The power of shaking hands

Dion Gijswijt

Although very simple to prove, the handshaking lemma can be a powerful tool in the hands of a combinatorialist. Here, I will show you some colourful applications and pose some challenges to you, dear reader.

Let me start by posing a simple question. Perhaps you already know the answer. If not, take a minute or two to see if you can solve it!

> *There are seven people at a party. Is it possible that each of them shakes hands with exactly three others?*

In the language of graph theory, we are asking for a graph[1] with 7 nodes in which every node has degree 3. The following simple observation will be a central idea in this article.

Lemma 1 (handshaking lemma)**.** *Let G be a graph. Then G has an* even *number of* odd *degree nodes.*

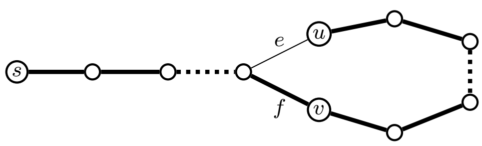
The proof is not hard. For all nodes u, count the edges incident to u. The total count equals the sum of the degrees of the nodes. It also equals twice the total number of edges, since every edge is counted twice. Hence, the sum of the degrees must be *even*. Therefore, the number of odd degree nodes must be even.

Hamiltonian paths

By a *Hamiltonian path* in a graph G, we mean a path that visits every node exactly once. Here, we will be interested in Hamiltonian paths between to given nodes s and t (we ignore the direction in which the path is traversed and only consider the set of edges on the path). Not every graph has a Hamiltonian path, and it is NP-hard to decide if a given graph has a Hamiltonian path between two given nodes. However, if we have one such path, we can sometimes conclude that there must be another.

Theorem 1 (Smith)**.** *Let G be a graph. Suppose that s and t are the only nodes of even degree. Then, the number of Hamiltonian paths between s and t is* even.

Proof. Make an auxilary graph H. The nodes of H are the Hamiltonian paths in G that end in node s. Given a Hamiltonian path P between s and node u, its neighbouring Hamiltonian paths in H are defined as follows. Take any of the $d(u)-1$ edges incident to u that are not in the path P, say edge e. Adding e to P will create a circuit. There is a unique edge $f \neq e$ in the circuit such that after removing f, we obtain a new Hamiltonian path $Q = P + e - f$ between s and a node v. Now Q is declared to be a neighbour of P in the graph H. Clearly, if Q is a neighbour of P, then also P is a neighbour of Q.



Observe that the number of neighbours of a Hamiltonian path P between s and u is equal to $d(u)-1$. It follows that such a Hamiltonian path has odd degree in H if and only if $u = t$. By the handshaking lemma, H has an even number of odd degree nodes, which means that G has an even number of Hamiltonian paths between s and t. □

As an example, consider the graph in Figure 1. Nodes s and t have degree 2 and all other nodes have degree 3. One Hamiltonian path between s and t is indicated. By Smith's theorem, there are an even number of such paths, hence there must be at least one other such path. Can you find it?

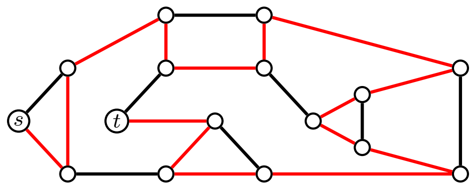


Figure 1: Find the second Hamiltonian s–t path.

The case that s and t have degree 2 and all other nodes have degree 3 is particularly interesting. Indeed, the degrees of the auxilary graph H will be either 1 or 2. The nodes of degree 1 correspond to Hamiltonian s–t paths. Hence, H just consists of disjoint paths linking nodes that correspond to Hamiltonian s–t paths. Simply following such a path from one end to the other gives an algorithm for finding a second Hamiltonian s–t path!
The catch is, that the graph H may be huge and the path we are following may have length exponential in the number of nodes of G. The theoretical complexity of this problem is unknown, but it belongs to a class of similar problems called PPA [2].

Sperner's lemma

Consider a triangulation of a triangle such as in Figure 2 (left). We want to colour the nodes with three colours (say red, green, and blue). A colouring is a *Sperner colouring* if the following are satisfied:

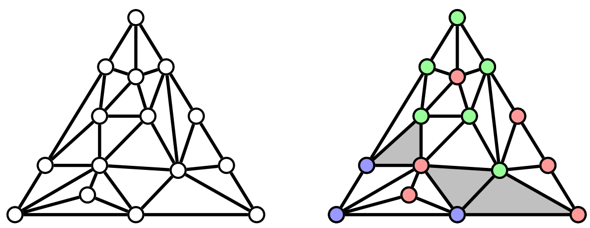


Figure 2: A triangulation (left) and a possible Sperner colouring (right).

(i) The three vertices of the large triangle have different colours.
(ii) The colour of a vertex of the large triangle does not occur on the opposite side of the triangle.

[1] All graphs will be finite and undirected

In Figure 2 (right), you see a Sperner colouring. Three of the smaller triangles are shaded because they are *complete*, meaning that their vertices have three different colours. Can you find a Sperner colouring with fewer than three complete triangles? How about a Sperner colouring with *no* complete triangles?

