Accelerating the Solution of Linear Systems Appearing in Two-Phase Reservoir Simulation by the use of Pod-Based Deflation Methods

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Motivation

* Solutions of systems of linear equations are required when simulating flow through subsurface porous media.
* The complex geometry and strongly heterogeneous rock properties of the porous medium makes the linear system ill-conditioned.
* Converge is hampered by large and ill-conditioned systems.
* For time-varying problems, it is required to compute a large number of simulations, which makes the solution of these problems expensive.

Solution Methods

Reduced Order Models (ROM) methods capture relevant information of a high-dimensional system and to project it into a lower-dimension space [1, 2, 3, 4, 5], easier to solve.

Krylov subspace iterative methods For these methods, the speed of convergence depends on the condition number and the right-hand side (*rhs*) of the system[6, 7]. If the condition number is large, preconditioning techniques can be used to transform the original system into a better conditioned one. Recycling strategies like augmentation and deflation [8] can also be implemented to accelerate the convergence rate.

Multigrid methods Employs grids of different mesh sizes, allowing to solve all wave-length components and provides rapid convergence rates.

Solution methods

Proper Orthogonal Decomposition (POD) is a ROM method that has recently been used to accelerate the solution of the linear pressure equation resulting from reservoir simulation [9, 10, 11, 12, 13], among other applications.
For the computation of the POD basis, two main approaches are used:

Training phase approach: a training simulation is run, and the solutions are stored as snapshots to obtain a POD basis. Especially suitable for solving problems with small changes in the input variables, e.g. the same well configurations but different flow rates or bottom hole pressures (*bhp*) [12, 9, 11].

Moving window approach: The basis is computed on-the-fly, using, e.g., the solution of the latest time steps [10, 9, 14]. With this approach, the basis has to be adapted during the simulation.

Solution methods

Convergence of the Krylov subspace methods takes place in three phases with different rates of convergence[15, 16, 17].

Convergence phases	How to address it
The method lacks information about the spectrum, and acts mainly on the information of the initial iteration error, leading to a short but rapid initial convergence rate, also known as the sublinear phase	Improve the initial guess so that the error is minimized
The influence of the spectral condition number becomes larger and the method enters the linear phase	Change the condition number of the system
A sufficient number of extreme eigenvalue components have been damped out, the method enters the so called superlinear convergence phase	Remove the influence of the smallest eigenvalues of the system matrix (e.g. with augmentation or deflation techniques)

State of the art

Article	Methodology
Markovinovic et al. [10]	Proposed the use of POD techniques to compute a good initial guess that accelerates the iterative method for simulation of two-phase flow through large-scale heterogeneous porous media.
Astrid et al. [9]	For the same type of problems, solving the problem in the small-scale domain, and projecting it back to the large-scale system.
Carlberg et al. [5]	Proposed a POD-augmented CG algorithm for Krylov-subspace recycling applied to solid-mechanics problems.
Pasetto et al. [2]	Suggested a preconditioner for the CG method, based on a POD basis for the solution of groundwater flow models.

Deflation techniques

Definition

Let $A \in \mathbb{R}^{n\times n}$ be an *SPD* matrix, and $Z \in \mathbb{R}^{n\times p}$ be a full rank matrix. The invertible Galerkin matrix, $E \in \mathbb{R}^{p\times p}$, the correction matrix, $Q \in \mathbb{R}^{n\times n}$ and the deflation matrix $P \in \mathbb{R}^{n\times n}$ are defined as [18, 19, 20]:

$$P = I - AQ, \qquad Q = ZE^{-1}Z^T, \qquad E = Z^TAZ. \tag{1}$$

where $Z \in \mathbb{R}^{n\times p}$ is called the *deflation* $-$ *subspace* matrix, and its columns are the *deflation* vectors or *projection* vectors[18, 21, 22, 23, 14, 24, 19, 25].

