Jan H. van Schuppen

Control and System Theory of Discrete-Time Stochastic Systems - Edition 1.E

January 24, 2022

Springer

Preface

This report is an extension of the book with the same name. The extension contains:

- References which were not included in the book.
- Various examples which were not included in the book mainly because of restrictions of space.
- Various exercises which were not included in the book.
- Proofs which were not included in the book because they are not too difficult.

Chapters in this booklet with these extensions are: Chapters 8, 12, 13, 14, 15, 18, and 19.

Amsterdam, *Jan H. van Schuppen*
January 24, 2022

Contents

Chapter 1
Control Problems

Abstract No extensions are described in this chapter.

Key words:

1.1 Control of a Mooring Tanker

1.2 Control of Freeway Traffic Flow

1.3 Control of a Shock Absorber

1.4 Further Reading

Chapter 2
Probability

Abstract No extensions are described in this chapter.

2.1 Probability Distribution Functions

2.2 Motivation of the Concept of a Probability Measure

2.3 Sets and Sigma-Algebras

2.4 Probability Measures

2.5 Random Variables

2.6 Expectation and the Characteristic Function

2.7 Gaussian Random Variables

2.8 Conditional Expectation

2.9 Conditional Independence

2.10 Computations

2.11 Exercises

2.12 Further Reading

Chapter 3
Stochastic Processes

Abstract No extensions are described in this chapter.

3.1 Concepts

3.2 Special Subsets of Stochastic Processes

3.3 Properties of Stochastic Processes

3.4 Gaussian Processes

3.5 Finite-Valued Stochastic Processes

3.6 Exercises

3.7 Further Reading

Chapter 4
Gaussian Stochastic Systems

Abstract No extensions are described in this chapter.

4.1 Modeling of Phenomena as a Stochastic System

4.2 The Concept of a Stochastic System

4.3 Time-Varying Gaussian Systems

4.4 Time-Invariant Gaussian Systems

4.5 Relation of Forward and Backward Gaussian System Representations

4.6 Stochastic Observability and Stochastic Co-Observability

4.7 Interconnections of Gaussian Systems

4.8 Stochastic Stability

4.9 Gaussian Factor Models and Gaussian Factor Systems

4.10 Computations

4.11 Exercises

4.12 Further Reading

Chapter 5
Stochastic Systems

Abstract No extensions are described in this chapter.

Key words:

5.1 Stochastic Systems and Probability Distributions

5.2 Output in Binary Set

5.3 Output in the Natural Numbers

5.4 Output in a Bounded Interval

5.5 Output in the Positive Real Numbers

5.6 Output-Finite-State-Polytopic Stochastic Systems

5.7 Sigma-Algebraic Stochastic System

5.8 The Multiple Conditional-Independence Relation

5.9 Technicalities

5.10 Further Reading

Chapter 6
Stochastic Realization of Gaussian Systems

Abstract No extensions are described in this chapter.

Key words: Stochastic realization. Gaussian systems.

6.1 Introduction to Realization Theory

6.2 Motivation

6.3 Weak Gaussian Stochastic Realization Problem

6.4 The Theorem

6.5 The Proof

6.6 Realization Procedures

6.7 State-Space Reduction of a Gaussian System

6.8 Special Stochastic Realizations-1

6.9 A Canonical Form

6.10 Exercises

6.11 Further Reading

Chapter 7
Stochastic Realization

Abstract No extensions are described in this chapter.

Key words:

7.1 The Conceptual Framework of Stochastic Realization

7.2 Stochastic Realization of a Tuple of Gaussian Random Variables

7.3 Stochastic Realization of a Tuple of Sigma-Algebras

7.4 Stochastic Realization of a Sigma-Algebra Family

7.5 Stochastic Realization of Output-Finite Stochastic Systems

7.6 Further Reading

Chapter 8
Filtering of Gaussian Systems

Abstract A reference for the Kalman filter and for the conditional Kalman filter is stated. References on the numerical analysis of Kalman filters are provided.

Key words: Kalman filter. Filter algebraic Riccati equation.

8.1 Problems of Filtering, Prediction, Smoothing, and Interpolation

8.2 Problem of Filtering

8.3 Time-Varying Kalman Filter

8.4 Time-Varying Kalman Filter and Stochastic Realization

8.5 Time-Invariant Kalman Filter

8.6 Approximations of a Time-Invariant Kalman Filter

8.7 Prediction

8.8 Interpolation

8.9 Conditional Kalman Filter

8.10 Exercises

8.11 Further Reading

Sections *Time-Varying Kalman Filter* and *Conditional Kalman Filter*.
Theorem 8.3.2 about the Kalman filter and Theorem 8.10.1 about the conditional Kalman filter are closely related to Theorem 13.3 on p. 63 and Theorem 13.4 on p. 65 of Volume II of the book by R.S. Liptser and A.N. Shiryayev, [1]. Those authors treat the case in which the variance of the measurement noise can be singular which special case is not treated in this book.

My apologies to the authors R.S. Liptser and A.N. Shiryayev for this lack of proper referencing.

