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Stability Criteria for the Vertical Boundary Layer Formed by Throughflow Near the Surface of a Porous Medium

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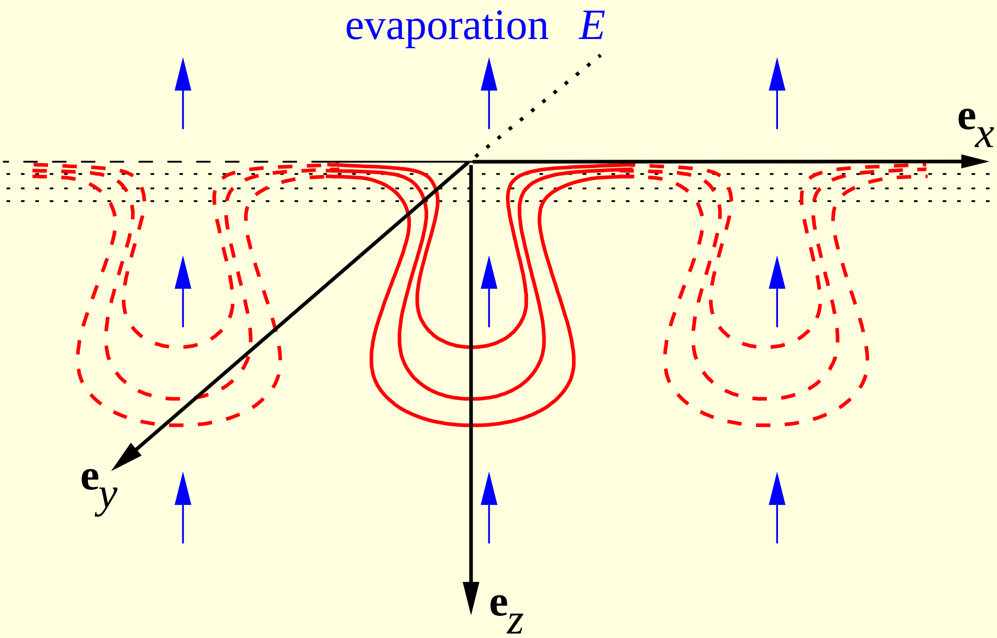
Salt Lakes: Lake Eyre Australia



$$\Omega = \{(x, y, z) : -\infty < x, y < \infty,\ z > 0\}$$

Equations for salt transport

- Conservation of the fluidum (groundwater)

$$\phi\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{q})=0,\quad (t,x,y,z)\in\mathbb{R}^{+}\times\boldsymbol{\Omega}.$$

- Conservation of salt dissolved in groundwater

$$\phi\frac{\partial(\rho\omega)}{\partial t}+\nabla\cdot(\rho\omega\mathbf{q}-\rho\mathbb{D}\nabla\omega)=0,$$
$$(t,x,y,z)\in\mathbb{R}^{+}\times\boldsymbol{\Omega}.$$

- Darcy's law with gravitation

$$\frac{\mu}{\kappa}\mathbf{q}+\nabla p-\rho g\mathbf{e}_z=\mathbf{0},\quad (t,x,y,z)\in\mathbb{R}^{+}\times\boldsymbol{\Omega}.$$

- Equation of state $\rho=\rho_{\mathrm{f}}\,\mathrm{e}^{\alpha\omega}$.

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Initial- and boundary conditions

- $\rho\big|_{t=0} = \rho_r$ in Ω,

- $\mathbf{q} = -E\mathbf{e}_z$ at $z = 0$ and for $t > 0$,

  (E is the evaporation rate)

- $\rho = \rho_m$ at $z = 0$ and for $t > 0$.

