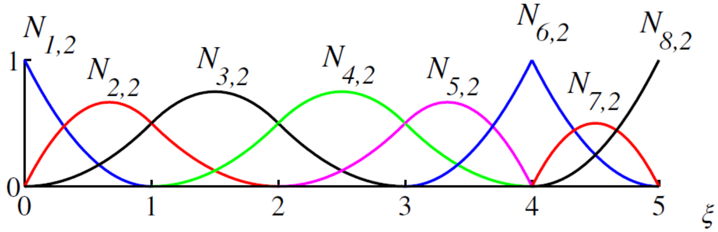
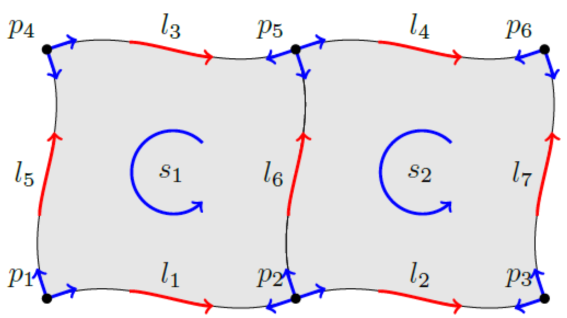
Midterm review:

Mimetic Isogemetric FEM



M.Sc. Thesis project

by Stevie-Ray Janssen



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Objective:

Combine ideas from isogeometric analysis and mimetic methods to develop a structure-preserving discretization for the Euler equations for incompressible fluids.

Project outline

- Planning:

Start Date: 1-9-2015 dinsdag

First Day of Week (Mon=2): 2

Tasks	Start	End	Duration (Days)	% Complete	Working Days	Days Complete	Days Remaining
Kickoff	9-1-15	9-30-15	30	100%	22	30	0
Reading/Orienting	9-1-15	9-18-15	18	100%	14	18	0
Write research proposal	9-19-15	9-30-15	12	100%	8	12	0
Kickoff	9-28-15	9-28-15	1	100%	1	1	0
Fundamentals, Phase I	10-1-15	1-3-16	95	100%	67	95	0
Reading	10-1-15	10-27-15	27	90%	19	24	3
Work, Implementation, and Analysis	10-28-15	11-23-15	27	90%	19	24	3
Writing	11-24-15	12-20-15	27	80%	19	21	6
Holiday	12-21-15	1-3-16	14	100%	10	14	0
Euler equations, Phase II	1-4-16	3-31-16	88	30%	64	26	62
Mid-term meeting	1-11-16	1-11-16	1	0%	1	0	1
Reading	1-4-16	2-2-16	30	30%	22	9	21
Work, Implementation, and Analysis	2-3-16	3-2-16	29	30%	21	8	21
Writing	3-3-16	3-31-16	29	20%	21	5	24
Iterative solving, Phase III	4-1-16	6-26-16	87	0%	61	0	87
Mid-term meeting	4-28-16	4-28-16	1	0%	1	0	1
Reading	4-1-16	4-29-16	29	0%	21	0	29
Work, Implementation, and Analysis	4-30-16	5-28-16	29	0%	20	0	29
Writing	5-29-16	6-26-16	29	0%	20	0	29
Hand in Literature Report	6-26-16	6-26-16	1	0%	0	0	1
Thesis & Graduation	6-26-16	8-31-16	67	0%	48	0	67
Holiday	6-26-16	7-8-16	13	0%	10	0	13
Verification/Validation of Results	7-9-16	7-21-16	13	0%	9	0	13
Writing	7-22-16	8-23-16	33	0%	23	0	33
Green light review	7-29-16	7-29-16	1	0%	1	0	1
Hand in Thesis	8-23-16	8-23-16	1	0%	1	0	1
Thesis defence	8-24-16	8-30-16	7	0%	5	0	7

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Project outline (cnt'd)

- Phase I questions:
  - How can we use IGA to solve PDE's?
  - What structures are facilitated in elliptic PDE's?
  - How can we preserve these structures?
  - Can we construct a MIMIGA method to discretize an elliptic PDE problem?

This presentation - literature review

- Introduction
  - Isogeometric Analysis & Mimetic Methods
- Approach for elliptic PDE’s
  - Exterior calculus
  - DeRham complex
  - Application: Scalar Poisson equation in 2D
- Conclusion
- Future work