It is to be noted that the Theorems 13.3 and 13.4 of [1] are based on a different system representation than that used in this book. Compare the equations (13.46) and (13.47) of that reference with the equations of Def. 4.3.1 of this book. It is well known that these two system representations cannot be converted into each other. Consequently, there exist two different system representations and two different corresponding Kalman filter representations. The system representation used in this book is in accordance with the literature for the system representation used in Europe and in North-America.

The proofs of the Theorems 8.3.2 and 8.10.1 of this book make use of Proposition 19.7.4 of this book. It is mentioned in Chapter 19 of this edition 1.E that Proposition 19.7.4 on p. 743 of this book is a special case of Theorem 13.3 and 13.4 of the book [1]. However, the proof of Proposition 19.7.4 of this book differs from the proofs of the mentioned theorems of the book [1]. The proofs of Theorem 8.3.2 and of Theorem 8.10.1 of this book are otherwise closely related to those of the book [1].

Numerical aspects of the Kalman filter. See also the references by M. Verhaegen and P.M. van Dooren, [2, 3].

References

1. R.S. Liptser and A.N. Shiryayev. *Statistics of random processes: I. General theory; II. Applications*. Springer-Verlag, Berlin, 1977,1978. 16, 33, 46
2. M. Verhaegen and P. Van Dooren. Numerical aspects of different Kalman filter implementations. *IEEE Trans. Automatic Control*, 31:907–917, 1986. 16
3. M.H. Verhaegen. Improved understanding of the loss-of-symmetry phenomenon in the conventional Kalman filter. *IEEE Trans. Automatic Control*, 34:331–333, 1989. 16

Chapter 9
Filtering of Stochastic Systems

Abstract No extensions are provided in this chapter.

Key words:

9.1 Problems of Estimation, Sequential Estimation, and of Filtering

9.2 Finite-Dimensional Filter Systems

9.3 Estimation Theory

9.4 Sequential Estimation

9.5 Filtering Theory

9.6 Filter of a Poisson-Gamma System

9.7 Filter of an Output-Finite-State-Finite Stochastic System

9.8 Further Reading

Chapter 10
Stochastic Control Systems

Abstract No extensions are provided in this chapter.

Key words:

10.1 Stochastic Control System

10.2 Gaussian Stochastic Control Systems

10.3 Stochastic Controllability and Stochastic Co-Controllability

10.4 State-Finite Stochastic Control Systems

10.5 Further Reading

Chapter 11
Stochastic Control Problems

Abstract No extensions are provided in this chapter.

Key words:

11.1 Control Problems of Stochastic Control

11.2 Control Laws

11.3 Closed-Loop Stochastic Control Systems

11.4 Stochastic Control Problems

11.5 Control Synthesis and Control Design

11.6 Statistical Decision Problems

11.7 Exercises

11.8 Further Reading

Chapter 12
Stochastic Control with Complete Observations on a Finite Horizon

Abstract Section 12.8 contains an example of a stochastic control problem with a state-finite stochastic control system.

Key words:

12.1 Control Problems

12.2 Problem Formulation

12.3 Explanation of Dynamic Programming

12.4 Digression on Optimization

12.5 Digression on Measurable Control Laws

12.6 Dynamic Programming for Additive Cost Functions

12.7 Control of a Gaussian Control System

12.8 Control of a State-Finite Stochastic Control System

The next example should be inserted in the book at the end of the section with the above title.

How to carry out the computations is illustrated by the following example of inventory control.

Example 12.8.1. *Inventory control.* Consider Example 11.1.3 of [1], of a shop selling radios. The problem is to determine an optimal control law for ordering radios of a shop such that the economic cost of the shop is minimized. The notation here is,

$x(t)$ Stock available at beginning of the t-th period
$u(t)$ Stock ordered and immediately delivered at beginning of the t-th period
$v(t)$ The random variable denoting the demand in the t-th period
h Holding cost per item per day
k Ordering cost per item per day
b Shortage cost per item per day for demand that cannot be met.

Let $T = \{0,1,2,3\}$, $X = \mathbb{N} = \{0,1,2,\ldots\}$, $U = \{0,1,2\}$, $x, v : \Omega \times T \to \mathbb{N}$. Suppose that $\{x_0, v(0), v(1), v(2), v(3)\}$ are independent random variables, and that $x_0 = 0$. Moreover, let v be an independent identically-distributed sequence with for any $t \in T$,

$$P(\{v(t) = 0\}) = 0.2,\ P(\{v(t) = 1\}) = 0.6,\ P(\{v(t) = 2\}) = 0.2.$$

To simplify the computations the condition is imposed that $x(t) + u(t) \leq 2$ for all $t \in T$. Hence $U(x) = \{u \in U |\ x + u \leq 2\}$, $U(0) = \{0,\ 1,\ 2\}$, $U(1) = \{0,\ 1\}$, and $U(2) = \{0\}$. Suppose further that excess demand, occuring if $v(t) - x(t) - u(t) > 0$, is lost to the shop. Take the numerical values $h = k = 1$, $b = 2$. The state dynamics are,

$$x(t+1) = \max\{0, x(t) + u(t) - v(t)\}. \tag{12.1}$$

The control objective is to minimize the cost rather than to maximize profit as in Example 11.1.3 This choice brings the problem in line with Problem 12.2.1, [1], of optimal control theory. The cost function is then,

$$\begin{aligned} J(g) = E \Bigg[\sum_{s=0}^{2} (ku(s) + b \max\{0, v(s) - x(s) - u(s)\} + \\ + h \max\{0, x(s) + u(s) - v(s)\}) + b_1(x(3)) |\ F^{x,u} \Bigg], \end{aligned}$$

$$b_1(x) = \begin{cases} 3,\ x = 0, \\ 2,\ x = 1, \\ 3,\ x = 2. \end{cases}$$

Due to the formulation of the cost function, the conditional expectation operation includes the cost rate. There is no terminal cost.