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Dimensionless equations

Introducing the scales

$$\mathbf{U} := \frac{\mathbf{q}}{u_c}\,, \qquad u_c := \frac{(\rho_m - \rho_r) g \kappa}{\mu}\,,$$

$$t := t\,\mathbb{D}/E\,, \qquad \{x,y,z\} := \{x,y,z\}\,\phi\mathbb{D}/E^2\,,$$

gives for $(t,x,y,z) \in \mathbb{R}^+ \times \boldsymbol{\Omega}$

$$(\text{P-I}) \begin{cases} \dfrac{\partial S}{\partial t} + \text{Ra}\,(\mathbf{U} \cdot \nabla S) = \Delta S\,, \\ \nabla \cdot \mathbf{U} = 0\,, \\ \mathbf{U} + \nabla P - S\mathbf{e}_z = \mathbf{0}\,, \end{cases}$$

with

$$\text{Ra} = \frac{(\rho_m - \rho_r) g \kappa}{\mu E} = \frac{u_c}{E}\,.$$

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Dimensionless initial- and boundary conditions

$$\text{(IBC)} \begin{cases} S\Big|_{t=0} = 0 \quad \text{in} \quad \Omega, \\[2ex] \mathbf{U} = -\dfrac{1}{\text{Ra}}\mathbf{e}_z \quad \text{at} \quad z = 0 \quad \text{and for} \quad t > 0, \\[2ex] S = 1 \quad \text{at} \quad z = 0 \quad \text{and for} \quad t > 0. \end{cases}$$

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Ground-state solution

The ground-state is characterized by the uniform upflow

$$\mathbf{U} = \mathbf{U_0} := -\frac{1}{\text{Ra}}\mathbf{e}_z ,$$

and the growing boundary layer ($z > 0$)

$$S = S_0(z,t) := \frac{1}{2}\,\mathrm{e}^{-z}\,\mathrm{erfc}\left(\frac{z-t}{2\sqrt{t}}\right) + \\ + \frac{1}{2}\,\mathrm{erfc}\left(\frac{z+t}{2\sqrt{t}}\right) .$$

We restrict the analysis to the equilibrium case only, i.e.

