Master Thesis
Literature Study Presentation

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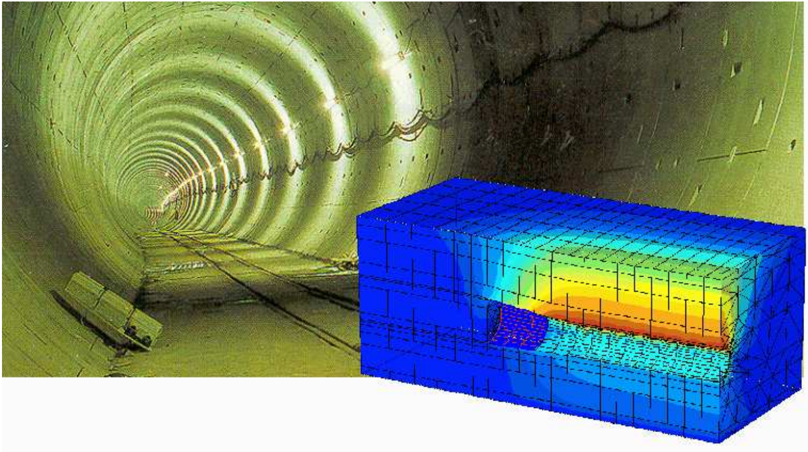
January 29, 2010

Plaxis

PLAXIS

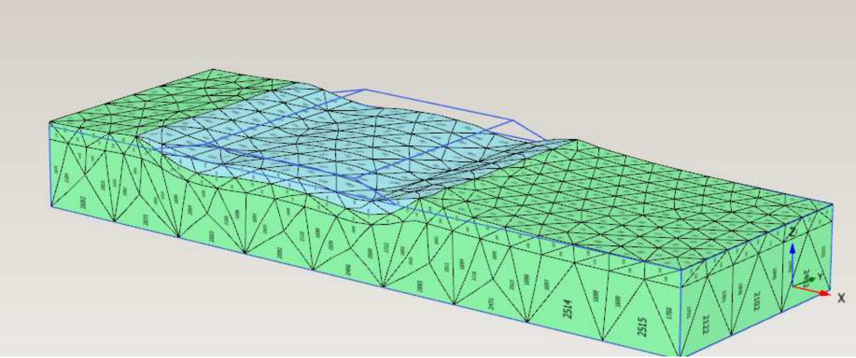
Plaxis B.V. is a company specialized in finite element software intended for 2D and 3D analysis of deformation, stability and groundwater flow in geotechnical engineering.



Finite Element Method

The Finite Element Method (FEM) is a numerical technique for finding approximate solutions of partial differential equations (PDE).

Example



Conjugate Gradient Method

Main problem

Solve:

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is SPD.

Basic Iterative Methods

Let us take

$$x_{i+1} = x_i + M^{-1}(b - Ax_i) \tag{2}$$

Which generates a sequence with the following property

$$x_i \in x_0 + \text{span}\{M^{-1}r_0, M^{-1}A(M^{-1}r_0), ..., (M^{-1}A)^{i-1}(M^{-1}r_0)\}. \tag{3}$$

Conjugate Gradient Method

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- $||x - x_i||_A = \min_{y \in K^i(A;r_0)} ||x - y||_A$, for all i.

Solution of the problem leads to the Conjugate Gradient Method.

Conjugate Gradient Algorithm

Choose x_0, set $i = 0$, $r_0 = b - Ax_0$.

WHILE $r_k \neq 0$ **DO**

$i := i + 1$

IF $i = 0$ **DO**

$p_1 = r_0$

ELSE

$$\beta_i = \frac{r_{i-1}^T r_{i-1}}{r_{i-2}^T r_{i-2}}$$

$p_i = r_{i-1} + \beta_i p_{i-1}$

ENDIF

$$\alpha_i = \frac{r_{i-1}^T r_{i-1}}{p_i^T A p_i}$$

$x_i = x_{i-1} + \alpha_i p_i$

$r_i = r_{i-1} - \alpha_i A p_i$

END WHILE (4)

Theorem

Let A and x be the coefficient matrix and the solution of (1), and let $(x_i, i = 0, 1, 2...)$ be the sequence generated by the CG method. Then, elements of the sequence satisfy the following inequality:

$$||x - x_i||_A \leq 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^i ||x - x_0||_A, \qquad (5)$$

where $\kappa(A)$ is the condition number of A in the 2 - norm.