The control objectives confict between (1) disappointing customers if there is no stock left and (2) high holding costs. Note that if the stock is zero at a particular day then all customers requesting products are disappointed. Moreover, if the shop is fully stocked at a day but less customers request products then a holding cost has to be paid at the end of the day. The optimal control law strikes a balance between these two conflicting control objectives.

Consider the past-state information structure and the corresponding set of control laws $g = \{g_0, g_1, g_2, \ldots, g_{t_1-1}\}$ with $g_t : X^{t+1} \rightarrow U$. The problem is then to solve $\inf_{g\in G} J(g)$.

The solution is determined by application of the dynamic programming procedure. First, $\forall\, x \in X$, define $V(3,0) = 3$, $V(3,1) = 2$, and $V(3,2) = 3$. Next,

$$
\begin{aligned}
V(2,x(2)) = & \min_{u(2)\in\{0,1,2|0\leq u(2)\leq 2-x(2)\}} \\
& \{E[ku(2) + b\max\{0, v(2) - x(2) - u(2)\} + \\
& + h\max\{0, x(2) + u(2) - v(2)\} + \\
& + V(3,\ \max\{0,\ x(2) + u(2) - v(2)\})|\, F^{x(2)}], \Big\}, \\
V(2,0) = & \min_{u(2)\in\{0,1,2\}} \ \{E[ku(2) + b\max\{0, v(2) - u(2)\} + \\
& + h\max\{0, u(2) - v(2)\} + \\
& + V(3,\ \max\{0,\ 0 + u(2) - v(2)\})|\, F^{x(2)}]\Big\} \\
= & \min_{u(2)\in\{0,1,2\}} \ \{ku(2) + \\
& + 0.2[\max\{0, u(2)\} + 3\max\{0, -u(2)\} + V(3,0)] \\
& + 0.6[\max\{0, u(2) - 1\} + 3\max\{0, 1 - u(2)\} + V(3,0)] \\
& + 0.2[\max\{0, u(2) - 2\} + 3\max\{0, 2 - u(2)\} + V(3,0)]\} \\
= & \min_{u(2)\in\{0,1,2\}} \{5.0,\ 4.4,\ 5.4\} = 4.4. \text{ Thus,} \\
g^*(2,0) = & \ 1. \\
V(2,1) = & \min_{u(2)\in\{0,1\}} \ \{E[ku(2) + b\max\{0, v(2) - 1 - u(2)\} + \\
& + h\max\{0, 1 + u(2) - v(2)\} + \\
& + V(3,\ \max\{0,\ 1 + u(2) - v(2)\})|\, F^{x(2)}]\Big\} \\
= & \min_{u(2)\in\{0,1\}} \ \{3.4,\ 4.4\} = 3.4, \text{ so,} \\
g^*(2,1) = & \ 0. \\
V(2,2) = & \ E[ku + b\max\{0, v(2) - 2\} + h\max\{0, 2 - v(2)\} + \\
& + V(3,\ \max\{0,\ 2 + u(2) - v(2)\})|\, F^{x(2)}] = 3.4, \\
g^*(2,2) = & \ 0;
\end{aligned}
$$

$$\begin{aligned}
V(1,0) &= \min_{u(1)\in\{0,1,2\}} \{E[ku(1)+b\max\{0,v(1)-u(1)\}+ \\
&\quad +h\max\{0,u(1)-v(1)\}+V(2,\max\{0,u(1)-v(1)\})|\,F^{x(1)}]\Big\}, \\
&= \min\{6.4,\ 5.8,\ 6.6\} = 5.8, \\
g^*(1,0) &= 1. \text{ Similarly,} \\
V(1,1) &= \min_{u(1)\in\{0,1\}} \{4.8,\ 5.6\} = 4.8,\ g^*(1,1) = 0, \\
V(1,2) &= \min_{u(1)\in\{0\}} \{4.6\} = 4.6,\ g^*(1,2) = 0. \\
V(0,0) &= \min_{u(0)\in\{0,1,2\}} \{E[ku(0)+h\max\{0,u(0)-v(0)\} \\
&\quad +b\max\{0,v(0)-u(0)\}+V(1,\max\{\,0,u(0)-v(0)\})|\,F^{x(0)}]\Big\} \\
&= \min\{7.8,\ 7.2,\ 7.96\} = 7.2, \\
g^*(0,0) &= 1; \\
V(0,1) &= \min_{u\in U(1)} \{6.20,\ 6.96\} = 6.2, \\
g^*(0,1) &= 0, \\
V(0,2) &= \min_{u\in U(2)} 5.96 = 5.96, \\
g^*(0,0) &= 0.
\end{aligned}$$