Lemma 2 (Sperner, two-dimensional case)**.** *Given a triangulation of a triangle, every Sperner colouring has an odd number of complete triangles.*

Proof. Make a graph G as follows. The nodes of G are the small triangles in the triangulation. We connect two nodes of G by an edge if the corresponding triangles share a side with a red and a blue vertex. Now we add one special node t corresponding to the large triangle. In G we make a edge from t to every small triangle that has both a red and a green vertex on a side of the large triangle.

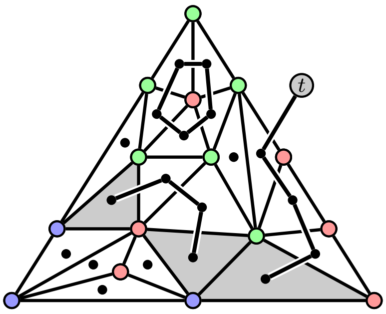


Figure 3: The graph corresponding to a Sperner colouring.

It is easy to check that a small triangle corresponds to a node of degree 0, 1, or 2. A small triangle has degree 1 if and only if it is complete. The degree of node t equals the number of colour changes along the red-green side of the large triangle. Hence, the degree of t is odd. By the handshaking lemma, G must have an even number of odd degree nodes. So besides t it must have an odd number of odd degree nodes. That is, the triangulation has an odd number of complete triangles. □

The general case of Sperners lemma deals with an n-simplex divided into small n-simplices and colouring the vertices with $n+1$ colours, concluding that there is an odd number of complete simplices (i.e. simplices with vertices of $n+1$ different colours). The proof is almost the same as in the two-dimensional case presented above.
Sperner's lemma can be used to prove Brouwer's fixed point theorem:

Theorem 2 (Brouwer's fixed point theorem)**.** *If $f : B^n \to B^n$ is a continuous map from the closed ball to itself, then $f(x) = x$ for some x.*

See the wonderful *Proofs from THE BOOK*[1] for a proof.
The many other applications of Sperner's lemma include fairy splitting rent and the possibility of dividing a birthday cake among n people in such a way that nobody prefers someone else's piece over their own. Here people may have different likes and dislikes regarding chocolat, marzipan, whipped cream and various pieces of fruit on the cake, of course [3].

Some puzzles

Now that we have seen some examples, the picture should be clear. To use the handshaking lemma, we first make a suitable auxilary graph. This graph should be such that the odd degree nodes correspond to the objects we are looking for. Here are three puzzles for you that can all be solved using the handshaking lemma. If you want to share a nice solution or other problem involving the handshaking lemma, don't hesitate to contact me!

Problem 1.

In Figure 4, you see a graph related to the board game Hex. The nodes N and S are coloured blue, the nodes E and W are coloured red. Suppose that we colour each of the other nodes blue or red. Prove that there is either a path from N to S using only blue nodes, or a path from E to W using only red nodes.
Hint. Consider the nodes that can be reached from N by a blue path, or from E by a red path. Colour those nodes green.

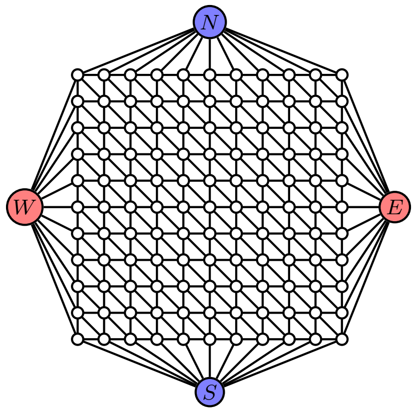


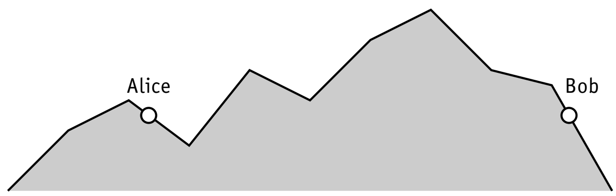
Figure 4: A graph related to the famous board game Hex.

Problem 2.

Let G be a graph. A set D of nodes is called *dominating* if every node of G has a neighbouring node in D, or is itself in D. Prove that G has an odd number of dominating sets.

Problem 3.

Let $f : [0,1] \to \mathbb{R}$ be a piecewise linear function such that $f(0) = f(1) = 0$, and $f(x) > 0$ for $0 < x < 1$. Two points (mountaineers) Alice and Bob are moving, in a continuous fashion, along the graph of f (a mountain). Alice starts at $(0,0)$, and Bob starts at $(1,0)$. At any moment in time, Alice and Bob must be at exactly the same height. Show that they are able to meet somewhere on the mountain.



References

[1] M. Aigner, G.M. Ziegler, *Proofs from the Book.* Vol. 274. Berlin: Springer, 2010

[2] C.H. Papadimitriou, *On the complexity of the parity argument and other inefficient proofs of existence.*, J. Comput. Systems Sci. 48 (1994): 498–532.

[3] F.E. Su, *Rental harmony: Sperner's lemma in fair division.* The American math. monthly 106.10 (1999): 930–942.

[4] G. Thomason, *Hamilton cycles and uniquely edge-colourable graphs.*, Ann. Discrete Math. 3 (1978): 259–268.