

# Deflated ICCG method applied to problems with extreme contrasts in the coefficients

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## Abstract

A Deflated ICCG method is applied to problems with extreme contrasts in the coefficients, such as those encountered in reservoir simulation. Two choices for the deflation vectors are considered. The first one is based on the physics of the problem, whereas the second one is algebraic, and is very suitable for parallel computing. Theoretical and numerical results are presented.

**Key words:** Deflation, porous media, incomplete factorizations, parallel computing, conjugate gradients.

**AMS subject classifications:** 65F10, 65Y05, 76S05.

## 1 Introduction

For an oil company it is important to know the fluid pressure history in the subsurface in order to predict the presence of oil and natural gas in reservoirs. A mathematical model for this prediction is given by a time-dependent diffusion equation. This equation is discretized by the finite element method leading to a large sparse system of linear equations. It is well known that the permeability coefficient can differ by orders of magnitude between various layers in the earth's crust [2]. This leads to a large condition number of the symmetric and positive definite coefficient matrix. When ICCG is used to solve the system one observes a bad convergence behavior and classical termination criteria are no longer reliable.

We simplify the problem considerably, taking care that its characteristic properties are preserved. Assume that we have to solve the stationary linear diffusion equation:

$$-\operatorname{div}(\sigma \nabla p) = 0 \text{ on } \Omega,$$

with boundary conditions  $p = f$  on  $\partial\Omega^D$  (Dirichlet) and  $\frac{\partial p}{\partial n} = g$  on  $\partial\Omega^N$  (Neumann), where  $\partial\Omega = \partial\Omega^D \cup \partial\Omega^N$ . The fluid pressure and permeability are denoted by  $p$  and  $\sigma$  respectively. The domain  $\Omega$  consists of a number of subdomains in each of which  $\sigma$  is constant. Two values for  $\sigma$  are considered:  $\sigma^h = 1$  for high-permeability subdomains and  $\sigma^l = \varepsilon$  for low-permeability subdomains (e.g. the permeabilities ratio for shale and sandstone:  $\varepsilon$  is of the order  $10^{-7}$  see [3]). The subdomains are denoted by the disjoint sets  $\Omega_i$ ,  $i \in \{1, \dots, k\}$ , which are such that:  $\cup_{i=1}^k \bar{\Omega}_i = \bar{\Omega}$  and neighboring sets have a different permeability coefficient (when  $\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset$  then  $\sigma_i \neq \sigma_j$ ).

**Definition 1.1** The high-permeability subdomains are numbered first:  $\Omega_i$ ,  $i \in \{1, \dots, k^h\}$ . Furthermore the first  $k^s$  high-permeability subdomains are such that  $\bar{\Omega}_i \cap \partial\Omega^D = \emptyset$ ,  $i \in \{1, \dots, k^s\}$ . ■

In this paper we prove that the number of small eigenvalues of the IC preconditioned matrix is equal to  $k^s$  the number of high-permeability layers that are not connected to a Dirichlet boundary. These eigenvalues lead to the high condition number.

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It appears that a Deflated ICCG method can be used to overcome the adverse effect of the eigenvalues on the convergence behavior and the termination criterion. We have to solve the system  $Ax = b$  with  $A \in \mathbb{R}^{n \times n}$ . We define the projection  $P$  by  $P = I - AZ(Z^T AZ)^{-1}Z^T$ ,  $Z \in \mathbb{R}^{n \times m}$ . Since  $x = (I - P^T)x + P^T x$  and since  $(I - P^T)x = Z(Z^T AZ)^{-1}Z^T Ax = Z(Z^T AZ)^{-1}Z^T b$  can easily be computed, we need only compute  $P^T x$ . Because  $AP^T = PA$ , we can solve the deflated system  $PAx = Pb$  [3]. We consider two different choices of the deflation vectors  $Z$ .

## 2 Physical deflation vectors

As a first choice we take the number of deflation vectors equal to the number of small eigenvalues of the IC preconditioned matrix. On physical arguments we choose the deflation vectors as follows: the value of the deflation vector is 1 in one high-permeability layer and 0 in all the other high-permeability layers. In the low-permeability layers the deflation vector is an approximate solution of the diffusion problem [4].

We assume that the finite element discretization is consistent, which means that the discretization error is zero for a constant function. The subdomains  $\Omega_i$  are approximated by polygons and each element is contained in only one polygon. Finally we assume that the off-diagonal elements of  $A$  are non-positive. Using these assumptions we prove the following theorem.

**Theorem 2.1** *The deflation vectors  $z_i$ ,  $1 \leq i \leq k^s$  are such that  $\|L^{-T}L^{-1}Az_i\|_2 = O(\epsilon)$ ,  $i \in \{1, \dots, k^s\}$ .*

From this result one can prove that the 'small' eigenspace of  $L^{-T}L^{-1}A$  is 'nearly' a subspace of the span $\{z_1, \dots, z_{k^s}\}$ . This suggests that the convergence of DICCG is independent of the ratio of the high and low permeability. This is confirmed by numerical experiments.