The optimal control law is then,

$$g^*(0,x(1)) = \begin{cases} 1, & \text{if } x(0)=0, \\ 0, & \text{if } x(0)=1, \\ 0, & \text{if } x(0)=2, \end{cases} \quad g^*(1,x(1)) = \begin{cases} 1, & \text{if } x(1)=0, \\ 0, & \text{if } x(1)=1, \\ 0, & \text{if } x(1)=2, \end{cases}$$

$$g^*(2,x(2)) = \begin{cases} 1, & \text{if } x(2)=0, \\ 0, & \text{if } x(2)=1, \\ 0, & \text{if } x(2)=2. \end{cases}$$

The optimal control law is such that for $t = 0,\ 1$, if the current state $x(t) = 0$ then one radio is ordered, $u(t) = 1$, while, if the current state $x(t) > 0$, then no radio is ordered, $u(t) = 0$. For $t = 2$ one radio is ordered for $x = 0$ and $x = 1$, none otherwise.

12.9 Invariance of a Subset of Value Functions

12.10 Relation of Optimal Control Law and State

12.11 Dynamic Programming for Multiplicative Cost Functions

12.12 Stochastic Control Problems of Economics and of Finance

12.13 Control via System Approximation

12.14 Exercises

12.15 Further Reading

References

1. Jan H. van Schuppen. *Control and system theory of discrete-time stochastic systems*. Springer Nature Switzerland AG, Cham, 2021. 24, 29, 35, 36

Chapter 13
Stochastic Control with Complete Observations on Infinite Horizon

Abstract Section 13.2 contains an example of a stochastic control problem with a state-finite stochastic control system and an average cost function. In the last section additional references are provided.

Key words: Stochastic control. Complete observations. Infinite-horizon.

13.1 Introduction to Control on an Infinite-Horizon

13.2 Average Cost

The following example is best placed at the end of Subsection 13.2.4 of the book [7].

Example 13.2.1. *Computation of an optimal control law by control law iteration.* Consider a finite stochastic control system with the representation

$$\begin{aligned} X &= \{x_1, x_2, x_3\}, \;\; U = \{u_1, \, u_2\}, \\ E[x(t+1)|F_t] &= A(u(t))x(t), \\ A(u_1) &= \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.6 & 0.8 & 0.0 \\ 0.2 & 0.0 & 0.6 \end{pmatrix}, \;\; A(u_2) = \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.8 & 0.0 \\ 0.6 & 0.0 & 0.6 \end{pmatrix} \in \mathbb{R}_{st}^{3\times 3}. \end{aligned}$$

The control system has three states. From state x_1 one proceeds either to state x_2 or to state x_3 or one stays at state x_1. From state x_2 and from state x_3 one returns to state x_1 with a probability, the probabilities differ by state. The latter property makes the system asymmetric. The user can influence the stochastic control system by the choice of the input at state x_1, with u_1 there is a higher probability to go to state x_2 than to state x_3, with the input u_2 the probabilities are reversed. The inputs have no effect at the states x_2 and x_3 to keep the problem simple. Note that for all $u \in U$, the stochastic matrix $A(u)$ is irreducible.

The average cost function is defined as

$$J(g) = \limsup_{t_1 \to \infty} \frac{1}{t_1} E[\sum_{s=0}^{t_1-1} b(x(s), u(s))],$$
$$b(x,u) = \begin{cases} 1, \text{ if } x = x_1, \ \forall\, u \in U, \\ 2, \text{ if } x = x_2, \ \forall\, u \in U, \\ 5, \text{ if } x = x_3, \ \forall\, u \in U. \end{cases}$$

The policy iteration procedure is applied.

Initialization. Let $g_a : X \to U$ be $g_a(x) = u_2$ for all $x \in X$.

Solve the equations

$$J(g_0) + V_{g_a}(x_1) = b(x, g_a(x)) + V_{g_a}^T A(g_a(x))x, \ \ \forall\, x \in X, \ V_{g_a}(x_3) = 0,$$

for $(J(g_a), V_{g_a})$. The condition $V_{g_a}(x_3) = 0$ is imposed because the solution is up to an additive constant.

Substract the equation of x_3 from that of x_1 so that one obtains

$$V_{g_a}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = V_{g_a}(x_1) - V_{g_a}(x_3)$$
$$= b(x_1, g_a(x_1)) - b(x_3, g_a(x_3)) + V_{g_a}^T A(g_a(x))x_1 - V_{g_a}^T A(g_a(x))x_3$$
$$= 1 - 5 + V_{g_a}^T \left[\begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0 \\ 0.6 \end{pmatrix} \right] \ \Leftrightarrow \ \begin{pmatrix} 1.2 & -0.2 & -1 \end{pmatrix}^T V_{g_a} = -4.$$

