Implementation of cellular traction forces in agent-based models

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Introduction

Introduction

Research on burn injuries and scars

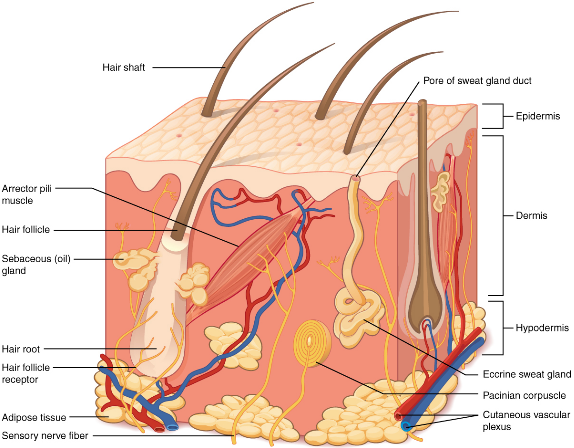
Elasticity model: Daan Smits

Agent-based model: Eline Kleimann

Special objective: obtain more understanding of burn contraction

Biology of burn injuries and contraction

Structure of the skin



Skin model

Wound contraction

The way to wound contraction:

Platelets → chemokines → (myo)fibroblasts → contraction

Plastic and elastic deformation

Model types

Elastic models:

- Purely elastic model
- Viscoelastic model
- Morphoelastic model

Carried out in one and two dimensions

Purely elastic model

Initializing quantities and more

Location $x(X, t)$
Langrangian location $X := x_0$
Displacement $u := x - X$
Velocity $v := \partial x / \partial t$
Stress $\sigma :=$ force per area
Strain $\varepsilon := \partial u / \partial x$

All quantities c can be expressed Eulerian: $c(x, t)$
as well as Lagrangian: $c(X, t)$

e.g. $u(x, t) = x - X(x, t)$ while $u(X, t) = x(X, t) - X$

Material derivative: $\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$

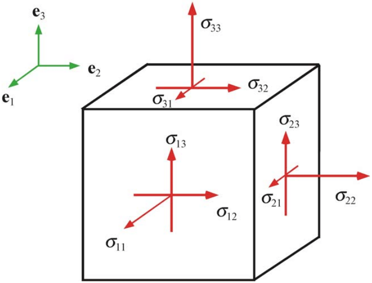
Stress and strain tensor

In more dimensions, stress and strain are tensors.

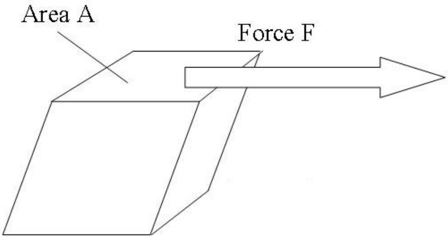
$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

normal stress: σ_{ii}
shear stress: σ_{ij} $(i \neq j)$

Normal and shear stress



Cube with normal and shear stress



Shear stress

Hooke's law (1D)

$$\sigma = \kappa\varepsilon = \kappa\frac{\partial u}{\partial x}$$

κ is called 'Young's modulus'.

Hooke's law in more dimensions (normal and shear modulus).

Cauchy momentum equation

Newton's second law: impulse is proportional with force

$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f}$

One dimension: $\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial \sigma}{\partial x} + f$

ρ: density,
v: velocity,
σ: stress, $\sigma = \kappa \frac{\partial u}{\partial x}$
f: internal force; caused by (myo)fibroblasts

Numerical aspects

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial \sigma}{\partial x} + f$$
$$\sigma = \kappa \frac{\partial u}{\partial x}$$

Finite Element Method with moving mesh

Euler Backward

$$\rho M^{k+1}\mathbf{v}^{k+1} = M^k\mathbf{v}^k + \Delta t S^{k+1}\mathbf{u}^{k+1} + \Delta t \mathbf{f}^{k+1}$$

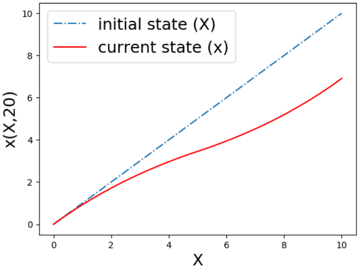
Approximate:

$x_i^{k+1} \approx x_i^k + \Delta t \cdot v_i^k$ (forward)

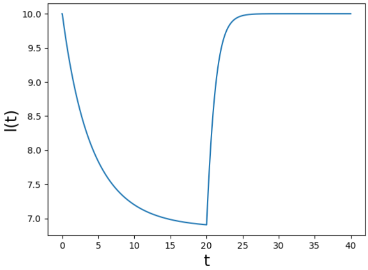
$x_i^{k+1} \approx x_i^k + \Delta t \cdot v_i^{k+1}$ (backward)

Results: purely elastic model

$$f(x,t) := 5.0 \cdot \begin{cases} 1 - \exp(-4 \cdot t/t_f) & \text{if } 0 \leq t < 20, \\ (1 - \exp(-4 \cdot t/t_f))\exp(-(t - t_f)) & \text{if } t \geq 20. \end{cases}$$



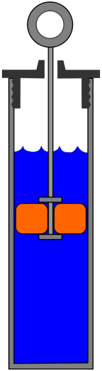
plot of current (red, x) against initial (blue, X)



plot of length against time (t)

Viscoelastic model

Viscosity



Suppose the blue fluid isn't water, but honey...