$$\lim_{t\to\infty} S_0(z,t) = \mathrm{e}^{-z}.$$

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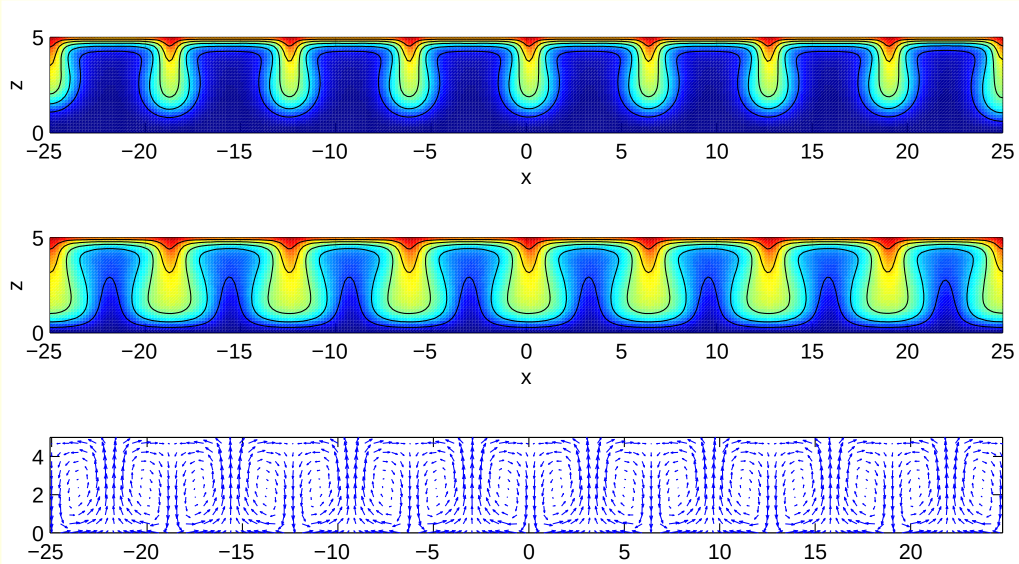
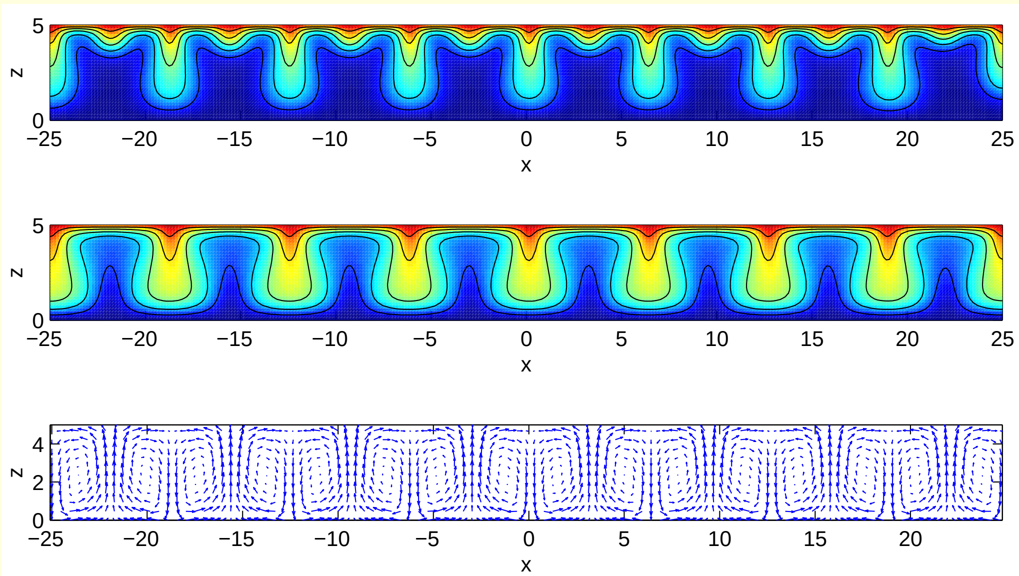
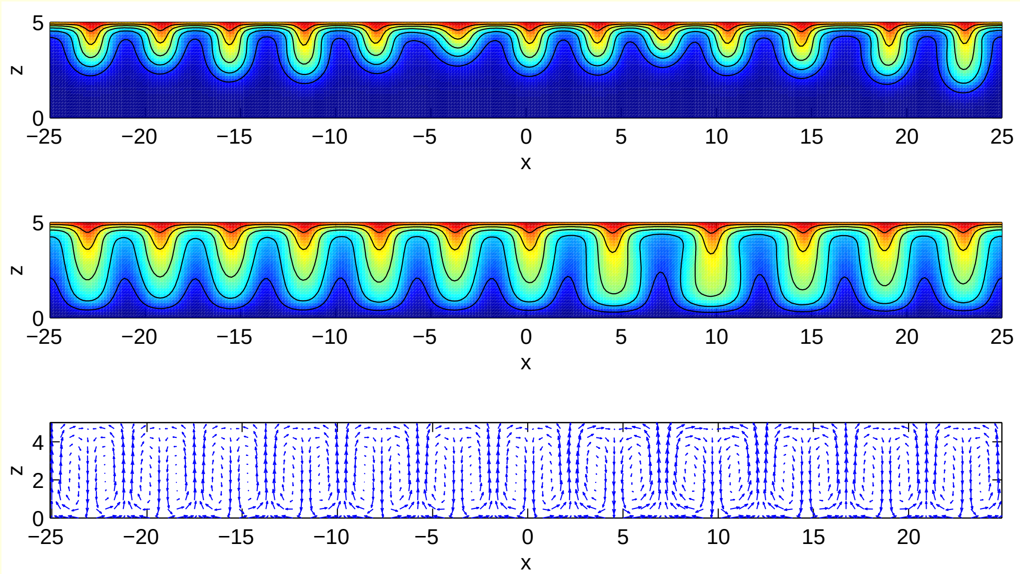
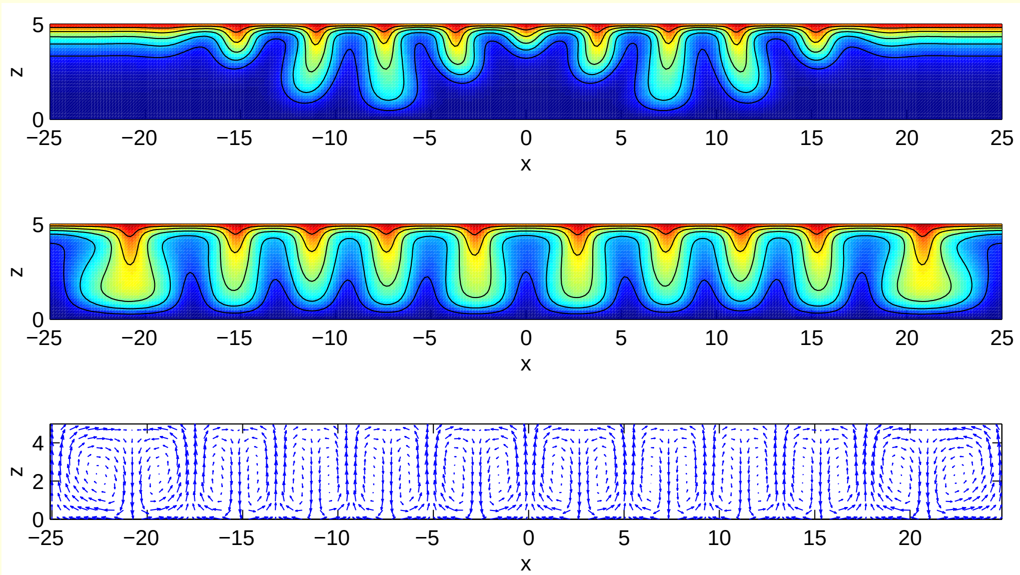
Numerical solutions I (Ra $= 35$)