Techniques used to select deflation vectors are based on:

- Eigenvectors or approximated eigenvectors of the system matriz [19, 25]
- Recycling vectors [23, 14]
- Subdomain deflation vectors [24]
- Multigrid and multilevel based deflation vectors [18, 21, 22]

Deflation vectors

Solving a linear system

$$Ax = b$$

Eigenvectors of the system matrix as deflation vectors

Lemma 1. Given

$$\sigma(A) = \{\lambda_1, \lambda_2 ... \lambda_p, \lambda_{p+1}, ..., \lambda_n\},$$

where

$$A\lambda_i = \lambda_1 v_i.$$

If $Z = [v_1, ..., v_p]$, then PA has the same eigenvectors as A and the spectrum is given by (ee the proof in [19, 25]):

$$\sigma(PA) = \{0, ..., 0, \lambda_{p+1}, ..., \lambda_n\}.$$

Recycled vectors as deflation vectors

Lemma 2. Given

$$x = \sum_{i=1}^{p} c_i x_i,$$

where the x_i's are linearly independent (l.i.) solutions such that $Ax_i = b_i$. If

$$Z = \left[x_1, ..., x_p\right],$$

the solution of the linear system is obtained within one iteration of DCG (see the proof in [14, 25]).

POD-based deflation method

We combine a POD basis with the Deflated Preconditioned Conjugate Gradient method preconditioned with Incomplete Cholesky (DICCG).

i) **Snapshots collection**. A set X of snapshots, vectors b_i with different configurations or at various time steps, is obtained for the computation of the POD basis.

Moving window approach: the window consists of a set of s snapshots updated for each time step.

$$X = [x^{t-s-1}, ..., x^{t-1}].$$

Training phase approach: a full pre-simulation is run.

$$X = [x^1, ..., x^n].$$

ii) **POD basis computation**.The previously obtained snapshots are used to construct a POD basis (Ψ).

iii) **Solution of the linear system**. The POD basis is used as subspace-deflation matrix (Z) in a deflation procedure for the solution of the linear system.

POD-based deflation method

Lemma 3. Let $Ax = b$, be a linear system with $A \in \mathbb{R}^{n\times n}$ symmetric with spectrum $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$, and eigenvectors $\sigma(A) = \{v_1, ..., v_n\}$, such that $v_i^T v_j = \delta_{ij}$.

We assume that the solution to this system can be written as $x = \sum_{i=1}^{p} a_i x_i$, where the $x_i's$ are linearly independent (l.i.) solutions to $Ax_i = b_i$. Rewriting the $x_i's$ as:

$$x_i = \sum_{k=1}^{n} c_k v_k, \qquad x_j = \sum_{m=1}^{n} \tilde{c}_m v_m,$$

and taking $c_\alpha \neq \tilde{c}_\beta$, and $|c_\alpha| >> |c_k|$, with $k \in [1, ..., \alpha-1, \alpha+1, ..., n]$, and $|\tilde{c}_\beta| >> |\tilde{c}_m|$, $m \in [1, ..., \beta-1, \beta+1, ..., n]$, the spectrum of the deflated system PA is given by:

$$\sigma(PA) = \{\lambda_1, ..., \lambda_{\alpha-1}, 0, \lambda_{\alpha+1}, ..., \lambda_{\beta-1}, 0, \lambda_{\beta+1}, ..., , \lambda_n\}.$$

See the proof in [14, 25].

Hence, if the unfavorable eigenvalues are captured in the l.i. solutions x_i and x_j, the behavior of the deflated method is the same as if these eigenvectors are used as deflation vectors.