For our numerical experiment we choose an oil flow problem. We consider a 9 layer problem. Five sandstone layers are separated by 4 shale layers. The layers vary in thickness and orientation. The IC preconditioned matrix of this problem has 4 small eigenvalues. The (D)ICCG results are given in Table 1. There is a large improvement in the number of iterations and CPU time. We also investigate the effect of the jump in permeabilities. We use a grid with 19665 nodal points, take the permeability in the sandstone layers equal to 1 and vary the permeability in the shale layers. The results in Table 2 confirm that DICCG is independent of the value of  $\sigma_{\text{shale}}$ .

nodal points	ICCG		DICCG		
	iterations	CPU	iterations	CPU	CPU construction
2760	47	1.19	10	0.37	0.12
19665	83	19.1	20	6.22	1.29
148185	189	350	44	108	12.7

Table 1: Number of iterations and CPU time for the 9 layer problem

$\sigma_{\text{shale}}$	ICCG		DICCG	
	$\lambda_{\min}$	iterations	$\lambda_{\min}$	iterations
$10^{-3}$	$1.5 \cdot 10^{-2}$	26	$6.9 \cdot 10^{-2}$	20
$10^{-4}$	$2.0 \cdot 10^{-3}$	39	$8.7 \cdot 10^{-2}$	19
$10^{-5}$	$2.2 \cdot 10^{-4}$	59	$7.7 \cdot 10^{-2}$	20
$10^{-6}$	$2.2 \cdot 10^{-5}$	73	$7.8 \cdot 10^{-2}$	20
$10^{-7}$	$2.3 \cdot 10^{-6}$	82	$7.7 \cdot 10^{-2}$	20

Table 2: The smallest nonzero eigenvalue and the number of iterations for the 9 layer problem

## 3 Coarse grid deflation vectors

As a second choice we take the number of deflation vectors equal to the number of layers. Now the deflation vectors are chosen equal to 1 in one layer and 0 in all other layers. We define  $B$  as the subdomain block-Jacobi submatrix of  $A$ . Let  $\Sigma(B)$  be the vector containing the row sums of  $B$  and  $A^* = B - \text{diag}(\Sigma(B))$ . The following theorem is proven in [1].

**Theorem 3.1** *Let  $A$  be symmetric and positive definite then*

$$\lambda_i(A^*) \leq \lambda_i(PA) \leq \lambda_i(A^*) + \lambda_{\max}(PC), \text{ where } C = A - A^*.$$

Moreover the effective condition number  $K_{eff}$  of  $PA$  is bounded by  $\frac{\lambda_n(A)}{\lambda_{m+1}(A^*)}$ .

We have also proven a preconditioned version of this theorem. An interpretation of this result is: subdomain deflation effectively decouples the original system into a set of independent Neumann problems on the subdomains with convergence governed by the “worst conditioned” Neumann problem. This implies that the effective condition number of the Deflated, Block IC preconditioned matrix is independent of the jumps in the permeabilities.

To illustrate this we consider a small diffusion problem with two layers in 1D. The permeability coefficient of the sand layer is equal to 1 and we vary the permeability coefficient ( $\epsilon$ ) of the shale layer. The results for a diagonal preconditioner are given in Table 3. Due to the diagonal preconditioning the smallest nonzero eigenvalue of  $D^{-1}A^*$  is independent of  $\epsilon$ . As expected the same property holds for  $D^{-1}PA$ . The smallest eigenvalue of  $D^{-1}A$ , however, decreases proportional to  $\epsilon$ . This leads to a large condition number and slow convergence of the conjugate gradient method when it is applied to  $D^{-1}Ax = D^{-1}b$ .

$\epsilon$	$\lambda_1(D^{-1}A)$	$\kappa(D^{-1}A)$	$\lambda_3(D^{-1}PA)$	$\kappa_{eff}(D^{-1}PA)$
1	$1.9 \cdot 10^{-2}$	$10^2$	$2.9 \cdot 10^{-1}$	6.5
$10^{-2}$	$3.3 \cdot 10^{-4}$	$6 \cdot 10^3$	$2.9 \cdot 10^{-1}$	6.8
$10^{-4}$	$3.3 \cdot 10^{-6}$	$6 \cdot 10^5$	$2.9 \cdot 10^{-1}$	6.8

Table 3: Condition numbers of  $D^{-1}A$  and  $D^{-1}PA$

This Block IC preconditioner and deflation are very suitable for parallel computing. To show this we present some timing results on the Cray T3E for a diffusion problem with permeability coefficient equal to 1. The results for a  $480 \times 480$  grid are given in Table 4. It appears that the number of iterations decreases when the number of blocks increases. This leads to an efficiency larger than 1. To explain this we note that when the number of blocks increases, more small eigenvalues are projected to zero which accelerates the convergence.

p	1	4	9	16	25	36	64
iterations	485	322	352	379	317	410	318
time	710	120	59	36	20	18	8
speedup	-	5	12	20	36	39	89

Table 4: Speedup of the iterative method using a  $480 \times 480$  grid

## 4 Conclusions

Deflated ICCG is an efficient method for problems with large jumps in the coefficients. Furthermore it can also be used as an efficient parallel iterative method for diffusion problems with a smoothly varying coefficient.

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