Numerical solutions II (Ra $= 35$)





Analysis of perturbation equations

The stability analysis is based on the expansion

$$S = S_0 + s \ ,$$

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{u} \ ,$$

$$P = P_0 + p \ ,$$

Substitution in (P-I) yields (in Ω and for all $t > 0$)

$$\nabla \mathbf{u} = 0 \ , \qquad (1)$$

$$\mathbf{u} + \nabla p - s\mathbf{e}_z = \mathbf{0} \ , \qquad (2)$$

$$\frac{\partial s}{\partial t} - \frac{\partial s}{\partial z} + \mathrm{Ra}\, w \frac{\partial S_0}{\partial z} + \mathrm{Ra}\, \mathbf{u} \cdot \nabla s = \Delta s \ . \qquad (3)$$

(1)–(2) give for s and w the linear relation

$$\Delta w = \Delta_\perp s \ , \qquad \Delta_\perp := \partial_{xx} + \partial_{yy} \ . \qquad (4)$$

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Methods

- Method of linearised stability
- Energy method (1966, D.D. Joseph)
- Nonmodal analysis (1993, D.S. Henningson et al.)

The method of linearised stability

We disregard the nonlinear term $\text{Ra}\,\mathbf{u}\cdot\nabla s$ in (3) and consider the *linear* evolution problem in Ω and for $t > 0$

$$\begin{cases} \dfrac{\partial s}{\partial t} - \dfrac{\partial s}{\partial z} + \text{Ra}\, w \dfrac{\partial S_0}{\partial z} = \Delta s\ , \\ \Delta w = \Delta_\perp s\ . \end{cases}$$

Furthermore we *assume*

$$\{s, w\} = \{s(z), w(z)\}\, \mathrm{e}^{\sigma t + i a_x x + i a_y y}\ .$$

Substitution gives

$$\text{(P-II)} \begin{cases} (D^2 + D - a^2 - \sigma)s = -\text{Ra}\, \mathrm{e}^{-z} w\ , \\ (D^2 - a^2)w = -a^2 s\ , \end{cases}$$

with $D = \dfrac{\mathrm{d}}{\mathrm{d}z}$ and $a^2 = a_x^2 + a_y^2$.

