Multigrid acceleration of a preconditioned Krylov method for the solution of the discretized vector wave equation

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Outline

About NLR

NLR

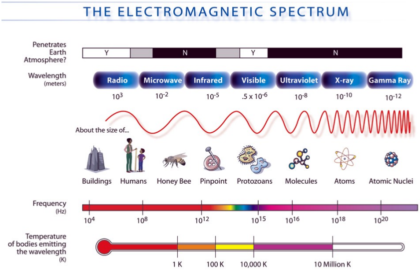
- NLR = Nationaal Lucht- en Ruimtevaartlaboratorium (in Dutch)
  National Aerospace Laboratory (in English)
- Consists of two residences: one in Amsterdam and one in NOP[1], Marknesse
- Independent technological institute
- About 700 employees

[1] NOP = Noordoostpolder

Problem description

- When the radar signature of a military platform cannot be determined experimentally, *numerical prediction* techniques are used
- In this thesis a faster and more memory efficient prediction method is proposed

- RADAR (RAdio Detection and Ranging): technology to detect military platforms (e.g. aircraft, ship or tank) by using electromagnetic waves



CLASS	FREQUENCY	WAVELENGTH	ENERGY
Y	300 EHz	1 pm	1.24 MeV
HX	30 EHz	10 pm	124 keV
	3 EHz	100 pm	12.4 keV
SX	300 PHz	1 nm	1.24 keV
EUV	30 PHz	10 nm	124 eV
NUV	3 PHz	100 nm	12.4 eV
NIR	300 THz	1 μm	1.24 eV
MIR	30 THz	10 μm	124 meV
FIR	3 THz	100 μm	12.4 meV
EHF	300 GHz	1 mm	1.24 meV
SHF	30 GHz	1 cm	124 μeV
UHF	3 GHz	1 dm	12.4 μeV
VHF	300 MHz	1 m	1.24 μeV
HF	30 MHz	10 m	124 neV
MF	3 MHz	100 m	12.4 neV
LF	300 kHz	1 km	1.24 neV
VLF	30 kHz	10 km	124 peV
VF/ULF	3 kHz	100 km	12.4 peV
SLF	300 Hz	1 Mm	1.24 peV
ELF	30 Hz	10 Mm	124 feV
	3 Hz	100 Mm	12.4 feV

Figure: RADAR – infrared detection – optic detection

RADAR

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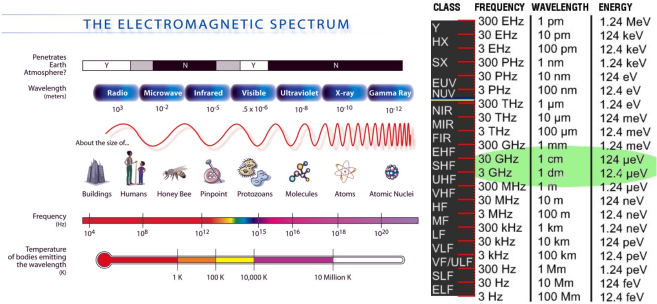

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Assumptions – mathematical model

- Radar excited from the front: jet engine air intake (of a modern fighter aircraft) accounts for the major part of the RCS for a large range of observation angles
- Maxwell's equations $\Rightarrow$ vector wave equation

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Figure: Jet engine air intake (left) closed by jet engine compressor fan (right) – together forms a large and deep open-ended cavity with varying cross section – *Dimensions:* $d \approx 30\lambda$ *and* $L \approx 200\lambda$ *for X-band excitation (10 GHz)*

RCS – definition

- Measure of detectability: radar cross section (RCS) (depends on observation angle, frequency, polarization)
- A platform is detected when the signal-to-noise ratio exceeds a certain threshold

RCS: measure or predict?

Measuring RCS not possible if:

- platform is still in design, development or procurement phase or belongs to a hostile party
- time consuming or measurements too expensive

$\Rightarrow$ Numerical prediction techniques must be used

Maxwell's equations

$$\nabla^* \times \mathcal{E}^* = -\frac{\partial^* \mathcal{B}^*}{\partial^* t^*}$$

$$\nabla^* \times \mathcal{H}^* = \frac{\partial^* \mathcal{D}^*}{\partial^* t^*} + \mathcal{J}^*$$

$$\nabla^* \cdot \mathcal{D}^* = \mathcal{Q}^*$$

$$\nabla^* \cdot \mathcal{B}^* = 0$$

$$\nabla^* \cdot \mathcal{J}^* = -\frac{\partial^* \mathcal{Q}^*}{\partial^* t^*}$$

The following (dimension full) variables are used with their S.I. unit between brackets:

$\mathcal{E}^*$ = electric field intensity $[\frac{V}{m}]$

$\mathcal{D}^*$ = electric flux density $[\frac{C}{m^2}]$

$\mathcal{H}^*$ = magnetic field intensity $[\frac{A}{m}]$

$\mathcal{B}^*$ = magnetic flux density $[\frac{Wb}{m^2}]$

$\mathcal{J}^*$ = electric current density $[\frac{A}{m^2}]$

$\mathcal{Q}^*$ = electric charge density $[\frac{C}{m^3}]$

t^* = time $[s]$

The vector wave equation (1/2)

Under additional assumptions:

field quantities above are harmonic oscillating functions with an angular frequency ω^* (*time-harmonic functions*)

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Maxwell's equations are rewritten in *phasor* quantities as:

$$\nabla^* \times \mathbf{E}^* = j\omega^* \mathbf{B}^*$$

$$\nabla^* \times \mathbf{H}^* = j\omega^* \mathbf{D}^* + \mathbf{J}^*$$

$$\nabla^* \cdot \mathbf{J}^* = -j\omega^* q^* \qquad j^2 = -1$$

The vector wave equation (2/2)

