

**ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL  
 EQUATIONS  
 ( CTB2400 )**

**Thursday June 26 2025, 13:30-16:30**

1. (a) The amplification factor is defined by

$$Q(\lambda\Delta t) = \frac{w_{n+1}}{w_n},$$

where  $w_{n+1}$  results from applying one step of the method to the test equation  $y' = \lambda y$ .  
 First we calculate  $k_1$  and use  $f(t, y) = \lambda y$ :

$$\begin{aligned} & k_1 = \lambda(w_n + \theta\Delta tk_1) \\ \Rightarrow & k_1 = \lambda w_n + \theta\lambda\Delta tk_1 \\ \Rightarrow & k_1 - \theta\lambda\Delta tk_1 = \lambda w_n \\ \Rightarrow & (1 - \theta\lambda\Delta t)k_1 = \lambda w_n \\ \Rightarrow & k_1 = \frac{\lambda}{1 - \theta\lambda\Delta t} w_n \end{aligned}$$

Then we calculate  $w_{n+1}$ :

$$\begin{aligned} w_{n+1} &= w_n + \Delta tk_1 \\ &= w_n + \frac{\lambda\Delta t}{1 - \theta\lambda\Delta t} w_n \\ &= \left(1 + \frac{\lambda\Delta t}{1 - \theta\lambda\Delta t}\right) w_n \\ &= \frac{1 + (1 - \theta)\lambda\Delta t}{1 - \theta\lambda\Delta t} w_n \end{aligned}$$

Finally division by  $w_n$  gives

$$Q(\lambda\Delta t) = \frac{1 + (1 - \theta)\lambda\Delta t}{1 - \theta\lambda\Delta t}.$$

- (b) The local truncation error for the test equation is given by

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t} y_n, \quad (1)$$

$e^{\lambda\Delta t}$  can be expanded by the use of Taylor expansions around  $\Delta t = 0$ :

$$e^{\lambda\Delta t} = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

$\frac{1}{1 - \frac{1}{2}\lambda\Delta t}$  can be expanded by the use of Taylor expansions around  $\Delta t = 0$ :

$$\begin{aligned} \frac{1}{1 - \frac{1}{2}\lambda\Delta t} &= 1 + \frac{1}{2}\lambda\Delta t + \left(\frac{1}{2}\lambda\Delta t\right)^2 + \mathcal{O}(\Delta t^3) \\ &= 1 + \frac{1}{2}\lambda\Delta t + \frac{1}{4}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3). \end{aligned}$$

This means the amplification factor can be rewritten to:

$$\begin{aligned}
Q(\lambda\Delta t) &= \left(1 + \frac{1}{2}\lambda\Delta t\right) \frac{1}{1 - \frac{1}{2}\lambda\Delta t} \\
&= \left(1 + \frac{1}{2}\lambda\Delta t\right) \left(1 + \frac{1}{2}\lambda\Delta t + \frac{1}{4}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3)\right) \\
&= 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3).
\end{aligned}$$

Substitution of the above in the local truncation error results in:

$$\begin{aligned}
\tau_{n+1} &= \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t} y_n \\
&= \frac{\left(1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3)\right) - \left(1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3)\right)}{\Delta t} y_n \\
&= \frac{\mathcal{O}(\Delta t^3)}{\Delta t} y_n \\
&= \mathcal{O}(\Delta t^2).
\end{aligned}$$

(c) For stability,

$$|Q(\lambda\Delta t)| \leq 1,$$

must hold for all eigenvalues of the linear initial value problem, with  $Q$  the amplification factor of the given method.

First, we determine the eigenvalues of the matrix  $A$ . Subsequently, the eigenvalues are substituted into the amplification factor.

The eigenvalues of the matrix  $A$  are given by  $\lambda_1 = -3$  and  $\lambda_2 = -1$ .

We first consider  $\lambda_1 = -3$ :

$$\begin{aligned}
Q(\lambda_1\Delta t) &= \frac{1 - \frac{3}{2}\Delta t}{1 + \frac{3}{2}\Delta t} \\
&= \frac{2 - 3\Delta t}{2 + 3\Delta t}.
\end{aligned}$$

Applying the stability criteria results in

$$-1 \leq \frac{2 - 3\Delta t}{2 + 3\Delta t} \leq 1,$$

and multiplying with the denominator gives

$$-2 - 3\Delta t \leq 2 - 3\Delta t \leq 2 + 3\Delta t.$$

First we solve the left inequality:

$$\begin{aligned}
& -2 - 3\Delta t \leq 2 - 3\Delta t \\
\Rightarrow & \qquad \qquad \qquad -2 \leq 2
\end{aligned}$$

As this inequality is always true, we obtain no new information.

Then we solve the right inequality:

$$\begin{aligned}
& 2 - 3\Delta t \leq 2 + 3\Delta t \\
\Rightarrow & \qquad \qquad \qquad 2 - 6\Delta t \leq 2 \\
\Rightarrow & \qquad \qquad \qquad -6\Delta t \leq 0
\end{aligned}$$

As this inequality is always true for  $\Delta t > 0$ , we obtain no new information.