Neutral stability

Proposition 1. *Let* $a > 0$ *and* Ra > 0 *be given. Furthermore, rewrite* (P-II) *in an eigenvalue problem for* σ*, i.e.*

$$\begin{cases} (D^2 + D - a^2 + \mathrm{Ra}\,\mathrm{e}^{-z}w = \sigma s \,, \\ (D^2 - a^2)w = -a^2 s \,. \end{cases}$$

Then $\sigma \in \mathbb{R}$ *for each* $a > 0$ *and* Ra > 0*.*

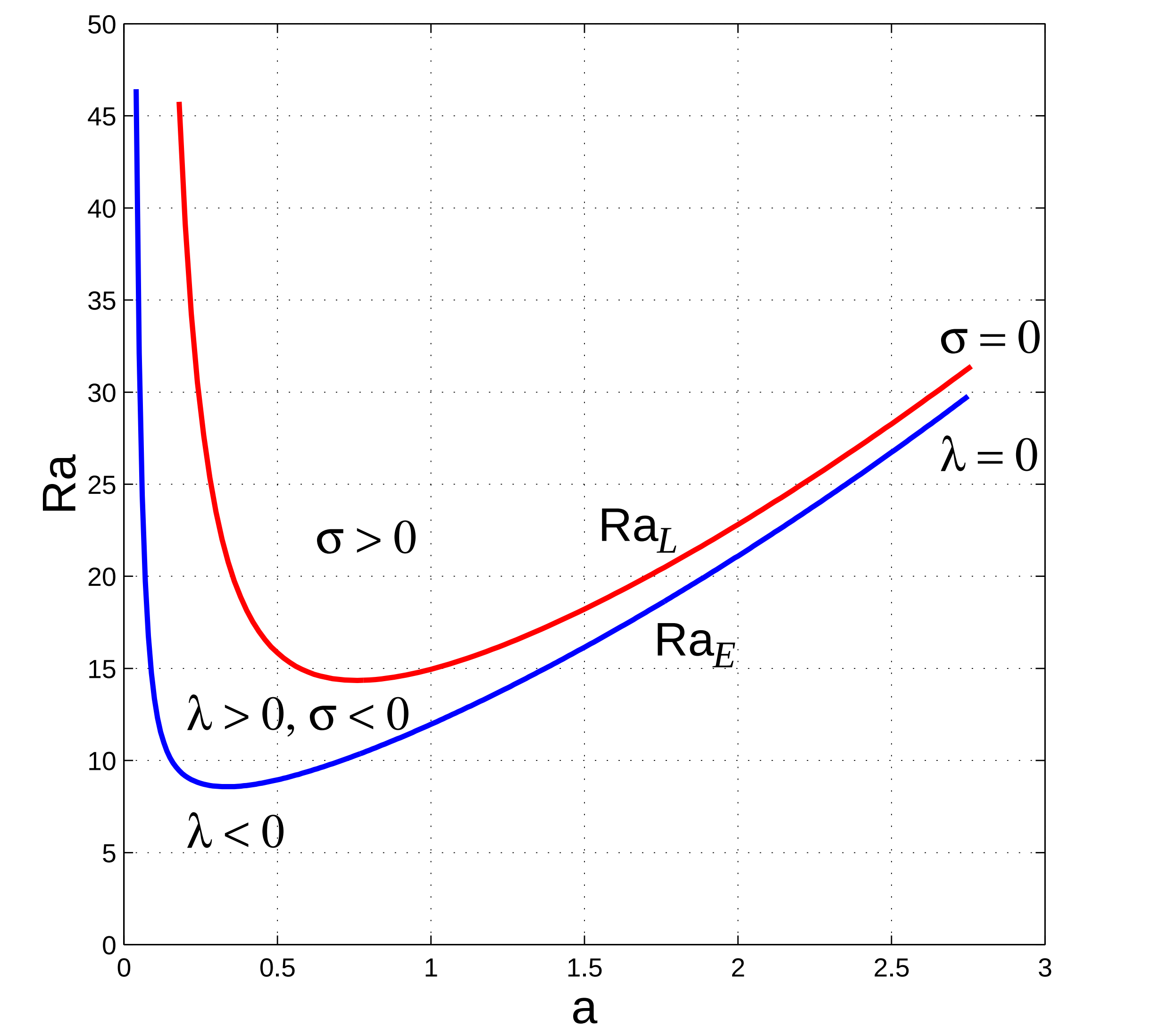
Theorem 1. *Given* $\sigma \in \mathbb{R}$*, let* $\mathrm{Ra}_1(a;\sigma)$*, for each* $a > 0$*, denote the smallest positive eigenvalue of problem* (P-II)*. Then*