Speed up?

Yes, with preconditioning.

Transform the system (1) into

$$A^{*}x^{*} = b^{*}, \tag{6}$$

where $A^{*} = P^{-1}AP^{-T}$, $x^{*} = P^{-T}x$ and $b^{*} = P^{-1}b$, where P is a non-singular matrix. The SPD matrix M defined by $M = PP^{T}$ is called the preconditioner.

Preconditioned Conjugate Gradient Algorithm

Choose x_0, set $i = 0$, $r_0 = b - Ax_0$.

WHILE $r_i \neq 0$ **DO**

$z_i = M^{-1} r_i$

$i := i + 1$

IF $i = 0$ **DO**

$p_1 = z_0$

ELSE

$$\beta_i = \frac{r_{i-1}^T z_{i-1}}{r_{i-2}^T z_{i-2}}$$

$p_i = z_{i-1} + \beta_i p_{i-1}$

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END WHILE (7)

There are several possibilities for the preconditioner.
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- Jacobi, M - diag(A).

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- Jacobi, M - diag(A).
- SSOR, $M = \frac{1}{2-\omega}(\frac{1}{\omega}D + L)(\frac{1}{\omega}D)^{-1}(\frac{1}{\omega}D + L)^T$
- Incomplete Cholesky
- Many, many more.

Second Simple Problem

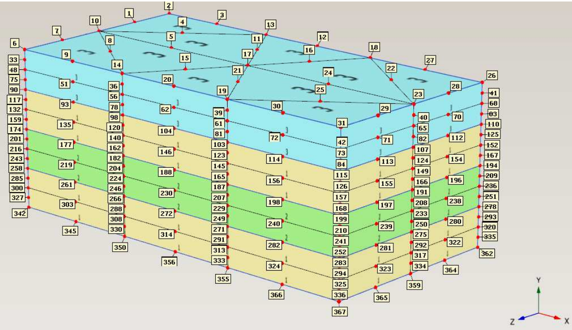

Figure: Second Simple Problem Representation

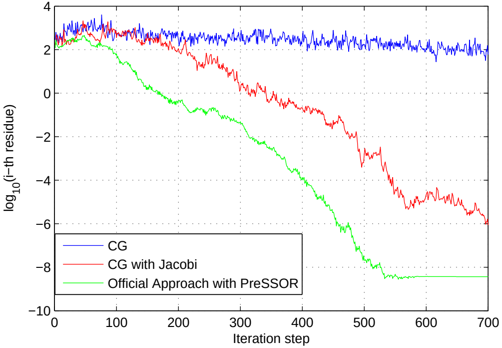

Figure: Plot of log_{10} of i-th residue

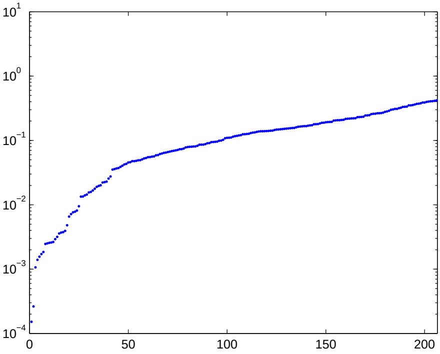

Figure: Plot of log_{10} of eigenvalues

DEFLATION

How to get rid of unfavorable eigenvalues that deteriorate the convergence of PCG?

Split

$$x = (I - P^T)x + P^T x, \tag{8}$$

where,

$$\begin{aligned} &P = I - AQ, \\ &Q = ZE^{-1}Z^T \\ &E = Z^T AZ \\ &\text{and } Z \in \mathbb{R}^{n \times m}. \end{aligned} \tag{9}$$

After some transformation we get an equivalent, "deflated" system:

$$PA\bar{x} = Pb, \tag{10}$$

which can be solved with CG or PCG.