Similarly for the second equation,

$$V_{g_a}^T \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = V_{g_a}(x_2) - V_{g_a}(x_3)$$
$$= b(x_1, g_a(x_2)) - b(x_3, g_a(x_3)) + V_{g_a}^T A(g_a(x))x_2 - V_{g_a}^T A(g_a(x))x_3$$
$$= 2 - 5 + V_{g_a}^T \left[\begin{pmatrix} 0.2 \\ 0.8 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0 \\ 0.6 \end{pmatrix} \right] \ \Leftrightarrow \ \begin{pmatrix} 0.2 & 0.2 & -0.4 \end{pmatrix}^T V_{g_a} = -3.$$

The combined set of equations then has the solution

$$\begin{pmatrix} 1.2 & -0.2 \\ 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} V_{g_a}(x_1) \\ V_{g_a}(x_2) \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \ V_{g_a} = \begin{pmatrix} -5 \\ -10 \\ 0 \end{pmatrix} \ \Rightarrow V_{g_a} = \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix};$$

where the latter vector is also a solution of the equation because V_{g_a} is unique upto an additive constant. Hence

$$\begin{aligned} J(g_a) &= -V_{g_a}(x_1) + b(x_1, g_a(x_1)) - A(g_a(x_1))x_1 \\ &= -5 + 1 + V_{g_a}^T \begin{pmatrix} 0.2 \\ 0.2 \\ 0.6 \end{pmatrix} = -5 + 1 + 7 = 3. \end{aligned}$$

The reader can verify that the solution $(J(g_a),\ V_{g_a})$ satisfies the three equations.

For V_{g_a} fixed, one determines a new control law g_b according to

$$\begin{aligned} h^*(x) &= \min_{\{u \in U = \{u_1, u_2\}} [b(x,u) + V_{g_a}^T A(u)x] \\ &= \min \left\{ b(x,u_1) + \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix}^T \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.6 & 0.8 & 0.0 \\ 0.2 & 0.0 & 0.6 \end{pmatrix}, \right. \\ &\qquad \left. b(x,u_2) + \begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix}^T \begin{pmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.8 & 0.0 \\ 0.6 & 0.0 & 0.6 \end{pmatrix} \right\}, \\ h^*(x_1) &= \min\{1+3,\ 2+7\} = 4,\ g_b^*(x_1) = 1, \\ h^*(x_2) &= \min\{2+1,\ 1+2\} = 3,\ g_b(x_2) = 1, \\ h^*(x_3) &= \min\{5+8,\ 5+8\} = 13,\ g_b(x_3) = 1, \end{aligned}$$

where it is noted that the control law at the states x_2 and x_3 can be either u_1 or u_2 though that has no effect on the dynamics as is clear from the stochastic control system.

Next the value function V_{g_b} has to be computed. Note that,

$$\begin{aligned} & V_{g_b}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = b(x_1, g_b(x_1)) - b(x_3, g_b(x_3)) + V_{g_b}^T \left[\begin{pmatrix} 0.2 \\ 0.6 \\ 0.2 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0.0 \\ 0.6 \end{pmatrix} \right] \\ \Leftrightarrow\ & \begin{pmatrix} 1.2 & -0.6 & -0.6 \end{pmatrix} V_{g_b} = -4; \\ & V_{g_b}^T \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = b(x_2, g_b(x_2)) - b(x_3, g_b(x_3)) + V_{g_b}^T \left[\begin{pmatrix} 0.2 \\ 0.8 \\ 0.0 \end{pmatrix} - \begin{pmatrix} 0.4 \\ 0.0 \\ 0.6 \end{pmatrix} \right] \\ \Leftrightarrow\ & \begin{pmatrix} 0.2 & 0.2 & -0.4 \end{pmatrix} V_{g_b} = -3; \end{aligned}$$

$$\begin{aligned} & \begin{pmatrix} 1.2 & -0.6 \\ 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} V_{g_b}(x_1) \\ V_{g_b}(x_2) \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \\ V_{g_b} &= \begin{pmatrix} -65/9 \\ -70/9 \\ 0 \end{pmatrix} \Rightarrow V_{g_b} = \begin{pmatrix} 5/9 \\ 0 \\ 70/9 \end{pmatrix}; \\ J(g_b) &= -V_{g_b}(x_1) + b(x_1, g_b(x_1)) + V_{g_b}^T A(g_b(x_1))x_1 \\ &= -5/9 + 1 + 15/9 = 2 + 1/9 < 3 = J(g_a). \end{aligned}$$

A further computation shows that g_b is the optimal control law with $(J^*, V) = (J(g_b), V_{g_b})$.

13.3 Discounted Cost

13.4 Minimum-Variance Control with Complete Observations

13.5 Exercises

13.6 Further Reading

Additional references are provided.

Existence of an optimal control law for an infinite horizon optimal stochastic control problem follows from the property that the set of control laws is compact. Sufficient conditions for the latter property are formulated and studied by Manfred Schäl, [6].

Dynamic programming on an infinite horizon with discounted cost (without cost constraints) is treated by Vivek S. Borkar in [1]. The case treated is for a stochastic control system with a countable state set, in fact $\mathbb{Z}_+$. The input sets are compact metric spaces. The stochastic system satisfies a continuity condition. It is further assumed that the state set of the stochastic system is a single communicating subset. The stochastic system and the cost function are such that the set of attainable probability measures on the product of the state set and the input set, is a compact set.