Difference between pure elasticity and viscoelasticity

Instead of

$$\sigma = \kappa\varepsilon,$$

we have

$$\sigma = \kappa\varepsilon + \mu\frac{D\varepsilon}{Dt}.$$

μ = viscosity rate

($\frac{D\varepsilon}{Dt}$ can also be written as $\frac{\partial v}{\partial x}$.)

Numerical aspects

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial \sigma}{\partial x} + f$$
$$\sigma = \kappa \frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial x}$$

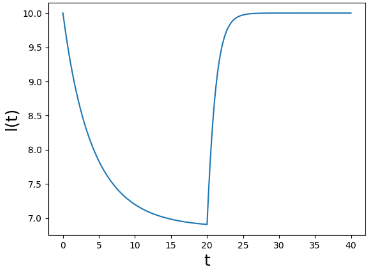
Finite Element Method

Euler Backward

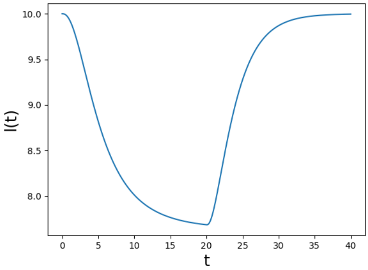
$$M^{k+1}\mathbf{v}^{k+1} = M^k\mathbf{v}^k + \Delta t S^{k+1}\mathbf{u}^{k+1} + \Delta t S^{k+1}\mathbf{v}^{k+1} + \Delta t \mathbf{f}^{k+1}$$

Results: comparing purely elastic and viscoelastic model

Plots of length against time (t)



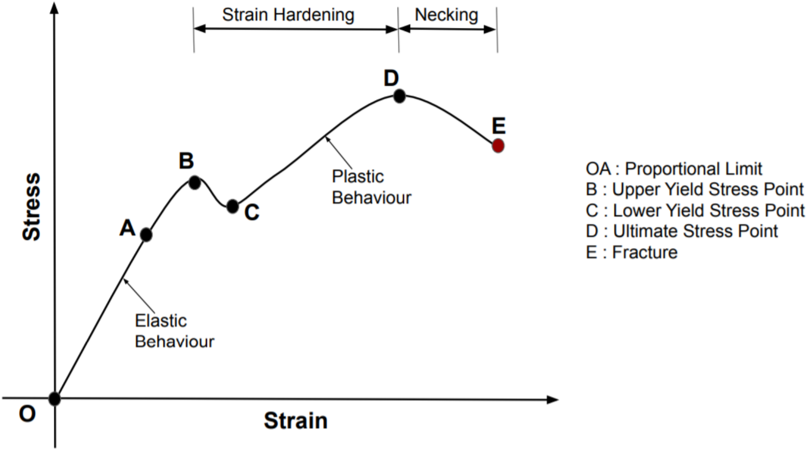
Purely elastic



Viscoelastic

Morphoelastic model

Elastic and plastic behaviour



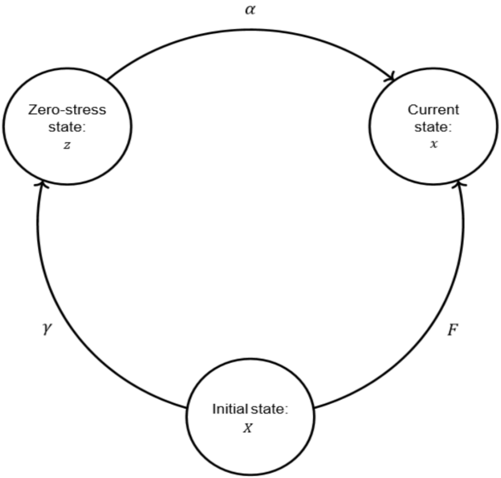
Elastic and plastic deformation

Deformation gradient $F := \frac{\partial x}{\partial X}$

$F = \frac{\partial x}{\partial z}\frac{\partial z}{\partial X} := \alpha\gamma$

α: elastic deformation
γ: plastic deformation
X: initial state
z: zero-stress state, equals X as long as $\frac{D\gamma}{Dt} = 0$
x: current state, equals z and X at $t = 0$

Elastic and plastic deformation (2)



Elastic and plastic deformation (3)

$\frac{D\gamma}{Dt} = Fg$

g: growth rate; $g = \xi\varepsilon$ (choice)