Problem Definition

Reservoir Simulation

Single-phase flow through porous media [26]

Darcy's law + mass balance equation

$$-\nabla \cdot \left[\frac{\alpha\rho}{\mu}\vec{\mathrm{K}}(\nabla \mathrm{p} - \rho g \nabla d)\right] + \alpha\rho\phi c_t \frac{\partial \mathrm{p}}{\partial t} - \alpha\rho \mathrm{q} = 0.$$

$$c_t = (c_l + cr),$$

α a geometric factor

ρ fluid density

μ fluid viscosity

p pressure

$\vec{\mathrm{K}}$ rock permeability

g gravity

d depth

ϕ rock porosity

q sources

c_r rock compressibility

c_l liquid compressibility

Experiments: Heterogeneous permeability layers

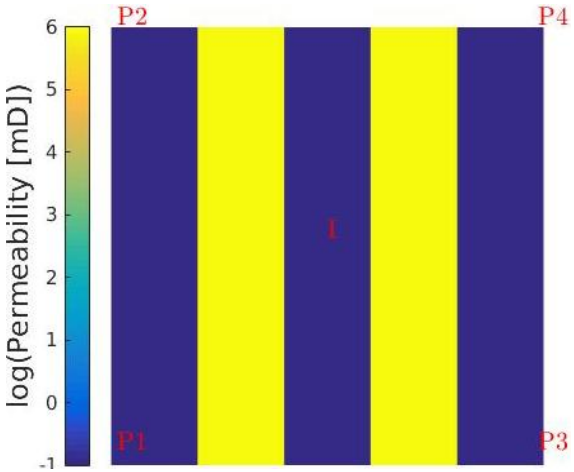
Water flooding with injection through the boundary and wells.

TC1. $c_\sigma = 10^{-1}$, no capillary pressure terms included.

TC2. $c_\sigma = 10^{-6}$, no capillary pressure terms included.

TC3. $c_\sigma = 10^{-1}$, capillary pressure terms included.

TC4. $c_\sigma = 10^{-6}$, capillary pressure terms included.



Rock permeability.

	Water	Oil	Units
μ	1	10	cp
ρ	1000	700	kg/m^3
k_r	$(S_w)^2$	$(1-S_w)^2$	
C_p	$10 * (1-S)$		bars
2D: 35 x 35 cells 3D: 25 x 25 x 25 cells.			

