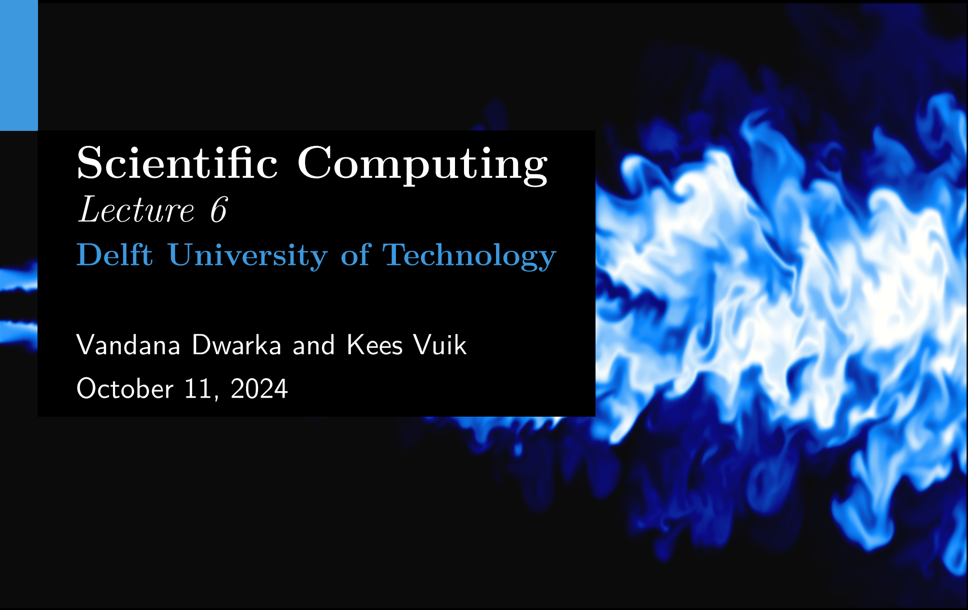

Scientific Computing

Lecture 6

Delft University of Technology

Vandana Dwarka and Kees Vuik

October 11, 2024

Today

- Remaining Chapter 4:
    - Partial Pivoting
    - Bandwidth
    - Cholesky
- Chapter 5:
    - Basic Iterative Methods + convergence
    - Richardson Method

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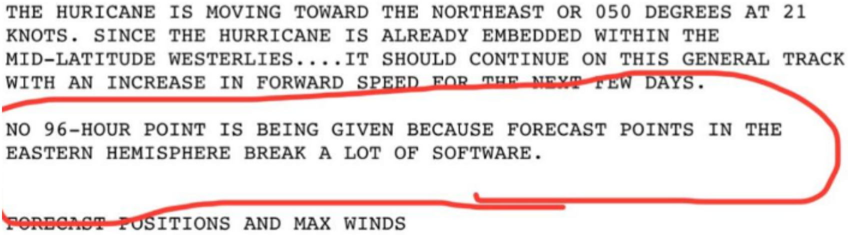

THE HURICANE IS MOVING TOWARD THE NORTHEAST OR 050 DEGREES AT 21 KNOTS. SINCE THE HURRICANE IS ALREADY EMBEDDED WITHIN THE MID-LATITUDE WESTERLIES....IT SHOULD CONTINUE ON THIS GENERAL TRACK WITH AN INCREASE IN FORWARD SPEED FOR THE NEXT FEW DAYS.

NO 96-HOUR POINT IS BEING GIVEN BECAUSE FORECAST POINTS IN THE EASTERN HEMISPHERE BREAK A LOT OF SOFTWARE.

FORECAST POSITIONS AND MAX WINDS

Recap

LU Decomposition for General Matrices

$A \in \mathbb{R}^{n\times n}$ $\quad A$ is non singular

$$A^{(k-1)} = M_{k-1}M_{k-2}\cdot M_1 A = \begin{pmatrix} A_{11}^{(k-1)} & A_{12}^{(k-1)} \\ 0 & A_{22}^{(k-1)} \end{pmatrix}$$

M_k is Gauss transformation: $M_k = I - a^{(k)}e_k^T$

$$e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } a^{(k)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{k+1,k}^{(k-1)}/a_{k,k}^{(k-1)} \\ \vdots \\ a_{n,k}^{(k-1)}/a_{k,k}^{(k-1)} \end{pmatrix}$$

$a^{(k)}$ is the Gauss vector and we must have pivot $a_{k,k}^{(k-1)} \neq 0$.