The author of this book prefers a condition in terms of stochastic controllability.

Dynamic programming on an infinite horizon with discounted cost and cost constraints is discussed in the references [3, 4, 5].

As already stated in the book, V. Borkar has treated the problem of existence of an optimal control law for a stochastic control problem on an infinite-horizon in the paper [2].

References

1. V. Borkar. A convex analytic approach to Markov decision processes. *Probability Theory and Related Fields*, 78:583–602, 1988. 32
2. Vivek S. Borkar. Control of Markov chains with long-run average cost criterion: The dynamic programming equations. *SIAM J. Control & Opt.*, 27:642–657, 1989. 32
3. Eugene A. Feinberg, Anna Jaskeiwicz, and Andrzej S. Nowak. Constrained discounted Markov decision processes with Borel state space. *Automatica*, 111:108582, 2020. 32
4. O. Hernández-Lerma and J. González-Hernández. Constrained Markov control processes in borel spaces: The discounted case. *Math. Methods of Oper. Res.*, 52:271–285, 2000. 32
5. A.B. Piunovskiy. *Optimal control of random sequences in problems with constraints*. Kluwer Academic Publishers, Dordrecht, 1997. 32
6. M. Schäl. On dynamic programming: Compactness of the space of policies. *Stochastic Process. Appl.*, 3:345–364, 1975. 32
7. Jan H. van Schuppen. *Control and system theory of discrete-time stochastic systems*. Springer Nature Switzerland AG, Cham, 2021. 24, 29, 35, 36

Chapter 14
Stochastic Control with Partial Observations on a Finite Horizon

Abstract A reference is provided in the section *Further Reading*.

Key words:

14.1 Motivation

14.2 Problem Formulation

14.3 Stochastic Realization of a Stochastic Control System

14.4 Control of a State-Finite Stochastic Control System

14.5 Exercises

14.6 Further Reading

Section *Stochastic realization of a Stochastic Control System*.
Theorem 14.4.3 on p. 562 is closely related to a special case of Theorem 3.4 on p. 65 of Volume II of the book of R.S. Lipster and A.N. Shiryayev, [1]. However, Theorem 3.4 of that reference is for a different system representation than that used in this book. Compare the equations (3.46) and (3.47) on p. 61 of that reference with those of Def. 4.3.1 on p. 89 of this book. These two system representations cannot be converted into each other. Therefore the formulas of Theorem 3.4 of that reference differ from those of Theorem 14.4.3 of this book.

References

1. R.S. Liptser and A.N. Shiryayev. *Statistics of random processes: I. General theory; II. Applications*. Springer-Verlag, Berlin, 1977,1978. 16, 33, 46

Chapter 15
Stochastic Control with Partial Observations on an Infinite Horizon

Abstract The proof of Theorem 15.2.7 of the book is provided, which below is described as the proof of Theorem 15.2.1.

Key words:

15.1 Problem Issues

15.2 Control of a Gaussian Stochastic Control System

The proof of Theorem 15.2.7 is provided below. It was not included in the book to save space. However, the proof is little subtle because of the integrability conditions and related arguments. For the reference of the reader, Theorem 15.2.7 of the book [1] is restated. However, below it has a different number.

Theorem 15.2.1. *[1, Th. 15.2.7]. Consider Problem 15.2.1 of optimal stochastic control with partial observations, reformulated in Problem 15.2.5 with respect to the information structure. Assume that:*

(1)Assumption 15.2.2(b) holds.
(2)The set $G_{tv,fc}$ of control laws with finite average cost is not empty.
(3)There exists a solution $(J_{ac}^*,\ V)$ of the Procedure 15.2.6 for dynamic programming with partial observations and with average cost.
(4)

$$\forall\, g \in G_{tv,fc},\ \forall\, t \in T,\ E|V(\hat{x}^g(t),\ Q_f^*)| < \infty.$$

Note that the function V is not necessarily positive.
(5)

$$\forall\, g \in G_{tv,fc},\ \lim_{t_1 \to \infty} \frac{1}{t_1}\, E\left[V(\hat{x}^g(t_1),\ Q_f^*)\right] = 0.$$

(a)Then $J_{ac}^* \leq J_{ac}(g)$ for all $g \in G_{tv,fc}$.
(b)If there exists a control law $g^* \in G_{tv,fc}$ as defined in Procedure 15.2.6 then there exists an optimal control law $g^* \in G$

$$J_{ac}^* = J_{ac}(g^*) = \inf_{g \in G_{tv,fc}} J_{ac}(g).$$

Proof. (a) (1) Note that $b : X \times U \to \mathbb{R}_+$ hence $\overline{b} : \hat{X} \times \mathbb{R}_{pds}^{n_x \times n_x} \times U \to \mathbb{R}_+$. Then, for all $g \in G_{tv,fc}$ and for all $t \in T \backslash \{t_1\}$, $E[\overline{b}(\hat{x}^g(t),\ Q_f^*),\ u^g(t))] < \infty$. Because, if the expectation is not finite then $J_{ac}(g) = +\infty$ contradicting that $g \in G_{tv,fc}$.