$u_z := x - z$

Strain evolution equation:

$\frac{D\varepsilon}{Dt} + (\varepsilon - 1)\frac{\partial v}{\partial x} = -g$

ε = new strain based on u_z, i.e. $\varepsilon = \frac{\partial u_z}{\partial x}$

Numerical aspects

Cauchy momentum: $\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial \sigma}{\partial x} + f$
Viscoelasticity: $\sigma = \kappa\varepsilon + \mu\frac{\partial v}{\partial x}$
Strain evolution: $\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} + (\varepsilon - 1)\frac{\partial v}{\partial x} = -g$

$$\Downarrow$$

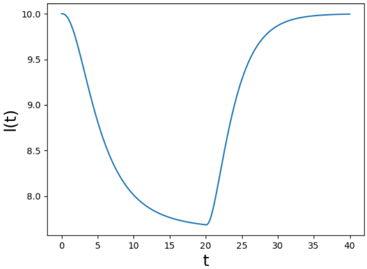
$$S^{k+1}\mathbf{w}^{k+1} = T^k\mathbf{w}^k + \Delta t\Phi^{k+1}$$

$$\mathbf{w}^k := \begin{pmatrix}\boldsymbol{\varepsilon}^k \\ \mathbf{v}^k\end{pmatrix} \text{ and } \Phi^k := \begin{pmatrix}\mathbf{0} \\ \mathbf{f}^k\end{pmatrix}$$

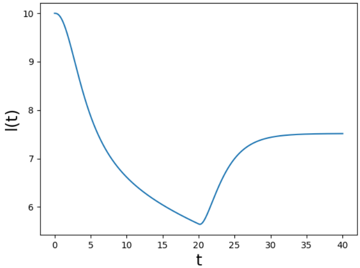
Equations combined in one Euler Backward FEM system.

Results: comparing visco- and morphoelastic model

Plots of length against time (t)



Viscoelastic



Morphoelastic

Two-dimensional models

More-dimensional equations

Cauchy momentum: $\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f}$

Viscoelasticity:

$$\underline{\underline{\sigma}} = \mu_1 \mathrm{sym}\left(\nabla \mathbf{v}\right) + \mu_2 \mathrm{Tr}\left(\nabla \mathbf{v}\right) \mathbf{I} + \frac{\kappa\sqrt{\rho}}{1+\eta}\left(\underline{\underline{\varepsilon}} + \frac{\eta}{1-2\eta}\mathrm{Tr}(\underline{\underline{\varepsilon}})\mathbf{I}\right).$$

Strain evolution:

$$\frac{\mathrm{D}\underline{\underline{\varepsilon}}}{\mathrm{D}t} + \underline{\underline{\varepsilon}}\,\mathrm{skw}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) - \mathrm{skw}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)\underline{\underline{\varepsilon}} + \left(\mathrm{Tr}\left(\underline{\underline{\varepsilon}}\right) - 1\right)\mathrm{sym}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) = -\underline{\underline{G}}.$$

Numerical aspects

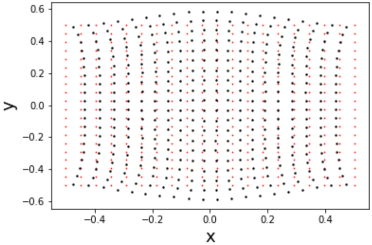
System to solve:

$$\overline{M}^{k+1}\mathbf{w}^{k+1} = M^k\mathbf{w}^k + \Delta t S^{k+1}\mathbf{w}^{k+1} + \Delta t\mathbf{f}^{k+1}(\mathbf{w}^{k+1})$$

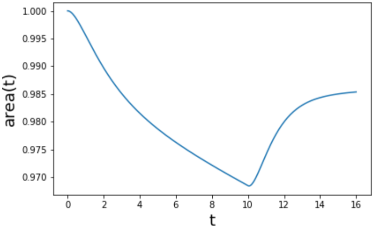
$$\mathbf{w}^k := \begin{pmatrix} \boldsymbol{\varepsilon}_{11}^k \\ \boldsymbol{\varepsilon}_{12}^k \\ \boldsymbol{\varepsilon}_{22}^k \\ \mathbf{v}_1^k \\ \mathbf{v}_2^k \end{pmatrix}$$

Non-linear system: use iterative method of Picard (in each B.E. iteration)

Results: morphoelastic model in 2D



current state (black) and initial state (red)



plot of area against time (t)

On programming

Specifying the path..

Python

$\Downarrow$

C++

$\Downarrow$

GPU (Cuda)

$\Rightarrow$ C++ (much) faster than Python

$\Rightarrow$ work of Eline Kleimann in C++ and on GPU

Future work

Future (or actually: current) work

Doing something on parameters...

Combining models (agent-based and elastic, in C++)

Triangulation

GPU-implementation?

Three-dimensions?

Questions?