$M_k = I - a^{(k)}e_k^T$

Recap

LU Decomposition for General Matrices

Gaussian elimination gives:

$$\begin{aligned} Au &= f \\ M_1 A u &= M_1 f \\ &\vdots \\ M_{n-1} \dots M_1 A u &= M_{h-1} \dots M_1 f \end{aligned}$$

Recap

LU Decomposition for General Matrices

Gaussian elimination gives:

$$Au = f$$

$$M_1 A u = M_1 f$$

$$\vdots$$

$$M_{n-1} \dots M_1 A u = M_{h-1} \dots M_1 f$$

$$M_{n-1} \dots M_1 A u = Uu = M_{n-1} \dots M_1 f \Rightarrow \underbrace{M_1^{-1} M_2^{-1} \quad M_{n-1}^{-1}}_{\text{is this } L?} Uu = f$$

Recap

LU Decomposition for General Matrices

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$$M_{n-1} \dots M_1 A u = Uu = M_{n-1} \dots M_1 f \Rightarrow \underbrace{M_1^{-1} M_2^{-1} \quad M_{n-1}^{-1}}_{\text{is this } L?} Uu = f$$

Yes! See Lemma 4.4.2, which states

$$\begin{aligned} (M_{n-1} \cdot \dots \cdot M_1)^{-1} &= M_1^{-1} \dots M_{n-1}^{-1} \\ &= \prod_{k=1}^{n-1} \left(I + \boldsymbol{\alpha}^{(k)} \mathbf{e}_k^T \right) \\ &= I + \sum_{k=1}^{n-1} \boldsymbol{\alpha}^{(k)} \mathbf{e}_k^T \end{aligned}$$

Example: LU w/o Pivoting (from *Numerical Linear Algebra*)

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

Step $k = 1$: select pivot $a_{11} = 2$ and construct $a^{(1)}$

$$a^{(1)} = \begin{pmatrix} 0 \\ \frac{a_{12}}{a_{11}} \\ \frac{a_{13}}{a_{11}} \\ \frac{a_{14}}{a_{11}} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \quad M_1 = I - a^{(1)} {e_1}^\top = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{bmatrix}$$

$$M_1 A = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix}$$

Example: LU w/o Pivoting

$$M_1A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix}$$

Step $k = 2$: select new pivot $a_{22} = 1$ and construct $a^{(2)}$

$$a^{(2)} = \begin{pmatrix} 0 \\ 0 \\ \frac{a_{32}}{a_{22}} \\ \frac{a_{42}}{a_{22}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \quad M_2 = I - a^{(2)}{e_2}^{\top} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -4 & & 1 \end{bmatrix}$$

$$M_2(M_1A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -3 & 1 & \\ & -4 & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & 3 & 5 & 5 \\ & 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & 2 & 4 \end{bmatrix}$$

Example: LU w/o Pivoting

$$M_2(M_1A) = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & {\color{red}2} & 2 \\ & & 2 & 4 \end{bmatrix}$$

Step $k = 3$: select new pivot ${\color{red}a_{33} = 2}$ and construct $a^{(3)}$

$$a^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{a_{43}}{a_{33}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad M_3 = I - a^{(3)} {e_3}^\top = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}$$

$$M_3(M_2M_1A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & 2 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{bmatrix} = U$$

Example: LU w/o Pivoting

Final step: the full LU decomposition requires $L = M_1^{-1}M_2^{-1}M_3^{-1}$. By lemma 4.4.2, the inverse of M_1 is M_1 itself, but with the sign of each entry below the diagonal switched (due to the plus sign in lemma 4.4.2):

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ -4 & & 1 & \\ -3 & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & & 1 & \\ 3 & & & 1 \end{bmatrix}$$

The inverses of M_2 and M_3 are obtained similarly. The product $M_1^{-1}M_2^{-1}M_3^{-1}$ is the unit lower-triangular matrix with the nonzero sub diagonal entries of M_1^{-1}, M_2^{-1} and M_3^{-1}. Thus we get:

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 3 & 1 & \\ 3 & 4 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 2 \\ & & & 2 \end{bmatrix}}_{U}$$

Recap

Example: LU Decomposition for General Matrices (from video lecture)

We start with $A, k = 2$, so $k = 1$ step done. Red block is $A_{2,2}^{(1)}$.