Because by assumption (4), $E|V(\hat{x}^g(t),\ Q_f^*)| < \infty$ for all $g \in G_{tv,fc}$ and for all $t \in T$, the next conditional expectation exists and its expectation equals,

$$\begin{aligned}
&E[V(\hat{x}^g(t), Q_f^*)|\ F_{t-1}^{y^g} \vee F_{t-1}^{u^g}],\\
&E[E[V(\hat{x}^g(t), Q_f^*)|\ F_{t-1}^{y^g} \vee F_{t-1}^{u^g}]] = E[V(\hat{x}^g(t), Q_f^*)].
\end{aligned}$$

(2) For $s \in T$,

$$\begin{aligned}
&\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s)) + E[V(\hat{x}^g(s+1),\ Q_f^*)|\ F_{s-1}^{y^g} \vee F_{s-1}^{u^g}]\\
&= \overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s)) +\\
&\quad + \int V(w_{\hat{x}},\ f_{FARE}(Q_f^*))\ p_G(w_{\hat{x}}; [A\hat{x}^g(s) + Bu^g(s)],\ Q_{K\overline{v}}(Q_f^*))\ dw_{\hat{x}}\\
&\geq \inf_{u_V \in U(\hat{x}^g(s),\ Q_f^*)} \Big[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u_V) +\\
&\quad + \int V(w_{\hat{x}},\ f_{FARE}(Q_f^*))\ p_G(w_{\hat{x}};\ [A\hat{x}^g(s) + Bu_V],\ Q_{K\overline{v}}(Q_f^*))\ dw_{\hat{x}}\Big]\\
&= J_{ac}^* + V(\hat{x}^g(s),\ Q_f^*),
\end{aligned}$$

by the dynamic programming procedure.

Next take expectation of the above expressions which expectation is well defined by Step (1) above. Sum then also from $s = 0$ to $s = t_1 - 1$. Then

$$\sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + \sum_{s=0}^{t_1-1} E[E[V(\hat{x}^g(s+1),\ Q_f^*)|\ F_{s-1}^{y^g} \vee F_{s-1}^{u^g}]]$$
$$= \sum_{s=0}^{t_1-1} \left(E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + E[V(\hat{x}^g(s+1),\ Q_f^*)] \right)$$
$$\geq \sum_{s=0}^{t_1-1} [J_{ac}^* + E[V(\hat{x}^g(s),\ Q_f^*)] = t_1\ J_{ac}^* + \sum_{s=0}^{t_1-1} E[V(\hat{x}^g(s),\ Q_f^*)].$$

Rearrangement of these terms and division by t_1 yields

$$\begin{aligned} J_{ac}^* &\leq \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + \\ &\quad + \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[V(\hat{x}^g(s+1),\ Q_f^*)] - \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[V(\hat{x}^g(s),\ Q_f^*)] \\ &= \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + \\ &\quad + \frac{1}{t_1} \sum_{r=1}^{t_1} E[V(\hat{x}^g(r),\ Q_f^*)] - \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[V(\hat{x}^g(s),\ Q_f^*)]. \end{aligned}$$

Note that all terms of the above expression are well defined because all terms are finite. Note also that the condition expectation on the σ-algebra $F_{s-1}^{y^g} \vee F_{s-1}^{u^g}$ has disappeared from the formulas. Then

$$\begin{aligned} J_{ac}^* &\leq \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + \\ &\quad + \frac{1}{t_1} E[V(\hat{x}^g(t_1),\ Q_f^*)] - \frac{1}{t_1} E[V(m_{x_0},\ Q_f^*)],\ \forall\ t_1 \in \mathbb{Z}_+, \\ &\quad \text{because } \hat{x}^g(0) = m_{x_0} \in \mathbb{R}^{n_x}, \text{ and by assumption (4);} \\ J_{ac}^* &\leq \limsup_{t_1 \to \infty} \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] + \\ &\quad + \limsup_{t_1 \to \infty} \frac{1}{t_1} E[V(\hat{x}^g(t_1),\ Q_f^*)] - \liminf_{t_1 \to \infty} \frac{1}{t_1} V(m_{x_0},\ Q_f^*), \\ &= \limsup_{t_1 \to \infty} \frac{1}{t_1} \sum_{s=0}^{t_1-1} E[\overline{b}(\hat{x}^g(s),\ Q_f^*,\ u^g(s))] = J_{ac}(g), \end{aligned}$$

where assumption (5) is used and because $V(m_{x_0},\ Q_f^*) < \infty$.
(b) If in the dynamic programming procedure 15.2.6 the infima are attained, if the functions $g_t^*(\hat{x}^g(t),\ Q_f^*)$ are defined, and if the candidate optimal control law $\{g^*,\ g^*,\ldots\}$ is such that g^* is a measurable function then the inequalities of the proof of (a) became equalities and $J_{ac}^* = J_{ac}(g^*)$. This with the result of (a) yields that $J_{ac}(g^*) = J_{ac}^* = \inf_{g \in G_{tv,fc}}\ J_{ac}(g)$. □

15.3 Minimum-Variance Control with Partial-Observations

15.4 Further Reading

References

1. Jan H. van Schuppen. *Control and system theory of discrete-time stochastic systems*. Springer Nature Switzerland AG, Cham, 2021. 24, 29, 35, 36

Chapter 16
Stochastic Control Theory

Abstract No extensions are provided in this chapter.