Repeating this for  $\lambda_2 = -1$  also results in no new information.

Therefore the time integration method applied to the initial value problem is stable for

$$\Delta t > 0.$$

(d) First we calculate  $\mathbf{k}_1$ , where we use  $\Delta t = 1$ :

$$\begin{aligned} \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \mathbf{k}_1 \right) + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{k}_1 \\ \Rightarrow \mathbf{k}_1 - \frac{1}{2} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{k}_1 &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix} \mathbf{k}_1 &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ \Rightarrow \mathbf{k}_1 &= \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ \Rightarrow \mathbf{k}_1 &= \begin{pmatrix} -0.9333 \\ 0.2667 \end{pmatrix}. \end{aligned}$$

Then we calculate  $\mathbf{w}_1$ , again with  $\Delta t = 1$ :

$$\begin{aligned} \mathbf{w}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -0.9333 \\ 0.2667 \end{pmatrix} \\ &= \begin{pmatrix} 0.0667 \\ 0.2667 \end{pmatrix} = \begin{pmatrix} \frac{1}{15} \\ \frac{4}{15} \end{pmatrix}. \end{aligned}$$

2. (a)  $p = \sqrt{3}$  is a fixed point of the function  $g$  if  $g(\sqrt{3}) = \sqrt{3}$ . We calculate  $g(p)$ :

$$\begin{aligned} g(p) &= g(\sqrt{3}) \\ &= -\frac{1}{3}(\sqrt{3})^2 + \sqrt{3} + 1 \\ &= -1 + \sqrt{3} + 1 \\ &= \sqrt{3}, \end{aligned}$$

So  $p = \sqrt{3}$  is indeed a fixed-point of the function  $g$ .

- (b) The function  $g$  is a polynomial and polynomials are continuous everywhere, so there for  $g$  also is continuous on the interval  $[1, 2]$ .
- (c) First note that  $g$  is a parabola opening to the bottom and therefore  $g$  has a maximum at the point where  $g'(x) = 0$ . We solve this equation:

$$\begin{aligned} & g'(x) = 0 \\ \Rightarrow & -\frac{2}{3}x + 1 = 0 \\ \Rightarrow & x = \frac{3}{2}, \end{aligned}$$

so the position of the maximum of  $g$  is located in the interval  $[1, 2]$  and attains the value  $g(3/2) = 7/4$ . Therefore we conclude

$$g(x) \leq 2 \quad \text{for } x \in [1, 2].$$

The function  $g$  attains its minimum on the boundary of the interval  $[1, 2]$ , so evaluation of  $g$  at these points gives

$$\begin{aligned} g(1) &= 5/3, \\ g(2) &= 5/3. \end{aligned}$$

Therefore we conclude

$$g(x) \geq 1 \quad \text{for } x \in [1, 2].$$

Putting everything together, we have found

$$1 \leq g(x) \leq 2, \quad \text{for } x \in [1, 2].$$

as requested.

- (d) The derivative of  $g$  is given by

$$g'(x) = -\frac{2}{3}x + 1,$$

which is a monotonous decreasing function. Therefore the minimum and maximum value are located on the boundary of the interval, leading to

$$\begin{aligned} & g'(2) \leq g'(x) \leq g'(1) \\ \Rightarrow & -1/3 \leq g'(x) \leq 1/3 \\ \Rightarrow & |g'(x)| \leq 1/3 \end{aligned}$$

So  $k = 1/3$ .

(e) *Remark: The final value of  $p_1$  should be given in 4 significant digits. Failure to do so results in a deduction of  $\frac{1}{4}$  point if  $p_1 = \sqrt[5]{3}$  is stated.*

*Remark: Calculation of  $p_2$  using  $p_1 = \sqrt[5]{3}$  is incorrect, and causes a deduction of  $\frac{1}{4}$  point.*

*Remark: The final value of  $p_2$  should be given in 4 significant digits. Failure to do so results in a deduction of  $\frac{1}{4}$  point if  $p_2 = \sqrt[47]{27}$  is stated.*

Straightforward application of the fixed point iteration gives

$$\begin{aligned} p_1 &= g(p_0) \\ &= g(2.000) \\ &= 1.667, \end{aligned}$$

and

$$\begin{aligned} p_2 &= g(p_1) \\ &= g(1.667) \\ &= 1.741. \end{aligned}$$

3. (a) Using central differences for the second order derivative at a node  $x_j = j\Delta x$  gives

$$y''(x_j) \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{\Delta x^2} =: Q(\Delta x). \quad (2)$$

Here,  $y_j := y(x_j)$ . Next, we will prove that this approximation is second order accurate, that is  $|y''(x_j) - Q(\Delta x)| = \mathcal{O}(\Delta x^2)$ .

