Propositions

supplementing the thesis

Matrix Algebras
and
Semidefinite Programming Techniques
for Codes

by Dion Gijswijt

Propositions

I

The *curling number* $C(w)$ of a word w is the largest integer k, for which w is a concatenation of the form $w = xy^k$, with y non-empty. For w equal the empty word, we set $C(w) := 1$. The *curling number transformation* $\mathcal{C}(b)$ of a sequence $b_1, b_2, b_3, \ldots$ is the sequence $c_1, c_2, c_3, \ldots$ defined by

$$c_n := C(b_1, \ldots, b_{n-1}). \tag{1}$$

The unique fixed point w of $\mathcal{C}$ begins with

1,1,**2**,
1,1,**2**,***2***,***2***,***3***,
1,1,2,1,1,**2**,**2**,**2**,**3**,***2***,
1,1,2,1,1,2,2,2,3,1,1,2,1,1,**2**,**2**,**2**,**3**,**2**,***2***,***2***,***3***,***2***,***2***,***2***,***3***,***3***,***2***, **. . .**

and contains all positive integers. The numbers 1, 2, 3, 4, and 5 have their first occurence in positions 1, 3, 9, 220, and (approximately) $10^{10^{23}}$. See: F. J. van de Bult, D. Gijswijt, J. P. Linderman, N.J.A. Sloane and A. R. Wilks, A Slow-Growing Sequence Defined by an Unusual Recurrence, *preprint*.

II

Let $G = (V, E)$ be a graph and let $b : E \to \mathbb{Z}_+$ be given capacities on the edges. Consider the system

$$\begin{array}{rll} \text{(i)} & x_v \geq 0 & \text{for every } v \in V, \\ \text{(ii)} & x(e) \leq b_e & \text{for every } e \in E, \\ \text{(iii)} & x(VC) \leq \lfloor \tfrac{1}{2} b(EC) \rfloor & \text{for every odd circuit } C. \end{array} \tag{2}$$

The following holds:

> system (2) is *totally dual integral* (TDI) for every $b : E \to \mathbb{Z}_+$, if and only if G has no 'bad K_4' as a subgraph.

See: D. Gijswijt, A. Schrijver, On the b-stable set polytope of graphs without bad K_4, *SIAM Journal on Discrete Mathematics* 16 (2003) 511–516.

III

Let V be a finite set of points on a circle, and let $A_1, \ldots, A_m$ be circular intervals with 'capacities' $c_1, \ldots, c_m \in \mathbb{Z}_+$. Define the set $X \subseteq \mathbb{Z}_+^V$ by

$$X := \{x \in \mathbb{Z}_+^V \mid x(A_i) \leq c_i \quad \text{for every } i = 1, \ldots, m\}.$$

Then for every positive integer k:

$$(k \cdot \text{conv.hull}(X)) \cap \mathbb{Z}^V = \{x_1 + \cdots + x_k \mid x_1, \ldots, x_k \in X\}.$$

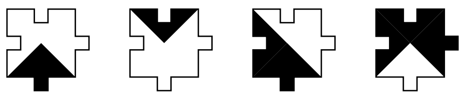
Furthermore, there is an algorithm that, given $x \in \mathbb{Z}_+^V$, finds a decomposition $x = x_1 + \cdots + x_k$ with $x_1, \ldots, x_k \in X$ whenever such a decomposition exists. The algorithm runs in time $O(n(n+m))$, where $n = |V|$. See: D. Gijswijt, Integer decomposition for polyhedra defined by nearly totally unimodular matrices, zal verschijnen in: *SIAM Journal on Discrete Mathematics.*

IV

Four is the smallest number of colours that suffice to colour every configuration of non-overlapping unit squares in the plane (squares sharing part of a side must receive different colours).

V

The question whether a template can be filled using copies of the following four pieces



(colours of adjacent pieces must match), is NP-hard. See: D. Gijswijt, Problemen, *Pythagoras* februari 2004.

VI

Orthogonal projection onto a matrix $*$-algebra preserves being positive semidefinite of a matrix. This can be used to reduce semidefinite optimisation problems.

VII

Let a finite number of red and blue points in the plane be given, no three on a line. Then there exists a line l such that the number of red points is the same on both sides of l, and also the number of blue points is the same on both sides.

VIII

A well-tried recipe for attracting math teachers has three main ingredients: curtain rings, polyethyleen cord, and above all: fire proof fingers.