ii} = \hat{d}_{ii}$

6. Numerical experiments

Linear elastic problem: $\frac{\partial^2 \rho \mathbf{u}}{\partial t^2} - div\sigma = \mathbf{F}$

σ stress tensor, $\mathbf{F}$ body force



Geometry and mesh for the bearing

Linear elastic problem

- (M)IC(D) breaks down without lumping
- $d_{ii} = a_{ii}/100$ breaks down

Lumped preconditioners

	iter	CPU
IC	125	0.10
ICD	109	0.06
MICD	106	0.06

Bi-CGSTAB

	iter	CPU
ILU	72	0.13
ILU (lumped)	58	0.10
ILUD	69	0.08
MILUD	64	0.07

Incompressible Navier Stokes

- finite volumes, unstructured grid
- staggered placement of unknowns
- pressure correction approach

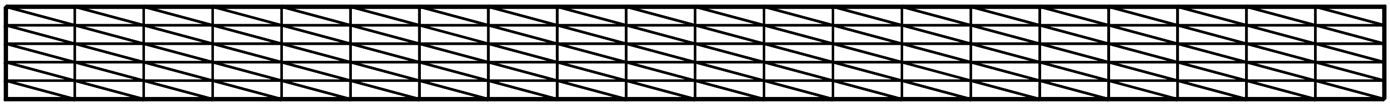
Method for pressure discretization

CI contour integral method
PI path integral method

Properties

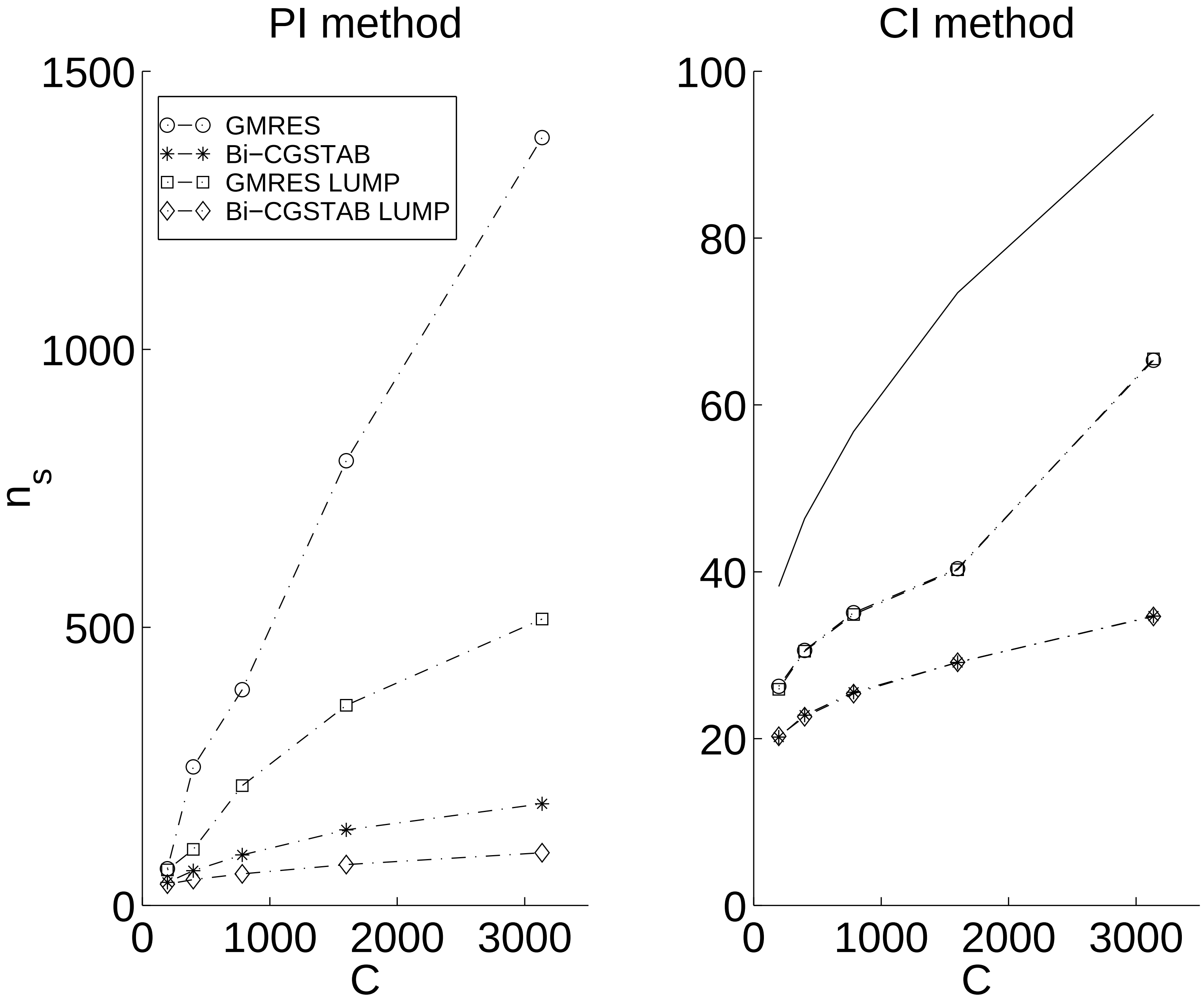
CI Laplace like matrix, pressure oscilations
PI non M-matrix