Results: Heterogeneous permeability layers

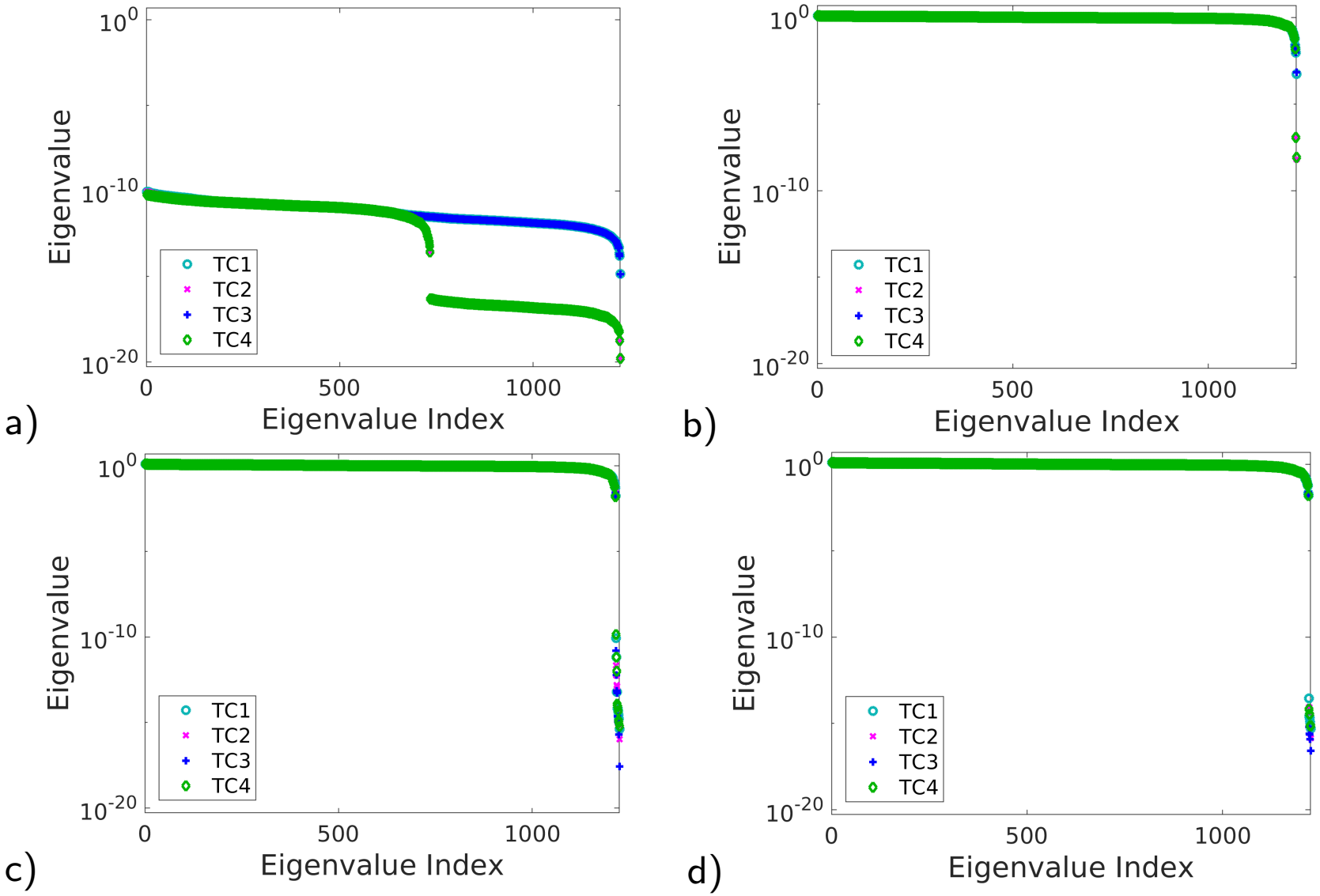


Figure: Eigenvalues of a)A , b)$M^{-1}A$, c)$P_{10}M^{-1}A$, d)$P_{POD_5}M^{-1}A$

Results: Heterogeneous permeability layers

	p	Total ICCG	DICCG ICCG	DICCG DICCG	Total DICCG	% of ICCG Iterations	% of ICCG Work
			2D Case				
TC1	10	23245	406	2830	3236	14	33.2
	5_{POD}			3929	4335	19	61.7
TC2	10	33834	194	3789	3983	12	28
	5_{POD}			5256	5450	16	47.6
TC3	10	23123	427	5792	6219	27	64
	5_{POD}			6296	6723	29	78.9
TC4	10	35507	220	6959	7179	20	48.1
	5_{POD}			7291	7511	21	55.2
			3D Case				
TC1	10	24231	498	3887	4385	18	37.7
	5_{POD}			4929	5427	22	56.5
TC2	10	20260	309	3342	3651	18	37.6
	5_{POD}			3117	3426	17	52.4
TC3	10	24557	513	8103	8616	35	73.1
	5_{POD}			8472	8985	37	78.1
TC4	10	19130	267	4180	4447	23	48.5
	5_{POD}			5084	5351	28	70.9

Results: Heterogeneous permeability layers

	p	Total ICCG	DICCG ICCG	DICCG DICCG	Total DICCG	% of ICCG Iterations	% of ICCG Work
			2D Case				
TC1	10	23245	406	2830	3236	14	33.2
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Results: Heterogeneous permeability layers