$$A^{(k-1)} = A^{(1)} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & \color{red}{2} & \color{red}{-1} \\ 0 & \color{red}{-1} & \color{red}{2} \end{pmatrix}$$

Pivot is $a_{2,2}^{(1)} = 2$, so Gauss vector becomes $a^{(2)} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$

$$M_2 = I - a^{(2)} {e_1}^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Recap

Example: LU Decomposition for General Matrices

$$M_2 = I - a^{(2)} {e_1}^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

This gives:

$$M_2 A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} = U$$

Recap

§ 4.5.2: Rounding Errors

$$\hat{L}, \hat{U} \Rightarrow \hat{L} \cdot \hat{U} = A + E, \|E\| \leqslant n\mu \|\hat{L}\| \|\hat{U}\|$$

Recap

§ 4.5.2: Rounding Errors

$$\hat{L}, \hat{U} \Rightarrow \hat{L} \cdot \hat{U} = A + E, \|E\| \leqslant n\mu \|\hat{L}\| \|\hat{U}\|$$

Example

Suppose we want to solve $Au = f$ on a computer which rounds to 3 digits:

$$\begin{pmatrix} 0.001 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Recap

§ 4.5.2: Rounding Errors

$$\hat{L}, \hat{U} \Rightarrow \hat{L} \cdot \hat{U} = A + E, \|E\| \leqslant n\mu\|\hat{L}\|\|\|\hat{U}\|$$

Example

Suppose we want to solve $Au = f$ on a computer which rounds to 3 digits:

$$\begin{pmatrix} 0.001 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Well conditioned problem but due to your LU decomposition becomes unstable.

Pivoting: Introduction

Suppose we want to solve $Au = f$ on a computer which rounds to 3 digits:

$$\begin{pmatrix} 0.001 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \hat{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u + \Delta u$$

§ 4.6: Pivoting

Gaussian Elimination with Partial Pivoting

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & & \\ \vdots & & \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

Determine $\hat{j}$ such that $\left|a_{\hat{j},1}\right| \geq |a_{j,1}|$ for $1 \leq j \leq n$.

Next, change row 1 with row $\hat{j}$:

$$\begin{pmatrix} a_{\hat{j}1} & \cdots & \cdots & a_{\hat{j}n} \\ a_{21} & \ldots & \cdots & a_{2n} \\ a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

§ 4.6: Pivoting

Gaussian Elimination with Partial Pivoting

Linear system to solve becomes:

$$\begin{pmatrix} a_{\hat{j}1} & \cdots & \cdots & a_{\hat{j}n} \\ a_{21} & \ldots & \cdots & a_{2n} \\ a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & & & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} f_{\hat{j}} \\ f_2 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Note: new multiplier becomes $\frac{|a_{i,1}|}{|a_{j,1}|}$ for $1 \leq i \leq n$.

We can show at the end that $PA = LU$, which leads to solving:

$$\begin{aligned} Au &= f & &\text{solve } y \text{ from } Ly = Pf \\ PAu &= Pf & &\text{solve } u \text{ from } Uu = y \end{aligned}$$

$$L \underbrace{Uu}_{y} = Pf$$

Example: LU with Pivoting

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ \color{red}{8} & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

Step $k = 1$: we start with selecting the largest value in the first column (in original matrix A, element $\color{red}{a_{13} = 8}$).

That will be our pivot so we switch rows 1 and 3 by multiplying with the permutation matrix P_1:

$$\underbrace{\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{P_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} \color{red}{8} & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

Example: LU with Pivoting

$$P_1A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

Step $k = 1$: select pivot new $a_{11} = 8$ and construct $a^{(1)}$

$$a^{(1)} = \begin{pmatrix} 0 \\ \frac{a_{12}}{a_{11}} \\ \frac{a_{13}}{a_{11}} \\ \frac{a_{14}}{a_{11}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}, \qquad M_1 = I - a^{(1)}{e_1}^\top = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ -\frac{1}{4} & & 1 & \\ -\frac{3}{4} & & & 1 \end{bmatrix}$$

$$M_1(P_1A) = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ -\frac{1}{4} & & 1 & \\ -\frac{3}{4} & & & 1 \end{bmatrix} \begin{bmatrix} 8 & 7 & 9 & 5 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix} =$$

$$\begin{bmatrix} 8 & 7 & 9 & 5 \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{bmatrix}$$

Example: LU with Pivoting

$$M_1 P_1 A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{bmatrix}$$

Step $k = 2$: we start with selecting the largest value in the second column, which is element $a_{24} = \frac{7}{4}$.