Stretched grid



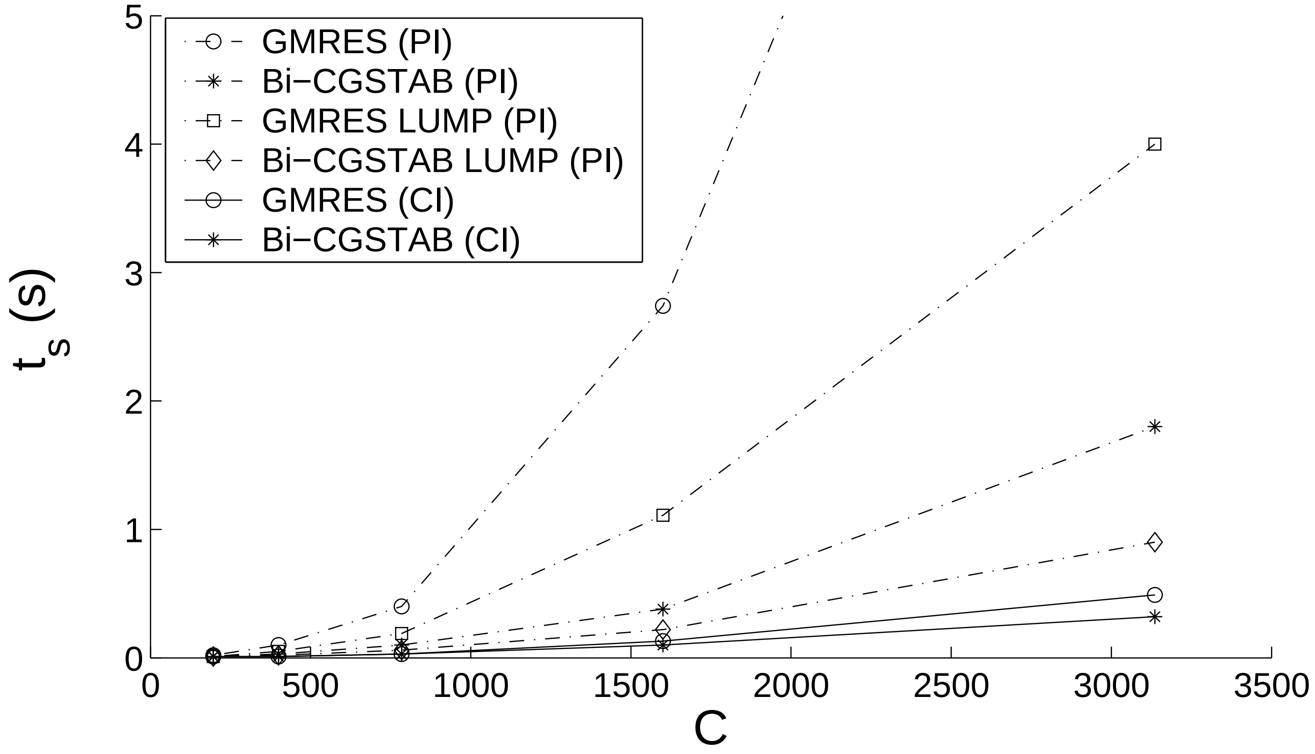
Various solvers

Number of iterations



Various solvers

CPU time



Convection diffusion equation

- square domain, diffusion coefficient $\mathbb{D}$
- angle α, central scheme, 100×100 grid

Number of matrix vector products
angle is 45^o

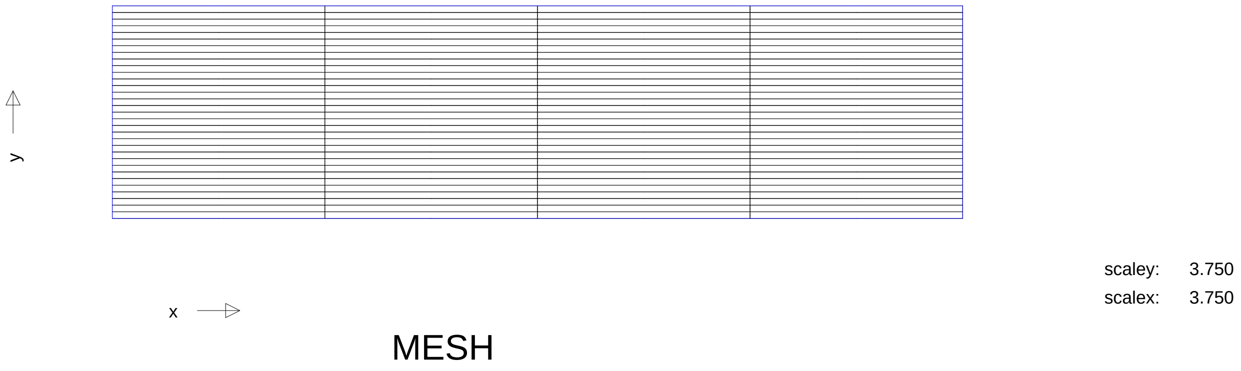
$\mathbb{D}$	Bi	Bi	GMRES	GMRESR(10)
10^{-3}	42	8	7	7
10^{-4}	nc	52	48	47
10^{-5}	nc	392	207	189
10^{-6}	nc	nc	438	350
10^{-7}	nc	nc	$>$ 500	389

Convergence for lumped ILU is only weakly dependent on the angle

Navier Stokes

- channel flow, quadratic finite elements
- coupled system, stretched grids

4×32 grid



Iterations for $n = 8$, ILUD, Bi-CGSTAB

m	no lumping	lumping
4	343	418
8	395	505
16	807	701
32	nc	1497

7. Conclusions

- Lumped preconditioners are robust
- For difficult problems, lumped preconditioners leads to (much) faster convergence
- There are problems, where lumped preconditioners are less efficient
- In many problems, a (lumped M)ILUD preconditioner is more efficient than a (lumped M)ILU preconditioner