Key words:

16.1 Research Problems of Control of Stochastic Systems

16.2 General Optimality Conditions

16.3 Stochastic Control via a Measure Transformation

16.4 Further Reading

Chapter 17
Appendix: Mathematics

Abstract No extensions are provided in this chapter.

Key words:

17.1 Algebra of Sets

17.2 Algebraic Structures

17.3 Linear Algebra and Linear Dependence

17.4 Matrices

17.5 Analysis

17.6 Geometry

17.7 Optimization

17.8 Further Reading

Chapter 18
Appendix: Positive Matrices

Abstract A reference has been added to this chapter.

Key words:

18.1 Problems

18.2 The Positive Real Numbers and a Positive Vector Space

18.3 Definitions of Positive Matrices

18.4 Geometry and Cones

18.5 Units

18.6 Similarity

18.7 Eigenvalues and Eigenvectors of Positive Matrices

18.8 Eigenvalues and Eigenvectors of Stochastic Matrices

18.9 Multiplicative Factorization

18.10 Computations

18.11 Further Reading

Doubly stochastic matrices which are irreducible and idempotent are characterized in [1].

References

1. Štefan Schwarz. A note on the structure of the semigroup of doubly-stochastic matrices. *Matematický časopis*, 17:308–316, 1967. 44

Chapter 19
Appendix: Probability

Abstract A reference is provided to the section Further Reading.

Key words: Probability theory. Conditional independence. Measure Transformations.

19.1 Sets and the Monotone Class Theorems

19.2 Probability Measures

19.3 Stable Subsets of Probability Distribution Functions

19.4 Gaussian Random Variables

19.5 Spaces and Sequences of Random Variables

19.6 Conditional Expectation and Conditional Probability

19.7 Conditionally Gaussian Random Variables

19.8 Conditional Independence Continued

19.9 Measure Transformations

19.10 The Family of Exponential Probability Distributions

19.11 Pseudo-Distances on the Set of Probability Measures

19.12 P-essential Infima

19.13 Further Reading

Section *Conditionally Gaussian Random Variables*.
Propositions 19.7.2, 19.7.3, and 19.7.4 on p. 742–743 are a special case of Theorem 13.3 on p. 63 and of Theorem 13.4 on p. 65 of Volume II of the book by R.S. Liptser and A.N. Shiryayev, [1]. However, the proof of Proposition 19.7.4 of this book differs from the corresponding proofs of Theorem 13.3 and of Theorem 13.4 of the book [1].

The concept of conditionally Gaussian random variables is referred on p. 764 of this book to Chapter 11 of the book by R.S. Liptser and A.N. Shiryayev, [1]. The Chapters 12 and 13 are also relevant and should also have been mentioned.

My apologies to R.S. Liptser and A.N. Shiryayev for this lack of proper referencing.

References

1. R.S. Liptser and A.N. Shiryayev. *Statistics of random processes: I. General theory; II. Applications*. Springer-Verlag, Berlin, 1977,1978. 16, 33, 46

Chapter 20
Appendix: Stochastic Processes

Abstract No extensions are provided in this chapter.

Key words:

20.1 Stochastic Processes and Filtrations

20.2 Martingale Theory

20.3 Stochastic Processes and Stopping Times

20.4 Supermartingale Convergence

20.5 Ergodicity

20.6 Further Reading

Chapter 21
Appendix: Control and System Theory of Deterministic Systems

Abstract No extensions are provided in this chapter.

Key words:

21.1 Deterministic Control Systems

21.2 Controllability

21.3 Observability

21.4 Geometric Approach to Linear Systems

21.5 Zero-Output Dynamics

21.6 Inverse of a Linear System

21.7 Canonical Factorization of a Deterministic Map

21.8 Realization Theory for Linear Systems

21.9 Stability

21.10 Further Reading

Chapter 22
Appendix: Matrix Equations

Abstract No extensions are provided in this chapter.

Key words:

22.1 Lyapunov Equation

22.2 Algebraic Riccati Equations of Filtering and of Control

22.3 Algebraic Riccati Equation of Gaussian Stochastic Realization

22.4 Further Reading

Chapter 23
Appendix: Covariance Functions and Dissipative Systems

Abstract No extensions are provided in this chapter.

Key words:

23.1 Definitions

23.2 Storage Functions

23.3 Relations

23.4 Algebraic Characterization of Dissipative Linear Systems

23.5 Further Reading

Chapter 24
Appendix: State-Variance Matrices

Abstract No extensions are provided in this chapter.

Key words:

24.1 Definition and Problem Formulation

24.2 Transformations

24.3 The Geometric Structure

24.4 Regularity

24.5 The Boundary of the Set of State-Variance Matrices

24.6 Singular Boundary Matrices

24.7 The Classification of State-Variance Matrices

24.8 Further Reading