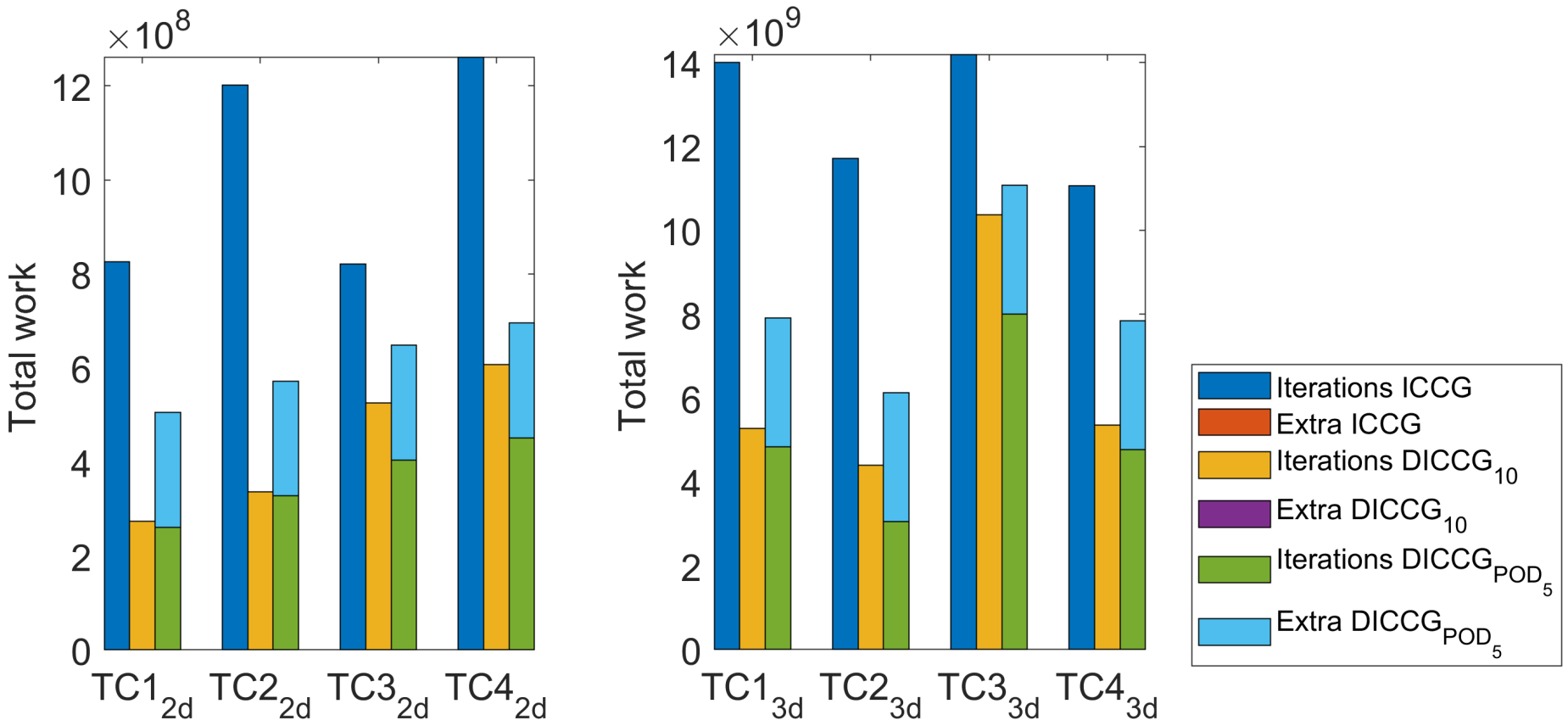


Figure: Total work for various methods, $\epsilon = 5 \cdot 10^{-7}$, Left: 2D cases, Right: 3D cases. The initial work is negligible in all cases.