That will be our pivot so we switch rows 2 and 4 by multiplying with the permutation matrix P_2:

$$\underbrace{\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix}}_{P_2} \begin{bmatrix} 8 & 7 & 9 & 5 \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \end{bmatrix} = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

Example: LU with Pivoting

$$P2(M_1 P_1 A) = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

Step $k = 2$: select pivot new $a_{22} = \frac{7}{4}$ and construct $a^{(2)}$

$$a^{(2)} = \begin{pmatrix} 0 \\ 0 \\ \frac{a_{32}}{a_{22}} \\ \frac{a_{42}}{a_{22}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{7} \\ -\frac{2}{7} \end{pmatrix}, \quad M_2 = I - a^{(2)} {e_2}^\top = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \frac{3}{7} & 1 & \\ & \frac{2}{7} & & 1 \end{bmatrix}$$

$$M_2(P_2 M_1 P_1 A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \frac{3}{7} & 1 & \\ & \frac{2}{7} & & 1 \end{bmatrix} \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & -\frac{3}{4} & -\frac{5}{4} & -\frac{5}{4} \\ & -\frac{1}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} =$$

$$\begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{2}{7} & \frac{4}{7} \\ & & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix}$$

Example: LU with Pivoting

$$M_2(P_2M_1P_1A) = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{2}{7} & \frac{4}{7} \\ & & {\color{red}-\frac{6}{7}} & -\frac{2}{7} \end{bmatrix}$$

Step $k = 3$: interchange rows 3 and 4, by multiplying by P_3 as largest value in magnitude is $a_{34} = -\frac{6}{7}$:

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}}_{P_3} \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{2}{7} & \frac{4}{7} \\ & & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & {\color{red}-\frac{6}{7}} & -\frac{2}{7} \\ & & -\frac{2}{7} & \frac{4}{7} \end{bmatrix}$$

Example: LU with Pivoting

$$P_3 M_2 P_2 M_1 P_1 A = \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & -\frac{2}{7} & \frac{4}{7} \end{bmatrix}$$

Step $k = 3$: select new pivot $a_{33} = -\frac{6}{7}$ and construct $a^{(3)}$

$$a^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{a_{43}}{a_{33}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}, \quad M_3 = I - a^{(3)} {e_3}^\top = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{1}{3} & 1 \end{bmatrix}$$

$$M_3(P_3 M_2 P_2 M_1 P_1 A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & -\frac{2}{7} & \frac{4}{7} \end{bmatrix} =$$

$$\begin{bmatrix} 8 & 7 & 9 & 5 \\ & \frac{7}{4} & \frac{9}{4} & \frac{17}{4} \\ & & -\frac{6}{7} & -\frac{2}{7} \\ & & & \frac{2}{3} \end{bmatrix} = U$$

§ 4.8: Diagonal Dominance and Pivoting

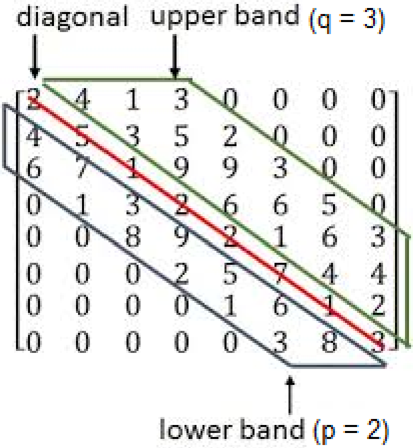
Is pivotting necessary?
If A is strictly diagonal dominant $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{j,i}|$, then partial pivoting is not needed.

§ 4.9: Cholesky Decomposition

If A SPD, then $A = CC^\top$:

- Symmetric: $A = A^\top$
- Positive Definite: $x^\top Ax > 0 \quad x \in \mathbb{R}^1 x \neq 0$

§ 4.10: Bandwidth



$$\begin{bmatrix} 3 & 4 & 1 & 3 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 5 & 2 & 0 & 0 & 0 \\ 6 & 7 & 1 & 9 & 9 & 3 & 0 & 0 \\ 0 & 1 & 3 & 2 & 6 & 6 & 5 & 0 \\ 0 & 0 & 8 & 9 & 2 & 1 & 6 & 3 \\ 0 & 0 & 0 & 2 & 5 & 7 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 8 & 3 \end{bmatrix}$$

Pivoting destroys the 'nice' bandwidth

Industrial application: unstructured grids, band structure also gone (§ 4.11).