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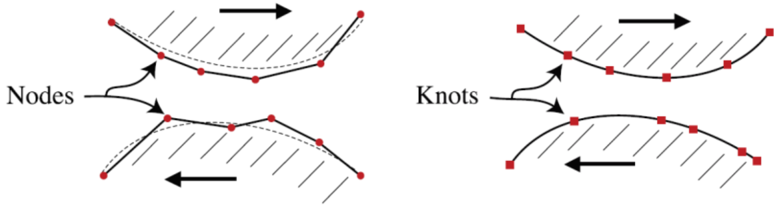
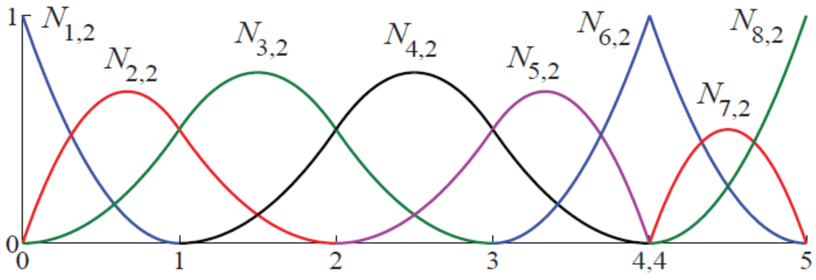
Introduction – Isogeometric Analysis

- Introduced by the Hughes group in 2005 to bridge the gap between CAD and FEM
- Isogeometric paradigm



- B-splines make an excellent basis for FEM



Introduction – Mimetic Methods

- PDE's facilitate physical structures and symmetries.
- Tools from exterior calculus and algebraic topology are used to capture these structures.
- Growing awareness: Disrete exterior calculus, discrete hodge theory, exterior finite element method, compatible methods, mimetic finite diference, etc

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Why exterior calculus?

- Structures become apparent.
- Distinction between topological and metric dependencies.
- Generalized for n dimensions.

Differential Forms; $\alpha^{(k)}$

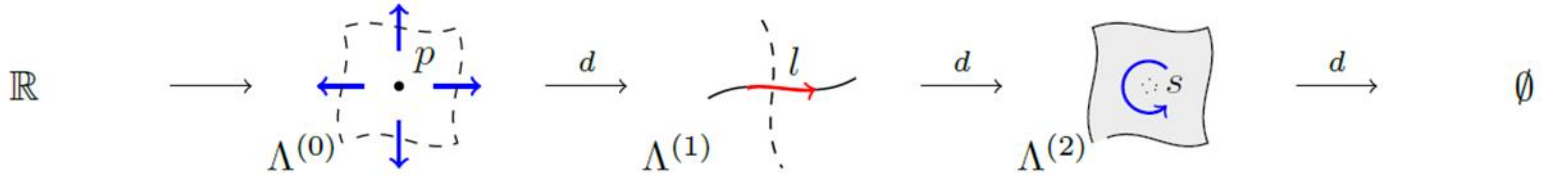
- Differential forms are elements from the dual vector space,
- Associated with geometric structure,
  - 0-form: $f^{(0)} = f(x, y)$
  - 1-form: $\alpha^{(1)} = \alpha_1(x, y)dx + \alpha_2(x, y)dy$
- "Measurement of physical variables,"
  - $M = \oiint \rho^{(2)} = \oiint \rho(x, y)dx \wedge dy$
- Space of k-forms: $\Lambda^{(k)}$

Exterior derivative; d

- Exterior derivative d generalizes $\nabla f,\ \nabla\times\underline{\omega},\ \nabla\cdot\underline{v}$

$$d\alpha^{(1)} = \left(\frac{\partial\alpha_2}{\partial x} - \frac{\partial\alpha_1}{\partial y}\right) dx\wedge dy$$

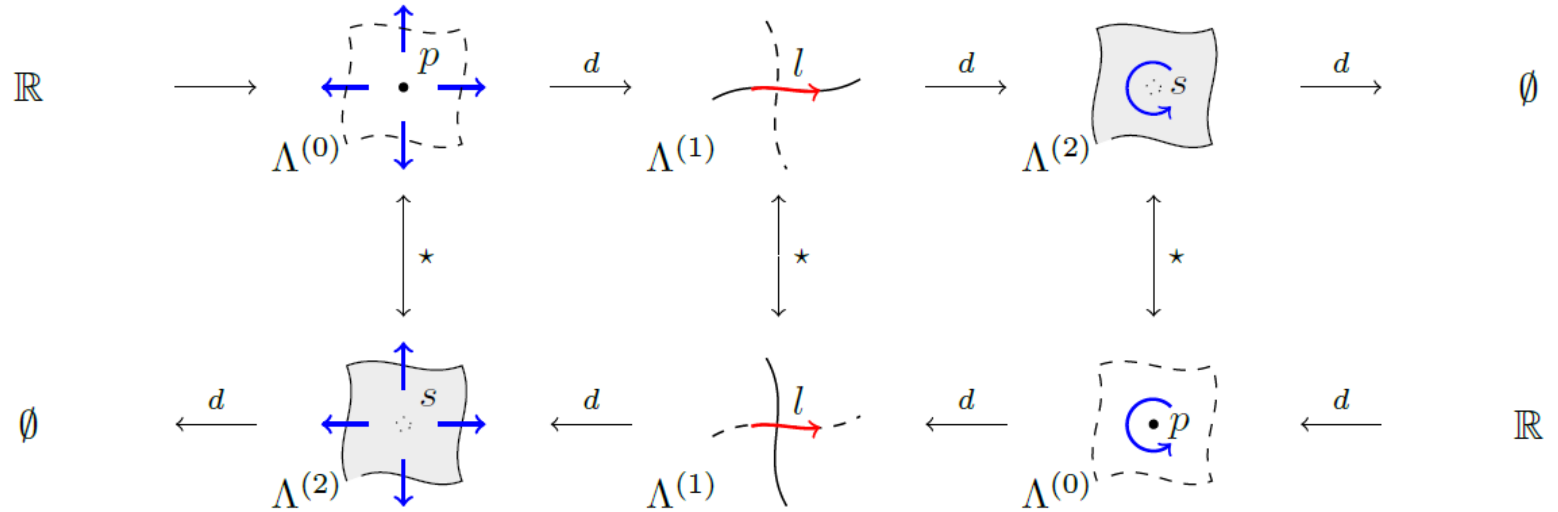
- $d: \Lambda^{(k)} \rightarrow \Lambda^{(k+1)}$