Combined with the *constitutive relations*[2], the dimensionless *vector wave equation* for the electric field can be derived as:

$$\bigtriangledown \times \bigtriangledown \times \mathbf{E} - k_0^2 \mathbf{E} = -jk_0 Z_0 \mathbf{J} \quad (1)$$

Additional boundary conditions to obtain a well-posed problem:

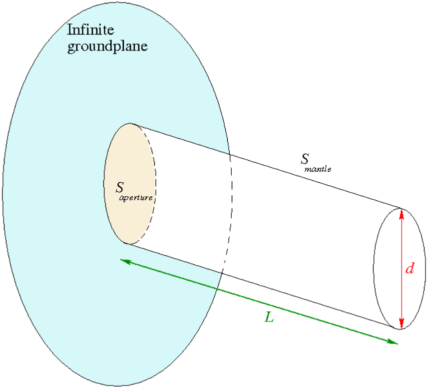
$$(\hat{n} \times \mathbf{E})_{S_{mantle}} = 0$$

$$(\hat{n} \times \mathbf{H}^{inc})_{S_{aperture}} = 4\hat{n} \times \left\{ \frac{\bigtriangledown^2 \cdot \mathbf{N} + k_0^2 \mathbf{N}}{j\omega\mu_0} \right\}$$

[2] used to describe the macroscopic properties of the medium of interest

Cavity layout

Figure: Jet engine air intake is abstracted to a cylindrical one-side open ended cavity



$$(\hat{n} \times \mathbf{E})_{S_{mantle}} = 0$$

$$(\hat{n} \times \mathbf{H}^{inc})_{S_{aperture}} = 4\hat{n} \times \left\{ \frac{\nabla^2 \cdot \mathbf{N} + k_0^2 \mathbf{N}}{j\omega\mu_0} \right\}$$

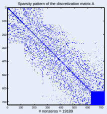
Linear system

FEM discretization

Equation (1) is discretized by a higher order edge based finite element discretization method, resulting in a large linear system:

$$Au = f$$

Properties linear system:



- A consists of a sparse part and a fully populated part
- A is ‘nearly’ symmetric but not Hermitian
- A is ill-conditioned
- A is ‘highly’ indefinite

Present implementation

- Dimension of system matrix $A : N \approx 1 \cdot 10^7$ (total number of unknowns)
- Iterative Krylov subspace method: *Generalized Conjugate Residual* (GCR) method
- Ill-conditioned matrix $\Rightarrow$ improve convergence GCR by *shifted Laplace preconditioner*
- GCR long recurrence method $\Rightarrow$ difficult to satisfy memory requirements for storing eigenvectors of Krylov basis

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Proposed solution:

Modify existing algorithm by using a *multigrid solution method* for the preconditioner system

Application of multigrid (MG)

General

- Iterative solvers used in a lot of applications where large (non)linear systems arise from discretization of PDEs
- Suitable for parallel computing
- When system matrix ill-conditioned, no efficient direct application
- Rate of convergence independent of mesh size h
- Linear system of N unknowns can be solved in cN arithmetic operations (c constant)

Application of multigrid – Helmholtz equation

Based on the work of Erlangga[3]:

- MG is a robust and efficient solver for the preconditioned Helmholtz equation in two and three dimensions
- Helmholtz and Maxwell's equation have similar properties

  $\Rightarrow$ expected that MG will also be very effective in current application

[3] See my literature report for the complete reference

Analogy Helmholtz and vector wave equation

Homogeneous Helmholtz equation	Vector Helmholtz equation
$-\nabla^2 u - k_0^2 u = 0$	$-\nabla^2 \mathbf{E} - k_0^2 \mathbf{E} = 0$
Shifted Laplace preconditioner	**Analogy of shifted Laplace preconditioner**
$\mathcal{M}_{\beta_1,\beta_2} = -\nabla^2 u - \hat{k}^2 u = 0$	$\mathcal{W}_{\beta_1,\beta_2} = -\nabla^2 \mathbf{E} - \hat{k}^2 \mathbf{E} = 0$

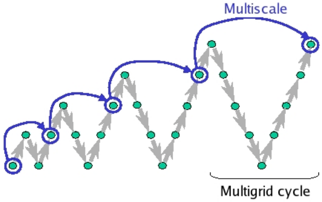
Recall: $\nabla \times \nabla \times \mathbf{E} - k_0^2 \mathbf{E} = -jk_0 Z_0 \mathbf{J}$

(the vector Helmholtz equation is obtained using the vector identity)

Application of multigrid

Current application

- *Indirect* application of MG in current problem: used as solver for preconditioner system
- Unstructured grid $\Rightarrow$ *algebraic multigrid*
- Using AMG solver $\Rightarrow$ constant preconditioner $\Rightarrow$ enables short recurrence method (e.g. BiCG-STAB)
- Short recurrence method $\Rightarrow$ considerable storage reduction



AMG solver

Limited time

No time available to develop a dedicated AMG solver

AMG solvers available

As there are several AMG black box solvers available, the task is to choose the one most suitable for the current application

$\Rightarrow$ formulate requirements for black box solver

Requirements for black box AMG solver

For minimal effort to incorporate in the existing solution algorithm, the AMG black box should:

- be parallelizable
- be suitable for the vector processor of the NEC SX-8R machine
- have a matrix free implementation: in the existing implementation the element matrices are not assembled into the system matrix
- use the element matrices as input-parameter: they are available in the FEM approach and contain useful information

Second part of the thesis

- choose suitable AMG black box solver
- incorporate AMG black box solver
- replace GCR by short recurrence method (e.g. BiCG-STAB)