Using Taylor's Theorem around  $x = x_j$  gives

$$\begin{aligned} y_{j+1} &= y(x_j + \Delta x) = y(x_j) + \Delta x y'(x_j) + \frac{\Delta x^2}{2} y''(x_j) + \frac{\Delta x^3}{3!} y'''(x_j) + \frac{\Delta x^4}{4!} y''''(\eta_+), \\ y_{j-1} &= y(x_j - \Delta x) = y(x_j) - \Delta x y'(x_j) + \frac{\Delta x^2}{2} y''(x_j) - \frac{\Delta x^3}{3!} y'''(x_j) + \frac{\Delta x^4}{4!} y''''(\eta_-). \end{aligned} \quad (3)$$

Here,  $\eta_+$  and  $\eta_-$  are numbers within the intervals  $(x_j, x_{j+1})$  and  $(x_{j-1}, x_j)$ , respectively. Substitution of these expressions into  $Q(\Delta x)$  gives

$$|y''(x_j) - Q(\Delta x)| = \mathcal{O}(\Delta x^2).$$

This leads to the following discretisation formula for internal grid nodes:

$$\frac{-w_{j-1} + 2w_j - w_{j+1}}{\Delta x^2} + (x_j + 1)w_j = 1. \quad (4)$$

Here,  $w_j$  represents the numerical approximation of the solution  $y_j$ . To deal with the boundary  $x = 0$ , we use a virtual node at  $x = -\Delta x$ , and we define  $y_{-1} := y(-\Delta x)$ . Then, using central differences at  $x = 0$  gives

$$0 = y'(0) \approx \frac{y_1 - y_{-1}}{2\Delta x} =: Q_b(\Delta x). \quad (5)$$

Using Taylor's Theorem, gives

$$\begin{aligned} Q_b(\Delta x) &= \\ &= \frac{y(0) + \Delta x y'(0) + \frac{\Delta x^2}{2} y''(0) + \frac{\Delta x^3}{3!} y'''(\eta_+)}{2\Delta x} \\ &\quad - \frac{y(0) - \Delta x y'(0) + \frac{\Delta x^2}{2} y''(0) - \frac{\Delta x^3}{3!} y'''(\eta_-)}{2\Delta x} \\ &= y'(0) + \mathcal{O}(\Delta x^2). \end{aligned}$$

Again, we get an error of  $\mathcal{O}(\Delta x^2)$ .

- (b) With respect to the numerical approximation at the virtual node, we get

$$\frac{w_1 - w_{-1}}{2\Delta x} = 0 \quad \Leftrightarrow \quad w_{-1} = w_1. \quad (6)$$

The discretisation at  $x = 0$  is given by

$$\frac{-w_{-1} + 2w_0 - w_1}{\Delta x^2} + w_0 = 1. \quad (7)$$

Substitution of equation (6) into the above equation, yields

$$\frac{2w_0 - 2w_1}{\Delta x^2} + w_0 = 1. \quad (8)$$

Subsequently, we consider the boundary  $x = 1$ . To this extent, we consider its neighbouring point  $x_{n-1}$  and substitute the boundary condition  $w_n = y(1) = y_n = 1$  into equation (4) to obtain

$$\frac{-w_{n-2} + 2w_{n-1}}{\Delta x^2} + (x_{n-1} + 1)w_{n-1} \quad (9)$$

$$= 1 \quad (10)$$

$$= 1 + \frac{1}{\Delta x^2}. \quad (11)$$

This concludes our discretisation of the boundary conditions. In order to get a symmetric discretisation matrix, one divides equation (8) by 2.

Next, we use  $\Delta x = 1/3$ . From equations (4, 8, 11) we obtain the following system

$$\begin{aligned} 9\frac{1}{2}w_0 - 9w_1 &= \frac{1}{2} \\ -9w_0 + 19\frac{1}{3}w_1 - 9w_2 &= 1 \\ -9w_1 + 19\frac{2}{3}w_2 &= 10. \end{aligned}$$

- (c) The Gershgorin circle theorem states that the eigenvalues of a square matrix  $\mathbf{A}$  are located in the complex plane in the union of circles

$$|z - a_{ii}| \leq \sum_{\substack{j \neq i \\ j=1}}^n |a_{ij}| \quad \text{where } z \in \mathbb{C} \quad (12)$$

For the  $3 \times 3$  matrix derived in part (b) we have

- For  $i = 1$ :

$$\left| z - 9\frac{1}{2} \right| \leq 9 \quad \Rightarrow \quad |\lambda_1|_{\min} \geq \frac{1}{2} \quad (13)$$

- For  $i = 2$ :

$$\left| z - 19\frac{1}{3} \right| \leq 18 \quad \Rightarrow \quad |\lambda_2|_{\min} \geq 1\frac{1}{3} \quad (14)$$

- For  $i = 3$ :

$$\left| z - 19\frac{2}{3} \right| \leq 9 \quad \Rightarrow \quad |\lambda_3|_{\min} \geq 10\frac{2}{3} \quad (15)$$

Hence, a lower bound for the smallest eigenvalue is  $\frac{1}{2}$ . For a symmetric matrix  $\mathbf{A}$  we have

$$\|\mathbf{A}^{-1}\| = \frac{1}{|\lambda|_{\min}} \leq 2 \quad (16)$$

This proves that the finite-difference scheme is stable, e.g., with constant  $C = 2$ .