The card game SET: a mathematical challenge

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The card game SET connects to a wealth of mathematical ideas. Here, we will explore recent developments on the *cap set problem*.

SET is an amazing card came invented by the population geneticist Marsha Falco in 1974. After the official release in the US in 1990, it has become hugely popular among children, students and mathematicians alike. Apart from being fun to play, the game connects to various mathematical problems. The title of the recently published book

The Joy of Set:
The Many Mathematical Dimensions of a Seemingly Simple Card Game[3]

is a pretty accurate description.

The rules

The game consists of 81 cards. Each card has four attributes:

Number:	each card has One, Two, or Three equal symbols.
Shape:	the symbol is either Diamond, Oval, or Squiggle.
Colour:	the symbol can be Red, Green, or Purple.
Filling:	the symbol can be Open, Striped, or Full.

For each combination of attributes, there is exactly one card. The objective of the game is to quickly find a SET. A SET is a triple of cards whose attributes match: for each of the four attributes, the three cards are all equal or all different. See Figure 1 for two examples.

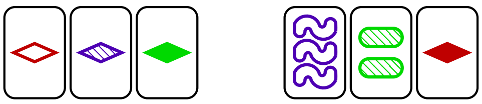


Figure 1: Two examples of a SET.
In the first, three attributes are all-equal and one is all-different.
In the second, all attributes are all-different.

The game begins by placing twelve cards face up on the table. The first player who spots a SET can take the corresponding three cards. The empty spots are filled with new cards. It may happen that there is no SET among the cards that are on the table. In that case three additional cards are laid out. The game ends when all cards are used up and no SET can be made using the remaining cards on the table. The one who has collected the most SETs wins.
Most toyshops sell the game, but you can also play online at `http://www.setgame.com/` (or one of the many unofficial sites).
Exercise 1. Can you find the six sets inside the twelve cards in Figure 2?

Enter geometry

The mathematical structure of SET becomes more transparent by encoding the cards in the following way. Let $\mathbb{F}_3 = \{0, 1, 2\}$ be the field of three elements (so we compute modulo 3). For each of the four attributes, we encode the three flavours by the three elements from $\mathbb{F}_3$. This way, the cards correspond to the 81 vectors in $\mathbb{F}_3^4$.
You can easily check that for three different 'cards' $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}_3^4$ the following are equivalent:

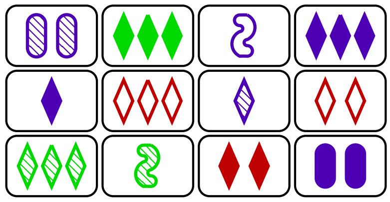


Figure 2: Find the six SETs.

- $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ form a SET,
- $\mathbf{x} + \mathbf{y} + \mathbf{z} = 0$,
- $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ form an affine line.

Being mathematicians, there is of course nothing stopping us from considering n-dimensional SET, where the cards are the elements of $\mathbb{F}_3^n$ and $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ form a SET if $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$.
Exercise 2. The average number of SETs contained in twelve cards is $\frac{220}{79}$. What is the average number of SET's contained in twelve cards from 5-dimensional SET?
Exercise 3. At the end of a game of SET, six cards are left on the table, see Figure 3. Determine the card represented by the question mark.

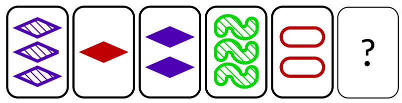


Figure 3: Six cards remaining. What card is missing?

Exercise 4. A *magic square* is a 3×3 array of SET cards in which all rows, colums and diagonals form a SET. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be three cards that do not form a SET. Show that we can always (uniquely) complete the following magic square:

x	**y**	?
z	?	?
?	?	?

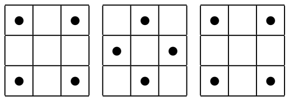
The resulting set of nine cards is an affine plane inside $\mathbb{F}_3^4$.

Caps

When playing SET, if often happens that there is no SET among the twelve cards on the table. On rare occasions, it may even happen that with an additional three cards there is still no SET! A collection of cards containing no SET is called a *cap*. The maximum size of a cap in n-dimensional SET is denoted $a(n)$.
Exercise 5. Show that $a(2) = 4$.

Proposition 1. *The maximum size of a cap in* $\mathbb{F}_3^3$ *is* $a(3) = 9$.

Proof. That a cap of size 9 exists can be seen in the figure below. The squares represent the $3 \times 3 \times 3$ elements of $\mathbb{F}_3^3$.



To show that $a(3) \leq 9$, suppose for contradiction that C is a cap of size 10. If we partition $\mathbb{F}_3^3$ into three parallel planes, then each contains at most $a(2) = 4$ points form C. Hence they also contain at least 2 points from C as $1 + 4 + 4 < 10$.

Let H_0 be a plane containing 2 or 3 points form C, say $a, b \in H_0 \cap C$. Let ℓ be the line through a and b. There are three planes H_1, H_2, H_3 that intersect in ℓ such that H_0 and H_1, H_2, H_3 together cover $\mathbb{F}_3^3$.

Since each H_i already contains a and b, it contains at most 2 other points from C. Also, H_0 contains at most 1 other point. This gives a total of $1 + 2 + 2 + 2 + 2 = 9$ points, a contradiction! □

The maximum size of a cap in $\mathbb{F}_3^4$ equals $a(4) = 20$. This is not so easy to show[1]. For a nice proof using Fourier transformations see the excellent overview paper [4]. Also for dimensions 5 and 6 the maximum size of a cap is known, see Table 1. Interestingly, up to dimension 6, there is (up to affine transformations) a unique cap of maximum size.