Results: Heterogeneous permeability layers

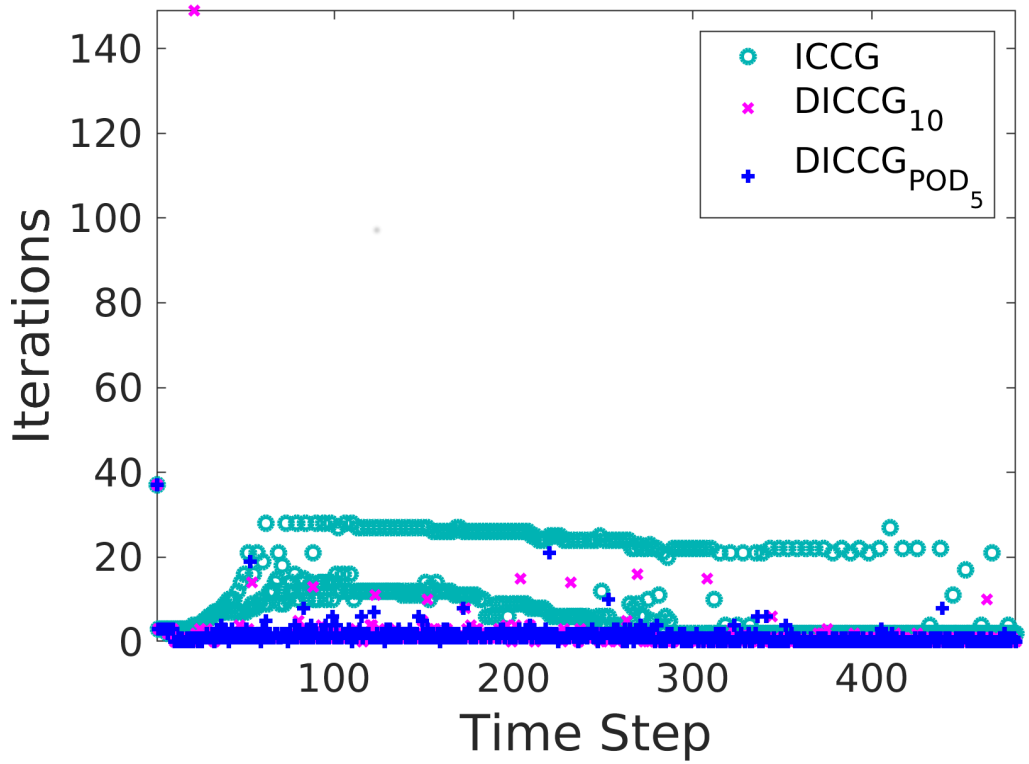
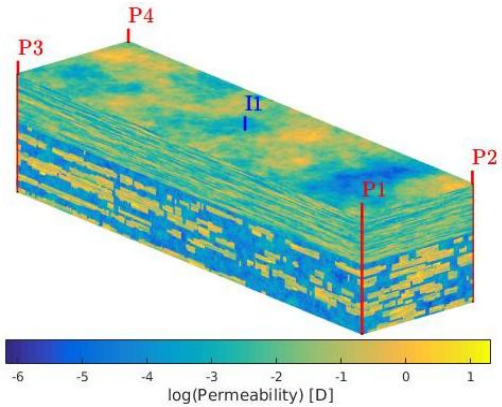


Figure: Number of iterations TC2 .

Experiments: SPE 10

Temporal parameters				Wells			
	2D	3D			Producers	Injector	
T_{steps}	480	240		MW	275	1100	bars
dT	15	1	day	TP training phase	random (137.5, 275)	1100	*bars*
				TP	experiments 200, 275, 400	1100	*bars*



$TC1_{MW}$. MW approach, 2D, no capillary pressure terms included, P: 275 [bars].

$TC2_{MW}$. MW approach, 2D, capillary pressure terms included, P: 275 [bars].

$TC3_{TP}$. TP approach, 2D, no capillary pressure terms included, P: 200 [bars].

$TC4_{TP}$. TP approach, 2D, capillary pressure terms included, P: 200 [bars].

$TC5_{TP}$. TP approach, 2D, no capillary pressure terms included, P: 275 [bars].

$TC6_{TP}$. TP approach, 2D, capillary pressure terms included, P: 275 [bars].

$TC7_{TP}$. TP approach, 2D, no capillary pressure terms included, P: 400 [bars].