$$\mathrm{Ra}_1(a;\sigma) \gtrless \mathrm{Ra}_1(a;0) \quad \text{if and only if} \quad \sigma \gtrless 0 \,.$$

$$\Downarrow$$

Sufficient: $\sigma \equiv 0$

Stabilitycurves



$$\mathcal{SC} = \left\{(a, \mathrm{Ra}) : a > 0,\ \mathrm{Ra}_E(a) < \mathrm{Ra} < \mathrm{Ra}_L(a)\right\}$$

The energy method

Aim:

$$\frac{\mathrm{d}}{\mathrm{d}t}||s||_2^2 < 0 \quad \Longrightarrow \quad \text{stability}$$

We assume x, y-periodicity $\longrightarrow$ analysis is restricted to the periodicity cell

$$\mathscr{V} = \left\{(x, y, z) : |x| < \pi/a_x, |y| < \pi/a_y, z > 0\right\}$$

Multiply (3) by s and integrate over $\mathscr{V}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathscr{V}} s^2 = -\int_{\mathscr{V}} |\nabla s|^2 + \mathrm{Ra}\int_{\mathscr{V}} sw\,\mathrm{e}^{-z} < 0\,.$$

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Maximization problem

$$\frac{1}{\mathsf{Ra}} = \sup_{(s,w)\in\mathscr{H}} \frac{\displaystyle\int_{\mathscr{V}} sw\,\mathrm{e}^{-z}}{\displaystyle\int_{\mathscr{V}} |\nabla s|^2}\,,$$

with

$$\mathscr{H} = \Big\{(s,w) : x,y\text{-periodic with respect to } \mathscr{V},$$
$$s = w = 0 \text{ at } z = 0,\infty, \text{ and } \Delta w = \Delta_\perp s\Big\}$$

After some tedious algebra we find the following sixth order eigenvalue problem for w:

$$(D^2 - a^2)^3 w - \frac{a^2\mathsf{Ra}}{2}\Big\{(D^2 - a^2)(\mathrm{e}^{-z}w) +$$
$$+\mathrm{e}^{-z}(D^2 - a^2)w\Big\} = 0\,. \quad (5)$$

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Comparison of eigenvalues

Proposition 2. *Let* $a > 0$ *be given. Let* $\mathrm{Ra}_L(a)$ *be the eigenvalue of the linearised approach and* $\mathrm{Ra}_E(a)$ *be the eigenvalue from the energy method with differential constraint. Then, for each* $a > 0$*,*

$$\mathrm{Ra}_E(a) \leqslant \mathrm{Ra}_L(a)\ .$$

Proof. From the linearised problem it follows that

$$\frac{1}{\mathrm{Ra}_L(a)} = \frac{\displaystyle\int_{\mathbb{R}^+} s_1 w_1 \,\mathrm{e}^{-z}}{\displaystyle\int_{\mathbb{R}^+} (Ds_1)^2 + a^2 s_1^2}$$

Clearly, $(s_1, w_1) \in \mathscr{H}$, which implies

$$\frac{1}{\mathrm{Ra}_L(a)} \leqslant \sup_{(s,w)\in\mathscr{H}} \frac{\displaystyle\int_{\mathbb{R}^+} sw \,\mathrm{e}^{-z}}{\displaystyle\int_{\mathbb{R}^+} (Ds)^2 + a^2 s^2} = \frac{1}{\mathrm{Ra}_E(a)}\ .$$