§ 5.2: Basic Iterative Methods (Introduction)

We assume that A is an invertable matrix so A^{-1} exits.
Iterative methods seek solutions to:

$$Au = f$$

by forming the sequences of iterates by

$$\left\{u^k\right\}_{k\geq 0} \text{ where } u^k \to u \text{ for } k \to \infty$$

where u_0 is the initial guess.

§ 5.3: Basic Iterative Methods

Notation: u^k is the k-th iteration. We define:

- Error: $e^k = u - u^k$
- Residual: $r^k = f - Au^k$
- Residual equation: $Ae^k = Au - Au^k = f - Au^k = r^k$

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We start with a splitting method: $A = M - N$.
$Au = f \Rightarrow (M - N)u = f = Mu = f + Nu$

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$Mu^{k+1} = f + Nu^k$
$u^{k+1} = M^{-1}f + M^{-1}Nu^k$

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$Au = f \Rightarrow (M - N)u = f = Mu = f + Nu$
$Mu^{k+1} = f + Nu^k$
$u^{k+1} = M^{-1}f + M^{-1}Nu^k$

Substituting $n = M - A$ we get:

$$\begin{aligned} u^{k+1} &= M^{-1}f + M^{-1}(M - A)u^k \\ &= u^k + M^{-1}(f - Au^k) \\ &= u^k + M^{-1}(r^k) \end{aligned}$$

§ 5.3: Basic Iterative Methods

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- Error: $e^k = u - u^k$
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Substituting $n = M - A$ we get:

$$\begin{aligned} u^{k+1} &= M^{-1}f + M^{-1}(M - A)u^k \\ &= u^k + M^{-1}(f - Au^k) \\ &= u^k + M^{-1}(r^k) \end{aligned}$$

Note: M^{-1} only for analysis!

§ 5.4.1: Convergence

Does this iterative scheme converge? I.e. $u^{k+1} \to u$ as k goes to infinity?

§ 5.4.1: Convergence

Does this iterative scheme converge? I.e. $u^{k+1} \to u$ as k goes to infinity?

$$\begin{aligned} u^{k+1} &= u^k + M^{-1}\left(f - Au^k\right) \\ u - u^{k+1} &= u - u^k - M^{-1}\left(Au - Au^k\right) \\ e^{k+1} &= e^k - M^{-1}Ae^k \Rightarrow e^{k+1} = \left(I - M^{-1}A\right)e^k \end{aligned}$$

§ 5.4.1: Convergence

Does this iterative scheme converge? I.e. $u^{k+1} \to u$ as k goes to infinity?

$$\begin{aligned} u^{k+1} &= u^k + M^{-1}\left(f - Au^k\right) \\ u - u^{k+1} &= u - u^k - M^{-1}\left(Au - Au^k\right) \\ e^{k+1} &= e^k - M^{-1}Ae^k \Rightarrow e^{k+1} = \left(I - M^{-1}A\right)e^k \end{aligned}$$

We define the iteration matrix $B = I - M^{-1}A$ such that:

$$e^{k+1} = B^k e^0$$

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For convergence: $\lim_{k\rightarrow\infty}\left\|B^k\right\|_2 = 0$, which using theorem 2.7.2 is equivalent to:

$$\lim_{k\rightarrow\infty}\left\|B^k\right\|_2 = 0 \Leftrightarrow \rho(B) < 1$$

§ 5.3.2: Richardson Method

One simple method is the Richardson Method: $M = I, N = I - A$, then $u^{k+1} = f + (I - A)u^k$

Hence, iteration matrix $B = I - M^{-1}A = I - A$.

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Hence, iteration matrix $B = I - M^{-1}A = I - A$.

$$u^0 = 0$$

$$u^1 = f$$

$$u^2 = f + (I - A)f$$

$$u^3 = f + (I - A)f + (I - A)^2 f$$

$$\vdots$$

$$u^{k+1} = \sum_{i=0}^{k} (I - A)^i f$$

Does this iterative scheme converge? I.e. k to infinity and $u^{k+1} \to u$, implies the sum $\sum_{i=0}^{k} (I - A)^i$ should go to A^{-1}.