If you like to experiment with finding caps in n-dimensional SET, you can use the online 'cap-builder' by Jordan Awan [1].

Challenge 1. Find (with or without computer assistance) the number $a(7)$.

n	1	2	3	4	5	6	7
$a(n)$	2	4	9	20	45	112	?

Table 1: Known values of $a(n)$. Sequence A090245 in the OEIS

Asymptotic bounds of cap size

How does the maximum cap size $a(n)$ grow when we let $n \to \infty$? Clearly, we have $2^n \leq a_n \leq 3^n$ since $\{0,1\}^n$ is a cap in $\mathbb{F}_3^n$. The *asymptotic solidity* of caps is defined as

$$\sigma := \sup_{n \in \mathbb{N}} \sqrt[n]{a_n}. \tag{1}$$

It is not hard to see that if C is a cap in $\mathbb{F}_3^n$ and D is a cap in $\mathbb{F}_3^m$, then $C \times D$ is a cap in $\mathbb{F}_3^{n+m}$. Hence $a_{n+m} \geq a_n a_m$. This implies, by Fekete's lemma, that in fact the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and equals σ.

When C is a cap in $\mathbb{F}_3^n$, then we immediately have a lower bound $\sqrt[n]{|C|}$ for σ. The current record is $\sigma \geq 2.217389$ obtained by Edel[7] by constructing a very large cap in $\mathbb{F}_3^{480}$.

The *cap set problem* asks whether $\sigma = 3$. See for example Terence Tao's blog post[6]. In May, Jordan Ellenberg and myself solved the problem by showing that in fact $\sigma \leq 2.75511$.

One of the interesting features of the proof is that it is very short and elementary. Although the proof is too long to give here (about 2 pages!), let me give a rough sketch here.

Idea of the proof. Let C be a cap in $\mathbb{F}_3^n$. We want to upper bound $|C|$. Now suppose that $f : \mathbb{F}_3^n \to \mathbb{F}_3$ is a function such that $f(\mathbf{x}) \neq 0$ if and only if $\mathbf{x} \in C$. Let $M(f)$ be the $\mathbb{F}_3^n \times \mathbb{F}_3^n$ matrix given by

$$M(f)_{\mathbf{x},\mathbf{y}} := f(-\mathbf{x} - \mathbf{y}). \tag{2}$$

[1] Donald Knuth has written a computer program to count the number of caps in $\mathbb{F}_3^4$ of each size using the group of affine transformations te reduce the search space. See [5].

Let us consider the submatrix N of $M(f)$ with rows and columns from C. Then, by construction, N is a diagonal matrix with nonzero's on the diagonal. This implies that $|C| = \operatorname{rank} N \leq \operatorname{rank} M(f)$. The idea, therefore, is to find such a function f for which the rank of $M(f)$ is small, say $O(2.75511^n)$.

It turns out that this does not quite work: demanding that f is nonzero on C is too restrictive. Instead, we just consider the linear space L of functions $f : \mathbb{F}_3^n \to \mathbb{F}_3$ that vanish on the complement of C. Note that the dimension of L is large if C is large. Now we suppose, for contradiction, that C is larger than our proposed bound. Then, it turns out, L is sufficiently large to find a function $f \in L$ that is nonzero on 'most' of C, while at the same time $M(f)$ has small rank. This implies that C is small after all, contradiction.

To guarantee that $M(f)$ has low rank, we view the functions $f : \mathbb{F}_3^n \to \mathbb{F}_3$ as polynomials in the n variables $x_1, \ldots, x_n$ with each monomial being of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i \in \{0,1,2\}$. Each function $f : \mathbb{F}_3^n \to \mathbb{F}_3$ is (uniquely) represented by such a polynomial.

One can show that if f has degree at most d, then $M(f)$ has rank no more that twice the number of monomials of degree at most $\frac{1}{2}d$. If we restrict to polynomials of degree at most $d = \frac{2n}{3}$ (two-third of the maximum possible degree), then we loose only a tiny fraction of the available monomials. At the same time, the number of monomials of degree at most $\frac{n}{3}$ (and hence rank $M(f)$) is quite small, see Figure 4.

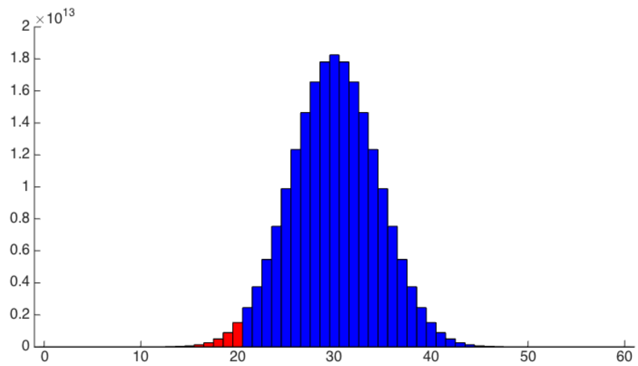


Figure 4: Counting the number of monomials of degree $0, 1, \ldots, 2n$, where $n = 30$. The red part corresponds to monomials of degree at most $\frac{2n}{3}$.

The details can be found in [2]. □

Although the cap set problem seems like an innocent question about a simple card game, it has connections to many other areas of mathematics. For instance, the fact that $\sigma < 3$ implies the so-called *Erdös-Szemerédi sunflower conjecture*. This in turn has implications for *Fast Matrix Multiplication*. If you want to know more about these connections, an good starting point is [8].

References

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