□

Nonmodal stability theory

Starting point is the linearised perturbation equation

$$\text{(P-III)} \begin{cases} \dfrac{\partial s}{\partial t} = \mathscr{L}_{\mathsf{Ra},a}\, s \;, \\[2ex] s\big|_{t=0} = s_0 \;, \end{cases}$$

with

$$\begin{aligned} \mathscr{L}_{\mathsf{Ra},a} &= D^2 + D - a^2 + a^2 \mathsf{Ra}\, \mathrm{e}^{-z} \mathscr{M}_a^{-1} \;, \\ \mathscr{M}_a &= -D^2 + a^2 \;. \end{aligned}$$

Again we consider the energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} ||s||_2^2 = (\mathscr{L}_{\mathsf{Ra},a}\, s, s) \leqslant \lambda ||s||_2^2 \;,$$

with

$$\lambda = \sup_{s \in \mathscr{F}} \frac{(\mathscr{L}_{\mathsf{Ra},a}\, s, s)}{(s, s)} \;. \qquad (6)$$

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Euler–Lagrange equation for (6) reads

$$\frac{1}{2}(\mathscr{L}_{\mathsf{Ra},a} + \mathscr{L}^*_{\mathsf{Ra},a})s = \lambda s \ , \qquad (7)$$

with

$$\mathscr{L}^*_{\mathsf{Ra},a} = D^2 - D - a^2 + a^2 \mathsf{Ra}\, \mathscr{M}_a^{-1}(\mathrm{e}^{-z}\ \cdot) \ .$$

Proposition 3 (Neutral stability). *Let* $\lambda \equiv 0$*. Then, for each* $a > 0$*, eigenvalue problem* (7) *for* Ra *is equivalent to eigenvalue problem* (5) *(energy method).*

As a consequence, the stability curves coincide.

Asymptotic bounds for $||s(t)||_2^2$

Proposition 4 (Initial energy growth). *Let* $(a_1, \mathsf{Ra}_1) \in \mathcal{SC}$ *and let* (s_1, λ_1), $\lambda_1 > 0$, *be the corresponding eigenpair of problem* (7). *Further, let*

$$s(z,t) = \mathrm{e}^{t\mathscr{L}_{\mathsf{Ra}_1,a_1}} s_1(z)$$

be the formal solution of $(\mathrm{ACP})_1$ *with* $s\big|_{t=0} = s_1$. *Then*

$$\frac{||s(t)||_2^2}{||s(0)||_2^2} = 1 + 2\lambda_1 t + \mathcal{O}(t^2)\ .$$

Proof. We have

$$\begin{aligned} ||s(t)||_2^2 &= \left(\mathrm{e}^{t\mathscr{L}_{\mathsf{Ra}_1,a_1}} s_1, \mathrm{e}^{t\mathscr{L}_{\mathsf{Ra}_1,a_1}} s_1\right) = \\ &= \left(\mathrm{e}^{t(\mathscr{L}_{\mathsf{Ra}_1,a_1} + \mathscr{L}^*_{\mathsf{Ra}_1,a_1})} s_1, s_1\right) = \\ &= ||s_1||_2^2 + \big((\mathscr{L}_{\mathsf{Ra}_1,a_1} + \mathscr{L}^*_{\mathsf{Ra}_1,a_1}) s_1, s_1\big) t + \mathcal{O}(t^2) = \\ &= ||s_1||_2^2 + 2\lambda_1 ||s_1||_2^2\, t + \mathcal{O}(t^2)\ . \end{aligned}$$