- Exact sequence, the DeRham complex
- Nilpotent, $dd\alpha^{(k)} = 0$
- Independent of metric

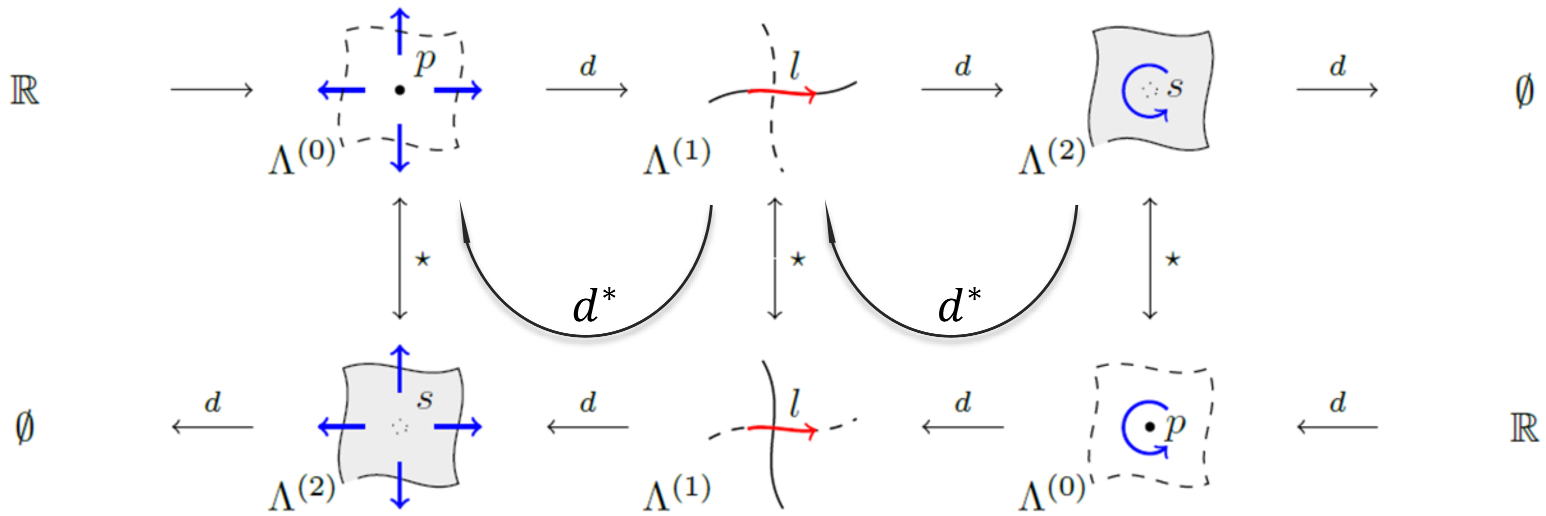
Hodge-⋆ operator;

- Maps forms to dual geometry,
- Metric dependent,
- Double DeRham complex,



Codifferential; d^*

- $d^* := \star\, d\, \star$
- Adjoint of d: $(\cdot, d^* \cdot) = (d\, \cdot, \cdot)$ - ∫ bc's
- Laplace operator: $\Delta = dd^* + d^*d$