Deflated Preconditioned Conjugate Gradient Algorithm

Choose $\bar{x}_0$, set $i = 0$, $\bar{r}_0 = P(b - A\bar{x}_0)$.

WHILE $\bar{r}_k \not= 0$ **DO**

$i := i + 1$

IF $i = 1$ **DO**

$$y_0 = M^{-1}\bar{r}_0$$

$$p_1 = y_0$$

ELSE

$$y_{i-1} = M^{-1}\bar{r}_{i-1}$$

$$\beta_i = \frac{\bar{r}_{i-1}^T y_{i-1}}{\bar{r}_{i-2}^T y_{i-2}}$$

$$p_i = y_{i-1} + \beta_i p_{i-1}$$

ENDIF

$$\alpha_i = \frac{\bar{r}_{i-1}^T \bar{r}_{i-1}}{p_i^T PAp_i}$$

$$\bar{x}_i = \bar{x}_{i-1} + \alpha_i p_i$$

$$\bar{r}_i = \bar{r}_{i-1} - \alpha_i PAp_i$$

END WHILE

$$x_{orginal} = Qb + P^T \bar{x}_{last} \quad (11)$$

How to choose Z?

Several choices for Z:

- Approximated Eigenvector Deflation

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- Subdomain Deflation
- Rigid Body Mode Deflation

Domain Decomposition

Definition

We will call a method a Domain Decomposition method, if its main idea will be based on the principle of divide and conquer applied on the domain of the problem.

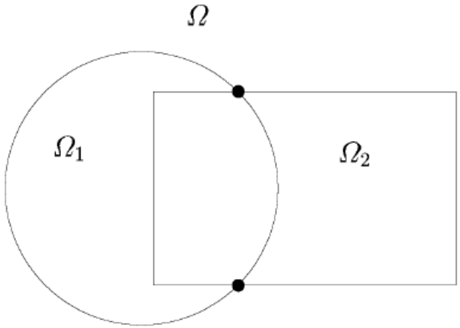

Figure: An example of domain decomposition

There are several ways to do it. The most known are:

- Schwarz Alternating Procedure
- Schur Complement

Lets consider a domain as shown in the last figure with two overlapping subdomains Ω_1 and Ω_2 on which we want to solve a PDE of the following form:

$$\begin{cases} Lu = b, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} . \tag{12}$$

Schwarz Alternating Procedure

Choose u_0

WHILE no convergence **DO**

FOR $i = 1, ...s$ **DO**

Solve $Lu = b$ in Ω_i with $u = u_{ij}$ in Γ_{ij}

Update u values on Γ_{ij}, $\forall j$

END FOR

END WHILE (13)

In our case, $s = 2$.

Within the SAP there are two distinguished variants

- Multiplicative Schwarz Method (MSM)
- Additive Schwarz Method (ASM)

Multiplicative Schwarz Method

$$u^{n+1/2} = u^n + \begin{bmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} (b - Au^n)$$
$$u^{n+1} = u^{n+1/2} + \begin{bmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{bmatrix} (b - Au^{n+1/2}) \qquad (14)$$

where A_{Ω_i} stays for the discrete form of the operator L restricted to Ω_i

Additive Schwarz Method

$$u^{n+1} = u^n + \left(\begin{bmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{bmatrix} \right) (b - Au^n) \qquad (15)$$

Additive Schwarz Method

Or in a more general:

Additive Schwarz Method

Choose $u_0, \ i = 0,$

WHILE no convergence **DO**

$r_i = b - Au^n$

FOR $i = 1, \dots s$ **DO**

$\delta_i = B_i r_i$

END FOR

$$u^{n+1} = u_n + \sum_{i=1}^{s} \delta_i$$

$i = i + 1$

END WHILE (16)

where $B_i = R_i^t A_{\Omega_i}^{-1} R_i$

Numerical Experiments

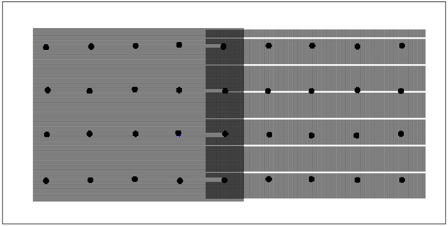

Figure: Domain Ω split into two subdomains, Ω_1 and Ω_2.