□

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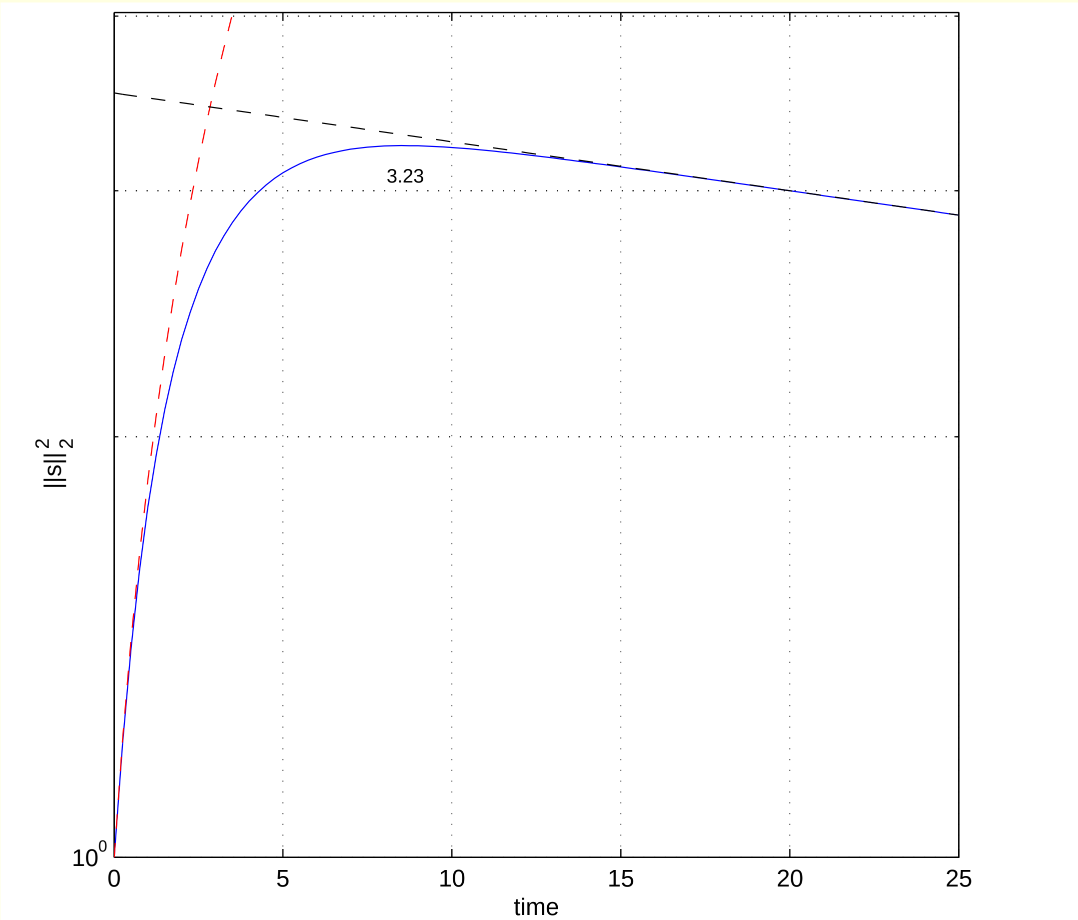
Proposition 5. *Under the same conditions of Proposition 4 we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{||s(t)||_2^2}{||s(0)||_2^2} = 2\,\sigma_{\max}\ ,$$

where $\sigma_{\max}$ *is the largest eigenvalue of operator* $\mathscr{L}_{\mathrm{Ra},a}$.



Conclusion: in the linear regime there exist perturbations for which the energy grow in time, whereas the linearised theory predicts stability, i.e. all growthrates σ lie in the stable halfspace.

Contradiction ?

We can write the solution of (ACP) formally as

$$s(z,t) = \sum_i A_i \, \mathrm{e}^{\sigma_i t} s_i(z) \ .$$

Operator $\mathscr{L}_{\mathrm{Ra},a}$ is *non-normal*, i.e. the eigenfunctions are not orthogonal:

$$(s_i, s_j) = \int_{\mathbb{R}^+} s_i s_j \neq 0 \ , \qquad i \neq j \ .$$

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Discussion

- The nonlinear term never plays a role in the analysis
- Energy methods depend on the choice of the norm
- Determination of the “threshold” amplitude of the perturbations