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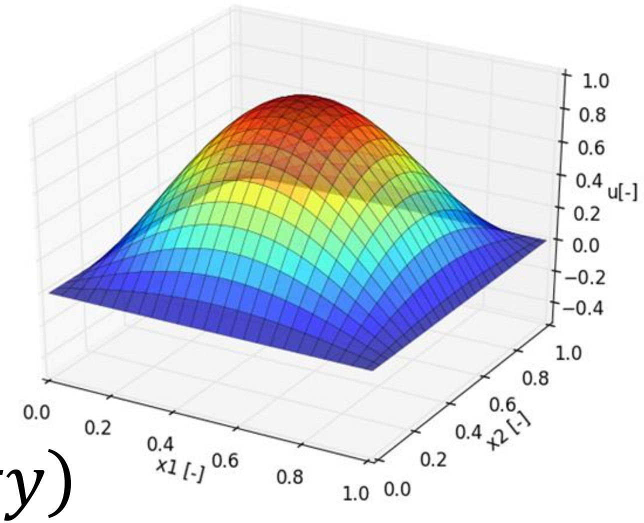
Scalar Poisson equation



- E.g. Potential flow, electrostatics,
- Given $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$
  find $\varphi(x,y)$ such that $\Delta\varphi = f$ on $\Omega = [0,1]^2$
  with $\varphi = 0$ on $\partial\Omega$

0-form,

Find $\varphi^{(0)}$ s.t. $d^* d\varphi^{(0)} = f^{(0)}$

2-form,

Find $\sigma^{(2)}$ s.t. $dd^*\sigma^{(2)} = f^{(2)}$

Same solution, different discretization

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0-form Poisson; $d^* d\varphi^{(0)} = f^{(0)}$

- Weak formulation,

$$\left(w^{(0)}, d^* d\varphi^{(0)}\right)_\Omega = \left(w^{(0)}, f^{(0)}\right)_\Omega$$

$$\Leftrightarrow$$

$$\left(dw^{(0)}, d\varphi^{(0)}\right)_\Omega = \left(w^{(0)}, f^{(0)}\right)_\Omega - \oint_{\partial\Omega} w^{(0)} \wedge \star d\varphi^{(0)}$$

- Well-posedness through Lax-Milgram,

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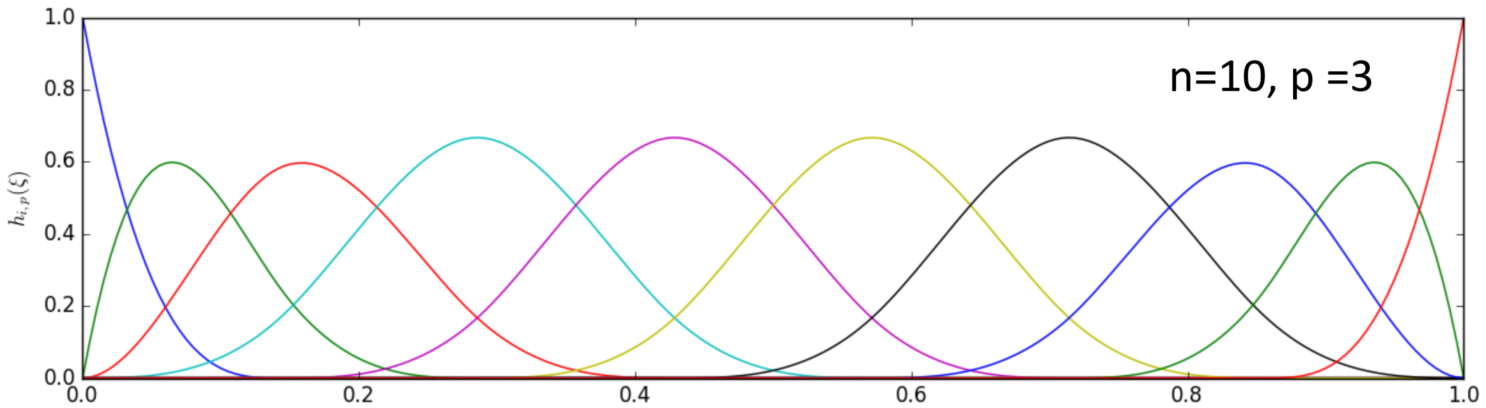
0-form Poisson; FEM

- Conforming FEM, take $\Lambda_h^{(k)} \subset \Lambda^{(k)}$
- Use B-spline spans $\Lambda_h^{(0)} = S^{p,p}$



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0-form Poisson; edge functions

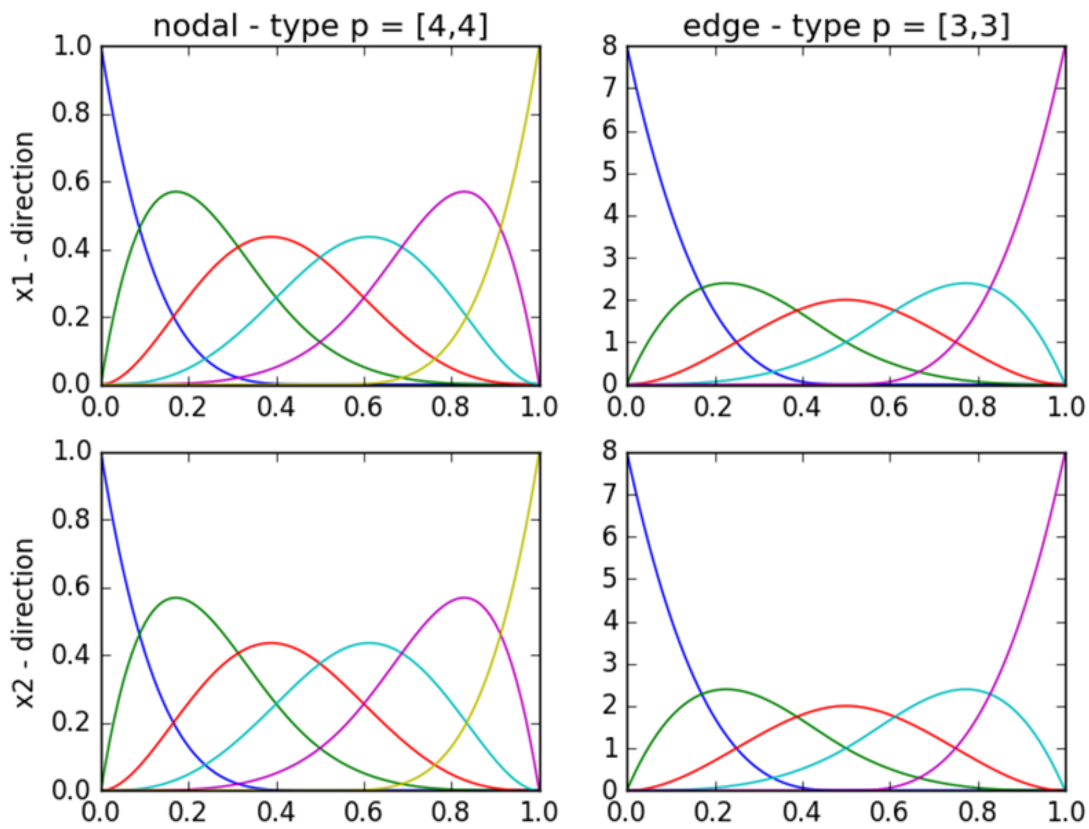
- Applying the exterior derivative (1D-example)
  - Nodal basis: $\varphi_h^{(0)} = \sum_{i=0}^{n} \varphi_i h_i^p(x) = \left(\underline{\varphi}\right)^T \underline{R}^0$
  - Then, $d\varphi_h^{(0)} = \sum_{i=1}^{n} (\varphi_i - \varphi_{i-1}) e_i^{p-1}(x) = \left(\mathbb{E}^{(10)}\underline{\varphi}\right)^T \underline{R}^1$

Differences of coefficients are captured in matrix using {-1,0,1}

New edge type basis function emerges with a polynomial degree less

0-form Poisson; edge functions (cnt'd)

- Extension to 2D using tensor products of nodal and edge type basis
- Nodal/edge
  - 0-form
  - 1-form
  - 2-form



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0-form Poisson, Matrices

Mass matrix $\mathbb{M}^{(11)}$

- $\left(dw_h^{(0)}, d\varphi_h^{(0)}\right)_\Omega = \underline{w}^T(\mathbb{E}^{10})^T\left(\int_\Omega \left(\underline{R}^{(1)}\right)^T \underline{R}^{(1)}\right)(\mathbb{E}^{10})\underline{\varphi}$

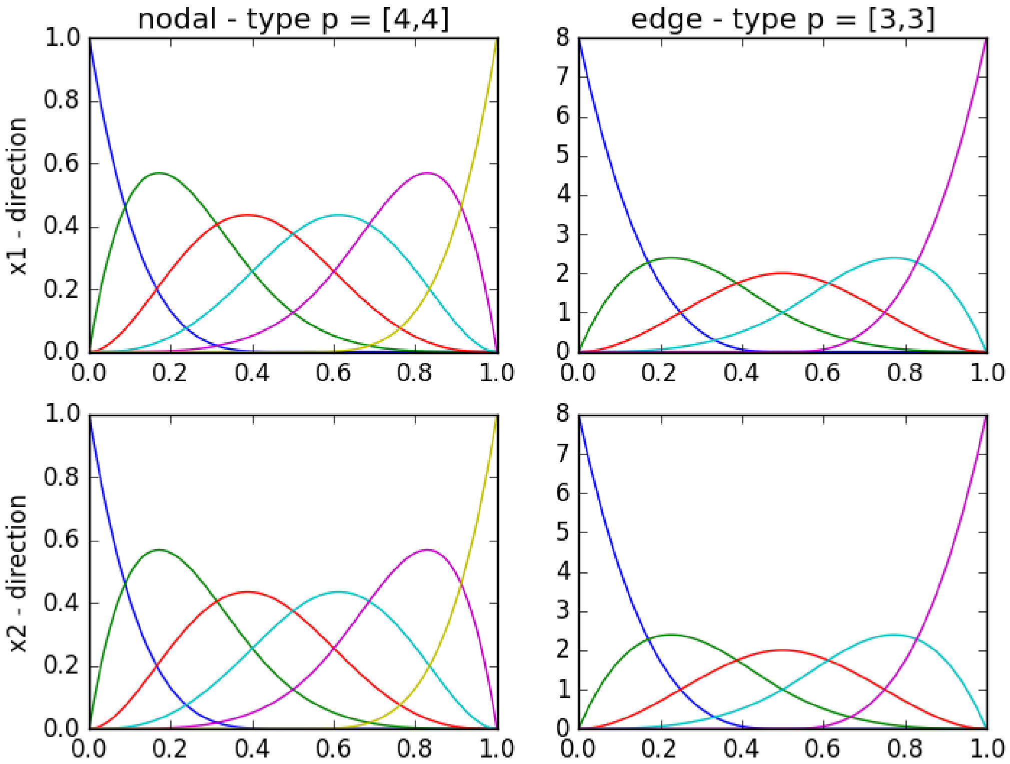
- Result: $\left(\mathbb{E}^{(10)}\right)^T \mathbb{M}^{(11)}\mathbb{E}^{(10)}\underline{\varphi} = \underline{f}$

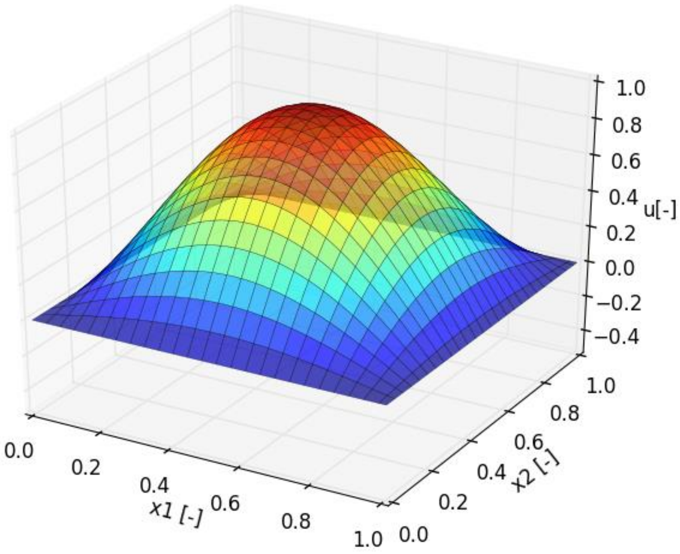
0-form Poisson, Matrices (cnt'd)

- Exact discretization of $v^{(1)} = d\varphi^{(0)}$ through incidence matrices, $\underline{v} = \mathbb{E}^{(10)}\underline{\varphi}$
- Incidence matrices are nilpotent $\mathbb{E}^{(21)}\mathbb{E}^{(10)} = \emptyset$, and satisfy the DeRham sequence

- Hodge-$\star$ operator (metric) is discretized through mass matrix $\mathbb{M}^{(11)}$

0-form Poisson, Results





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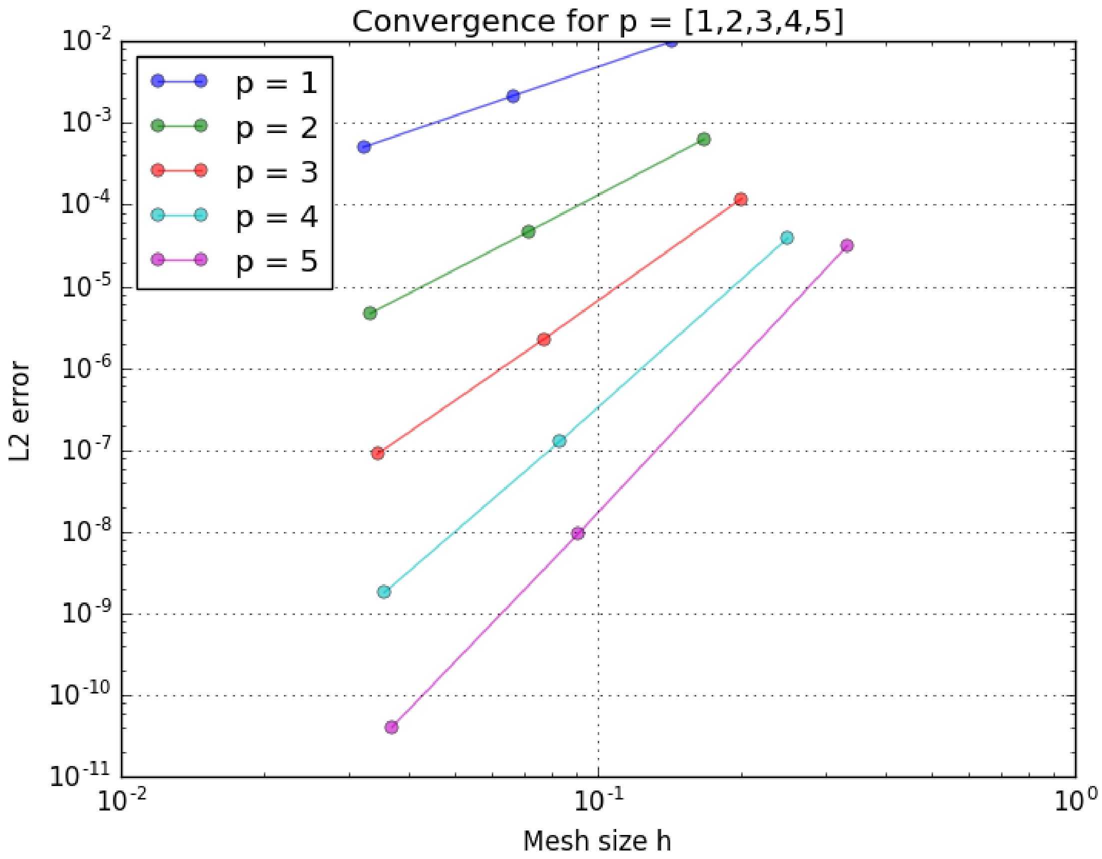
0-form Poisson, Results (cnt'd)



P	Slope
1	1.9998
2	3.0444
3	4.0945
4	5.1311
5	6.1736

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2-form Poisson; $dd^*\sigma^{(2)} = f^{(2)}$

- Weak formulation;

$$\left(w^{(2)}, dd^*\sigma^{(2)}\right)_\Omega = \left(w^{(2)}, f^{(2)}\right)_\Omega$$

- Integration by parts? No, take mixed formulation:

$$\begin{cases} d^*\sigma^{(2)} = \psi^{(1)} \\ d\psi^{(1)} = f^{(2)} \end{cases}$$

- Weak form:

$$\begin{cases} \left(dq^{(1)}, \sigma^{(2)}\right)_\Omega = \left(q^{(1)}, \psi^{(1)}\right)_\Omega - \oint_{\partial\Omega} q^{(1)} \wedge \star\sigma^{(2)} \\ \left(w^{(2)}, d\psi^{(1)}\right)_\Omega = \left(w^{(2)}, f^{(2)}\right)_\Omega \end{cases}$$

- Well posedness through Inf-Sup conditions

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2-form Poisson; FEM

- Can we take,
  - $\Lambda_h^{(1)} = S^{p,p}$?
  - $\Lambda_h^{(2)} = S^{p,p}$?
- No, well-posedness depends on the DeRham sequence. We take
  - $\Lambda_h^{(1)} = S^{p-1,p} \times S^{p,p-1}$
  - $\Lambda_h^{(2)} = S^{p-1,p-1}$
- Which satisfy exact sequence

$$S^{p,p} \xrightarrow[\mathrm{d}]{\mathbb{E}^{(10)}} S^{p-1,p} \times S^{p,p-1} \xrightarrow[\mathrm{d}]{\mathbb{E}^{(21)}} S^{p-1,p-1}$$

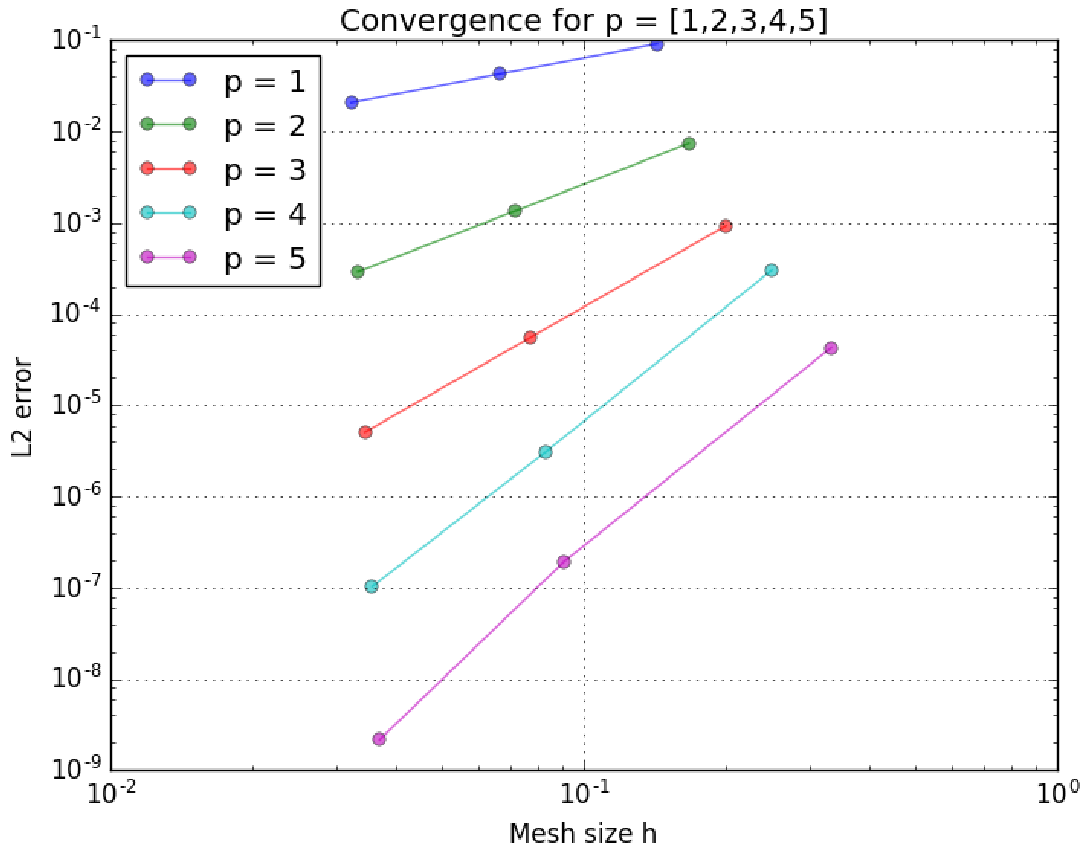
2-form Poisson; Matrices

$$\begin{cases} -\left(q^{(1)}, \psi^{(1)}\right)_\Omega + \left(dq^{(1)}, \sigma^{(2)}\right)_\Omega = 0 \\ \left(w^{(2)}, d\psi^{(1)}\right)_\Omega = \left(w^{(2)}, f^{(2)}\right)_\Omega \end{cases}$$

$$\begin{bmatrix} -\mathbb{M}^{(11)} & \left(\mathbb{M}^{(22)}\mathbb{E}^{(21)}\right)^T \\ \mathbb{M}^{(22)}\mathbb{E}^{(21)} & \emptyset \end{bmatrix} \begin{bmatrix} \underline{\psi} \\ \underline{\sigma} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{f} \end{bmatrix}$$

- Or $\left(\mathbb{M}^{(22)}\mathbb{E}^{(21)}\right)^T \left(\mathbb{M}^{(11)}\right)^{-1} \left(\mathbb{M}^{(22)}\mathbb{E}^{(21)}\right)\underline{\psi} = \underline{f}$

2-form Poisson; Results



P	Slope
1	0.9977
2	2.0150
3	2.9672
4	4.1104
5	4.4811

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Conclusion

- Elliptic problems can be discretized using mass matrices and incidence matrices.
- Solution spaces are chosen such that they satisfy the DeRham complex.

Conclusion (cnt'd)

- Comparison 0-form & 2-form Poisson:

0-form	2-form
$\left(\mathbb{E}^{(10)}\right)^T \mathbb{M}^{(11)} \mathbb{E}^{(10)}$	$\begin{bmatrix} -\mathbb{M}^{(11)} & \left(\mathbb{M}^{(22)}\mathbb{E}^{(21)}\right)^T \\ \mathbb{M}^{(22)}\mathbb{E}^{(21)} & \emptyset \end{bmatrix}$
Obtain solution $\varphi^{(0)}$	Obtain solutions $\sigma^{(2)}, \psi^{(1)}$
- Dirichlet is essential - Neumann is natural	- Dirichlet is natural - Neumann is essential
Gradient exact $\mathbb{E}^{(10)}\underline{\varphi} = \underline{v}$	Divergence exact $\mathbb{E}^{(21)}\underline{\psi} = \underline{0}$ i.e. $\nabla \cdot v = 0$

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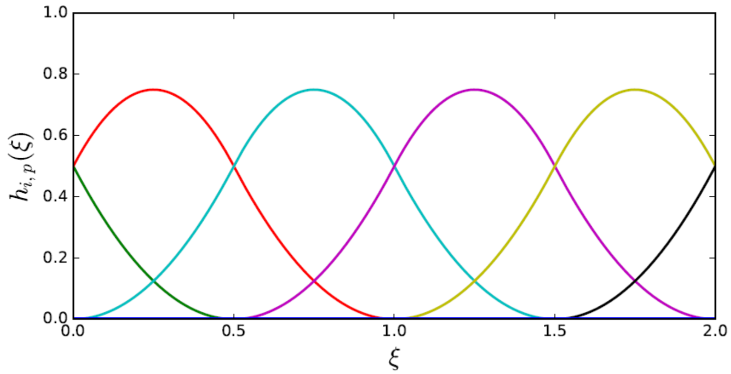
Future Work

- Towards the incompressible Euler equations:
  - Extend to hyperbolic problems,
  - Linear advection equation.
  - Construction of periodic domain.



  - Staggering velocity and vorticity in time?

Questions