MSM

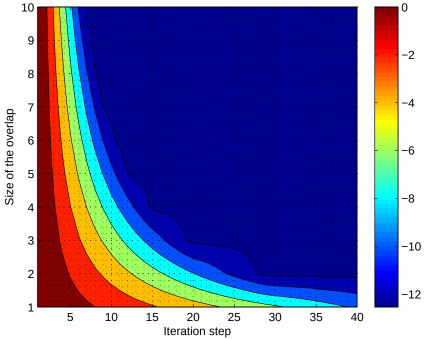

Figure: Contour plot of the log_{10} (i-th residue) for MSM.

ASM

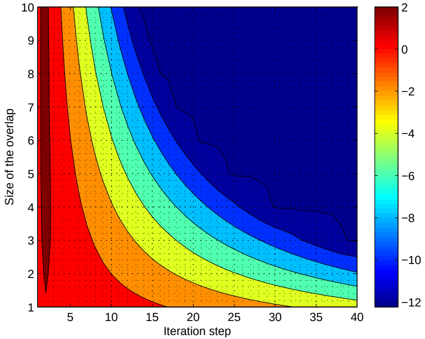

Figure: Contour plot of the log_{10} (i-th residue) for ASM

Research

Choice of the method

DPCG with Additive Schwarz as the preconditoner and physics based decomposition.

First Simple Problem

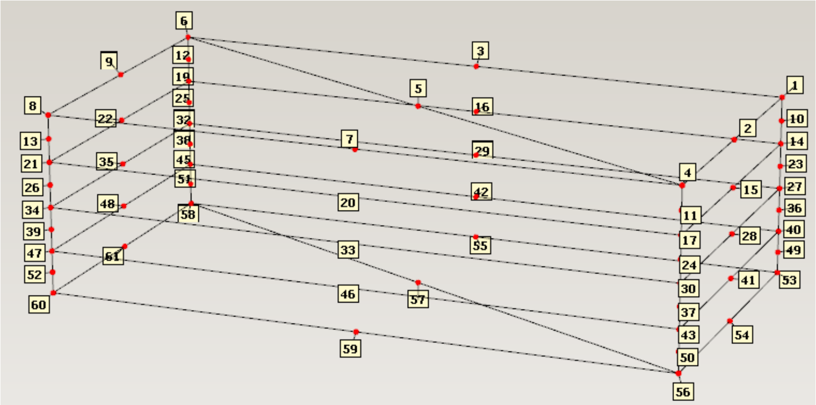

Figure: First Simple Problem Representation

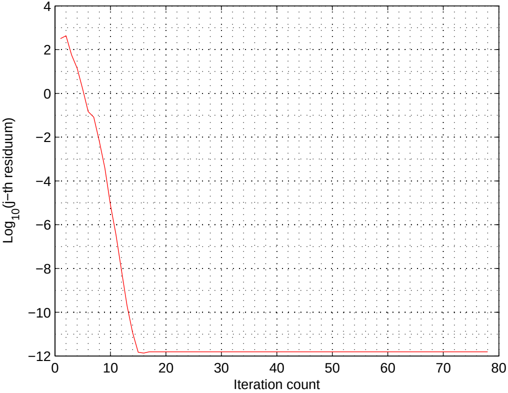

Figure: First Simple Problem convergence behavior for PCG

Analysis of the decomposition position

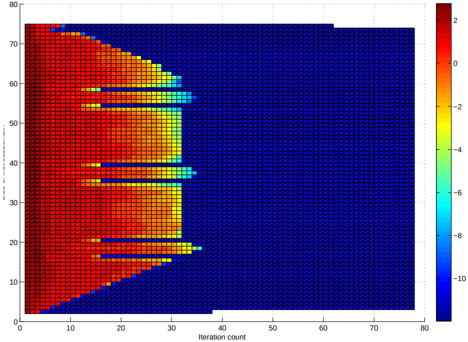

Figure: First Simple Problem distribution of error in PCG

Second Simple Problem

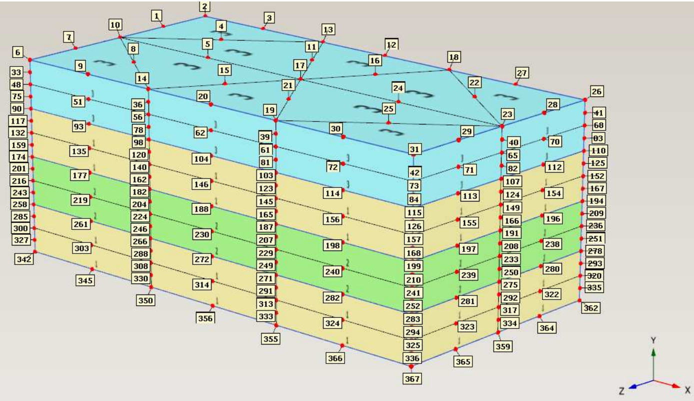

Figure: Second Simple Problem Representation

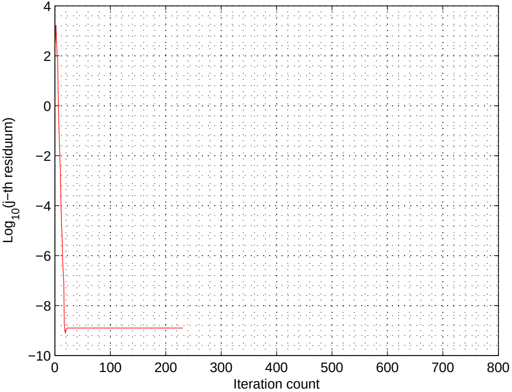

Figure: Second Simple Problem convergence behavior for PCG with 2 subdomains

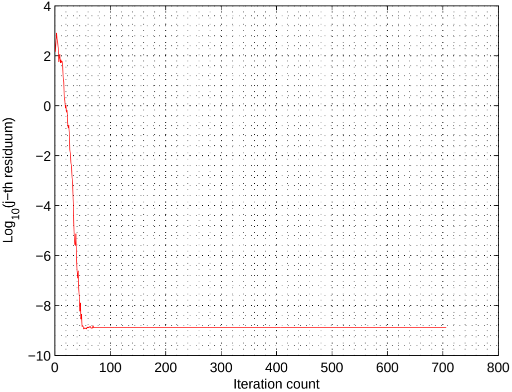

Figure: Second Simple Problem convergence behavior for PCG with 4 subdomains

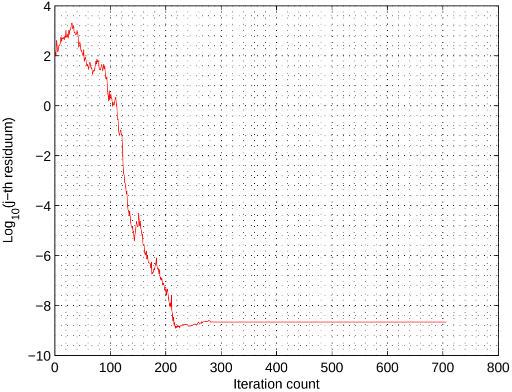

Figure: Second Simple Problem convergence behavior for PCG with 8 subdomains

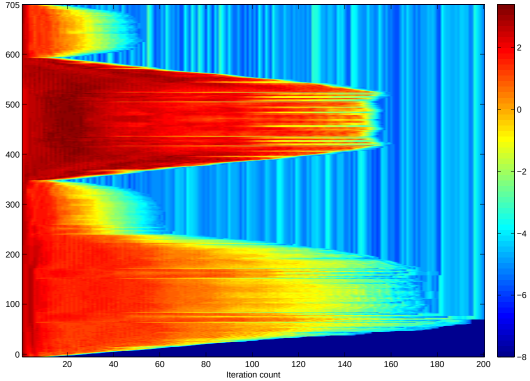

Figure: Second Simple Problem distribution of error in PCG

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- Introduction of deflation.

Research Goals

Research Questions

- Subdomain overlap.

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- Subdomain overlap.
- Performance of Deflation.

The End