$TC8_{TP}$. TP approach, 2D, capillary pressure terms included, P: 400 [bars].

$TC9_{MW}$. MW approach, 3D, no capillary pressure terms included, P: 275 [bars].

$TC10_{MW}$. MW approach, 3D, capillary pressure terms included, P: 275 [bars].

$TC11_{TP}$. TP approach, 3D, no capillary pressure terms included, P: 275 [bars].

$TC12_{TP}$. TP approach, 3D, capillary pressure terms included, P: 275 [bars].

Results: SPE 10

	p	Total ICCG	DICCG ICCG	DICCG DICCG	Total DICCG	% of ICCG Iterations	% of ICCG Work
2D case							
$TC1_{MW}$	10	80764	2241	5691	7932	10	23.4
	5_{POD}			6667	8908	11	27.1
$TC2_{MW}$	10	78950	2256	5334	7590	10	22.9
	5_{POD}			6018	8274	10	26.3
$TC3_{TP}$, P: 200[bars]	10_{POD}	80764			12044	15	35.5
	5_{POD}				20281	25	42.5
$TC4_{TP}$, P: 200[bars]	10_{POD}	78950			11746	15	35.4
	5_{POD}				17659	22	37.8
$TC5_{TP}$, P: 275[bars]	10_{POD}	83972			12935	15	37
	5_{POD}				21439	26	38.3
$TC6_{TP}$, P: 275[bars]	10_{POD}	83279			13078	16	37
	5_{POD}				19045	23	38.3
$TC7_{TP}$, P: 400[bars]	10_{POD}	72158			10279	14	33.9
	5_{POD}				17858	25	41.8
$TC8_{TP}$, P: 400[bars]	10_{POD}	70819			9824	14	33
	5_{POD}				15353	22	36.7
3D case							
$TC9_{MW}$	10	87789	5280	14798	20078	23	35.2
	5_{POD}			17084	22364	25	42.3
$TC10_{MW}$	10	76878	5323	14285	19608	26	39.3
	5_{POD}			15945	21268	28	46.1
$TC11_{TP}$	10_{POD}	87789			25410	29	44.6
	5_{POD}				40959	47	71.9
$TC12_{TP}$	10_{POD}	76878			22740	30	45.6
	5_{POD}				35617	46	71.4

Results: SPE 10

	p	Total ICCG	DICCG ICCG	DICCG DICCG	Total DICCG	% of ICCG Iterations	% of ICCG Work
2D case							
$TC1_{MW}$	10	80764	2241	5691	7932	10	23.4
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	5_{POD}				35617	46	71.4

Conclusions

- The use of a POD basis as deflation vectors requires some extra work, however, it makes the deflation method more robust.
- The total work is reduced up to 23% the ICCG work when using the DICCG method, for the 2D case, and up to 35% for the 3D case.
- The MW approach showed a better performance than the TP approach for the studied cases.
- A slightly better performance is observed if no capillary pressure terms are included.
- For the TP approach, a single basis can be used for alike problems, resulting in similar performance of the DICCG method.

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Thank you!!

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ORIGINAL PAPER

Accelerating the solution of linear systems appearing in two-phase reservoir simulation by the use of POD-based deflation methods

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Abstract
We explore and develop a Proper Orthogonal Decomposition (POD)-based deflation method for the solution of ill-conditioned linear systems, appearing in simulations of two-phase flow through highly heterogeneous porous media. We accelerate the convergence of a Preconditioned Conjugate Gradient (PCG) method achieving speed-ups of factors up to five. The up-front extra computational cost of the proposed method depends on the number of deflation vectors. The POD-based deflation method is tested for a particular problem and linear solver; nevertheless, it can be applied to various transient problems, and combined with multiple solvers, e.g., Krylov subspace and multigrid methods.

Keywords Deflation · Krylov Methods · Porous media · Two-phase reservoir simulation

More info: