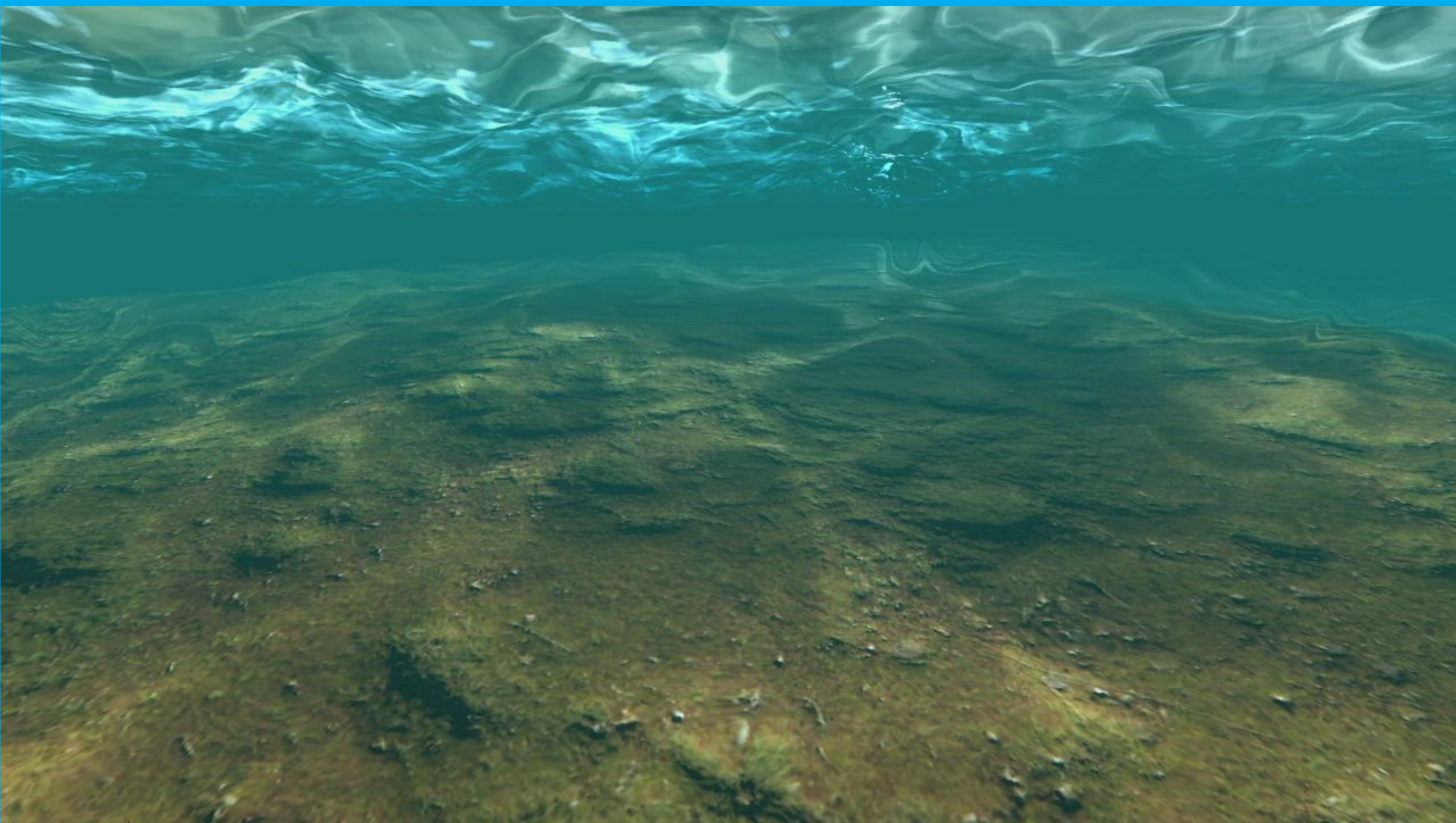
Compressible vs. incompressible pore water in fully-saturated poroelastic soil

Literature Report

F.P.M. Klein



TU Delft

Rijkswaterstaat
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Compressible vs. incompressible pore water in fully-saturated poroelastic soil

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by

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Abstract

Especially in a country like the Netherlands it is important to have well-maintained levees, since a large part of the Netherlands is below sea level. Therefore, researching the factors that have an impact on the status of these levees is of importance. One of these factors is the hydrodynamic load acting on the seabed due to waves. The distribution of this pressure may impact the levee. Therefore, it is important to have a model that describes the behaviour of hydrodynamic loads acting on the seabed due to waves sufficiently. The aim of this literature report is to describe such a model and solve it numerically using the finite-element method for discretisation in space. We use Backward-Euler for discretisation in time.

To describe the behaviour of hydrodynamic loads acting on the seabed due to waves it is currently common to use Biot's model. This model assumes that the effective stresses of the porous soil are zero at the surface and the load due to the waves is completely carried by the pressure at the surface. An other important assumption made in Biot's model is that the pore water must be compressible. Therefore, the model of Biot is in line with the effective stress principle of Terzaghi. However, the assumption of compressible water raises some questions.

Recently a new model tried to handle these questions. In the model of Van Damme and Den Ouden-Van der Horst the stresses are absorbed by both the pore water particles and the soil particles. Thus in this new model the pore water does not carry the full load in this model. Instead of setting the effective stresses to zero at the surface, the vertical momentum balance equation is used as a boundary condition at the surface. So Terzaghi's stress principle is not used as a boundary condition at the surface in the new model. Furthermore, the compressibility of the pore water does not need to be assumed in the model of Van Damme and Den Ouden-Van der Horst, since it includes the (in)compressibility into account. So we may choose the (in)compressibility later on instead of assuming it beforehand. We do find that both models can describe the behaviour of the seabed subjected to waves. However, both models make different assumptions, for example when choosing the boundary conditions and assumption of the (in)compressibility of water. Therefore, it depends on what kind of problem the model is used for and the corresponding physics.

Nomenclature

Table 1: Directions and their symbols.

Direction	Symbol
Horizontal	x
Vertical	z

Table 2: Symbols and their definitions and units.

Definition	Symbol		Units
Boundary domain	$\mathrm{d}\Omega$		-
Compressibility of the pore water	β		-
Density of the soil	ρ_s		kg/m^3
Density of the pore water	ρ_f		kg/m^3
Displacement of the soil particles in the i-direction	u_i	(for $i = x, z$)	m
Displacement of the pore water in the i-direction	w_i	(for $i = x, z$)	m
Domain	Ω		-
Effective size of grains	d_{10}		m
Effective stress tensor	σ'_{ij}	(for $i, j = x, z$)	Pa
Elasticity modulus	E		Pa
Hydraulic conductivity	K_s		m/s
Functions in time	F_{xz}, F_{zz}		N
Lamé's constants	λ, μ		Pa
Length in i-direction	n_i	(for $i = x, z$)	m
Normal unit vector to the boundary	η		-
Poisson ratio	v_p		-
Pore water pressure	P		Pa
Porosity of the soil	p		
Specific weight of the pore water	γ_w		Pa
Stopping time	t_{end}		s
Strain tensor for soil	ϵ_{ij}	(for $i, j = x, z$)	-
Time	t		s
Time step	Δt		s
Total stress tensor	σ_{ij}	(for $i, j = x, z$)	Pa
Volumetric strain of the soil particles	ϵ_{vol}		-
Vorticity of the soil particles	ω		Hz

Contents

1 Introduction

For countries like the Netherlands it is important to have levees and keep them well-maintained in order to protect the people and land. This way rivers and the sea are prevented from flooding [1]. However, since several countries as the Netherlands, Egypt, India and China are located in lower reaches of rivers and the fact that the water level is rising, it is important to get an indication about the influence of the waves on a levee and to know the interaction between the pore water and the deformation of the soil of the seabed [1]. An important factor is the water pressure, since the levees can be damaged by the oscillations of the pressure travelling through the ground nearby the levees.

In this literature report we assume that this seabed laying before the levees is a fully-saturated poroelastic medium. Describing the behaviour of such fully-saturated poroelastic soil is important for different kinds of Civil Engineering. It is common to use Biot's model. This model describes the deformation of such fully-saturated poroelastic media [2] and has been studied extensively [3], [4]. However, in this model it is often assumed that the pore water is compressible [2], because otherwise the results of Biot's model do not agree with the reality [5]. Unfortunately, this assumption of compressibility of pore water can have a significant impact on the distribution of the effective stress of the soil and thus also the deformation of the soil [6]. However, recent research has found a similar model for which both compressible and incompressible water can be assumed [6].

The aim of this literature report is to work out a description of the behaviour of a seabed subjected to waves using numerical methods. Two models will be described of which one is old and well-known (Biot, 1955) [2] and the other one recently published (Van Damme and Den Ouden-Van der Horst, 2023) [6]. For both models the acceleration terms and body forces like gravity will be ignored unless stated otherwise [6]. A main difference is that the boundary conditions at the surface in Biot's model are in line with the effective stress principle of Terzaghi which states that the total stress acting on a porous medium has to be equal to the pore water pressures added tot he effective stresses [2], [3], [6] and sets the following boundary conditions: the full hydrodynamic load due to the waves is carried by the pressure and the effective stresses at the surface of the porous soil are zero. The new model is based on making sure that the momentum balance equations are all valid at the computational domain and its boundaries [6]. The new approach takes instead the following boundary conditions into account: the stress and the gradient of the stress. The stress gradient follows from one of the momentum balance equations and therefore, the model follows D'Alembert's principle of minimisation of virtual work [6]. However, the effective stress principle of Terzaghi is not valid at the surface in case of hydrodynamic load which would be valid for linear and static load [6].

Solutions of the two models will be derived by using numerical methods. In this literature report, we will assume one or two homogeneous layers of soil as seabed and use the Finite-Element Method (FEM) for discretising in space. For discretising in time the Backward Euler Method is used. In this literature report, we want to find a numerical model for Biot's model and Van Damme and Den Ouden-Van der Horst. We would like to answer the following questions:

1. Do Biot's model and Van Damme and Den Ouden-Van der Horst model differ in (numerical) solution?

2. Do the two models have a unique solution?

3. Do the results of the two models get impacted by the assumption of (in)compressibility?
4. What is the impact on the behaviour of the soil when having two different homogeneous layers of soil compared to one homogeneous layer of soil?

To understand the interaction between the pore water and the soil, it is important to understand the stress and strain relations first. Therefore, we start Chapter 2 with deriving Biot's model in two dimensions and its numerical discretisation. In Section 2.1, we will determine the basic equations for a linear elastic medium. In Section 2.2 these equations will be modified into constitutive equations describing the deformation of fully-saturated poroelastic media which is better known as Biot's model [2] where pore water is often assumed to be compressible. In Section 2.3 we will look into the application of the finite-element method and in Section 2.4 we will look into the application of the backward-Euler method on the model of Biot in order to discretise the governing equation in space and time, respectively. In this literature report, the corresponding two-dimensional results will not be shown yet. In Chapter 3 Biot's model and its numerical model is derived in one dimension for first analysis, since it is a simplified form of the two-dimensional model described in Chapter 2. In Section 3.1 the (numerical) model for one type of soil is a unique solution derived and the results are presented. For a seabed with two layers of two different types of soil the (numerical) model with a unique solution is derived and the results are presented in Section 3.2. In Chapter 4 the new model of Van Damme and Den Ouden-Van der Horst will be described in two dimensions. In Section 4.1 the governing equations are determined. In Sections 4.2 and 4.3 these equations are discretised in space by the finite-element method and in time by the backward-Euler method, respectively. In this literature report, the corresponding two-dimensional results will not be shown yet. In Chapter 5 the Van Damme and Den Ouden-Van der Horst model and its numerical model is derived in one dimension which is a simplified form of the two-dimensional model described in Chapter 4. This is done for one layer of soil having infinitely many solutions, one layer of soil having a more strict boundary condition and thereby a unique solution, and two layers of different types of soil assuming the more strict boundary condition. In Section 5.1 the (numerical) model with infinitely solutions is derived and the results are presented for one type of soil. In Section 5.2 the (numerical) model with only one solution is derived and the results are presented for one type of soil. In Section 5.3 the (numerical) model with only one solution is derived and the results are presented for a seabed with two layers having two different types of soil. Lastly, in Chapter 6 the conclusions are made and ideas for further research are discussed.

2

Biot's model (2D)

In the next sections we will derive the governing equations for Biot's model in two dimensions and discretise them first in space and second in time. For discretising in space we will use the finite-element method and for discretising in time we will use the Backward-Euler method.

2.1. Linear elastic medium

We will begin with deriving the basic equations for a linear elastic medium, using Cartesian coordinates x, z. We can write the stresses and strains as [2], [4]

$$\bar{\bar{\sigma}} := \begin{bmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{bmatrix} \text{ and} \tag{2.1}$$

$$\bar{\bar{\epsilon}} := \begin{bmatrix} \epsilon_{xx} & \epsilon_{xz} \\ \epsilon_{xz} & \epsilon_{zz} \end{bmatrix}, \tag{2.2}$$

where σ_{ij} is the stress and ϵ_{ij} is the strain for $i, j = x, z$. We will refer to $\bar{\bar{\sigma}}$ and $\bar{\bar{\epsilon}}$ as the stress tensor and strain tensor, respectively. We assume a linear elastic medium of which a volume of the solid system will be represented by a rectangle of size $n_x \times n_z$, which is shown in Figure 2.1.

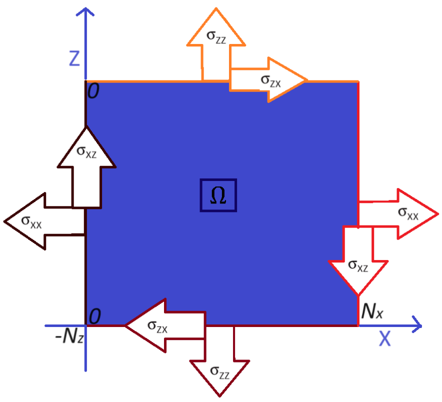

Figure 2.1: Stress components of σ acting on the computational domain Ω which is a rectangle.

Then we can use geometric equations, equations of motion and constitutive equations to represent the strain-displacement relations and stress-strain relations. In tensor form the geometric equations are given by [2], [4]

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) \quad \text{for } i,j = x,z, \tag{2.3}$$

where u_i denotes the displacement of the solid in the i-direction and $u_{i,j}$ means derivative of u_i with respect to the j-th component.

According to [4], the constitutive equation which will be given in tensor form of Hooke's law:

$$\sigma_{ij} = -\sum_{k=x,z}\sum_{l=x,z} c_{ijkl}\epsilon_{kl} \quad \text{for } i,j = x,z, \tag{2.4}$$

where c_{ijkl} are components of a fourth-rank tensor including 81 components. Since stress tensors and strain tensors are symmetric, we have first minor symmetry and second minor symmetry, i.e. $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{ijlk}$. Furthermore, for a homogeneous medium, we also have major symmetry, i.e. $c_{ijkl} = c_{klij}$ [4]. Because of these symmetries, the number of independent components decreases to 2 so that Equation (2.4) becomes [3]

$$\sigma_{ij} = -2\mu\epsilon_{ij} - \lambda\delta_{ij}\epsilon_{\text{vol}} \quad \text{for } i,j = x,z,$$

where $\epsilon_{\text{vol}} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ for $i = x,y,z$, δ_{ij} is the kronecker delta ($\delta_{ij} = 1$ if $i = j$, otherwise 0) and λ and μ are Lame's constants. Lamé's constants are defind as

$$\lambda = \frac{v_p E}{(1+v_p)(1-2v_p)}, \tag{2.5}$$

$$\mu = \frac{E}{2(1+v_p)}, \tag{2.6}$$

where E represents Young's modulus and v_p Poisson's ratio [1], [4]. Rewriting this in matrix-vector form [4] gives

$$\boldsymbol{\sigma} = -C\boldsymbol{\epsilon},$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix}\epsilon_{xx}\\ \epsilon_{zz}\\ \epsilon_{xz}\end{bmatrix}, \quad C = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}.$$

Note that matrix C is written this way, since it should be a non singular and invertible matrix.

2.2. Governing equations

Recall that the above relations of stresses and strains are for a linear elastic medium. We now look into the governing equations of a fully saturated poroelastic medium. First consider a volume of a solid-fluid system which will be described by a unit size cube and assume that the solid-fluid system is statistically isotropic which implies that the principal stress and strain directions are the same. Furthermore, a fully saturated poroelastic medium without the acceleration terms of the fluid relative to the solid is assumed for now. The body forces are also ignored.

In the next sections we will describe the stress and strain relations again, but now for this fluid-solid system. This way the governing equations of Biot's model will be derived in two dimensions. The corresponding boundary conditions will also be described. For deriving Biot's model we follow the steps presented by Verruijt [1], [3].

2.2.1. Conservation of mass equation

First of all, we need the equations of conservation of mass of the solids and the pore water. The resulting mass balance equation for the pore water (fluid) is given by [3]

$$\frac{\partial p\rho_f}{\partial t} + \frac{\partial}{\partial x}\left(p\rho_f \frac{\partial w_x}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\rho_f \frac{\partial w_z}{\partial t}\right) = 0, \tag{2.7}$$

where ρ_f is the density of the fluid, w_x is the displacement of the pore water in x-direction, w_z is the displacements of the pore water in z-direction and p the porosity of the medium. The mass balance equation for the soil is given by [3]

$$\frac{\partial(1-p)\rho_s}{\partial t} + \frac{\partial}{\partial x}\left([1-p]\rho_s \frac{\partial u_x}{\partial t}\right) + \frac{\partial}{\partial z}\left([1-p]\rho_s \frac{\partial u_x}{\partial t}\right) = 0, \tag{2.8}$$

where ρ_s is the density of the soil, u_x is the displacement of the soil in x-direction and u_z is the displacements of the soil in z-direction.

According to [1], we can rewrite the mass balance equation of the pore water given by Equation (2.7) as

$$\frac{\partial p}{\partial t} + p\beta\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(p\rho_f \frac{\partial w_x}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\rho_f \frac{\partial w_z}{\partial t}\right) = 0, \tag{2.9}$$

where P denotes the pore water pressure and β the compressibility. Note that the water is incompressible if $\beta = 0.0$ and compressible if $\beta \in (0, 1]$. We assume that the soil particles are incompressible. Then we have that the density ρ_s is constant. Therefore, we have that Equation (2.8) becomes

$$-\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left([1-p]\rho_s \frac{\partial u_x}{\partial t}\right) + \frac{\partial}{\partial z}\left([1-p]\rho_s \frac{\partial u_z}{\partial t}\right) = 0. \tag{2.10}$$

When adding Equations (2.9) and (2.10), we get [3]

$$p\beta\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(p\frac{\partial(w_x - u_x)}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\frac{\partial(w_z - u_z)}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial t}\right) = 0. \tag{2.11}$$

Using $\epsilon_{\text{vol}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}$, we can write Equation (2.11) as [3]

$$p\beta\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(p\frac{\partial(w_x - u_x)}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\frac{\partial(w_z - u_z)}{\partial t}\right) + \frac{\partial \epsilon_{\text{vol}}}{\partial t} = 0. \tag{2.12}$$

Using Darcy's law we get that $q_i = \frac{K_s}{\gamma_w}\frac{\partial P}{\partial i}$ [3], where q_i is the quantity in Darcy's law for fluid motion for $i = x, z$, K_s denotes the hydraulic conductivity and γ_w. This quantity can also be given by the porosity multiplied by the difference of the velocities of the soil and pore water particles, i.e. $q_i = \frac{\partial p(u_i - w_i)}{\partial t}$ [3]. Therefore, we have that

$$-\nabla\cdot\left(\frac{K_s}{\gamma_w}\nabla P\right) = \nabla\cdot\left(p\frac{\partial(\boldsymbol{u} - \boldsymbol{w})}{\partial t}\right), \tag{2.13}$$

where $\nabla\cdot = \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}$ for all vectors $\boldsymbol{v} = \begin{bmatrix} v_x \\ v_z \end{bmatrix}$. After substituting Equation (2.13) into Equation (2.12) and assuming K_s and γ_w are constants, we get

$$\frac{\gamma_w}{K_s}p\beta\frac{\partial P}{\partial t} - \nabla^2 P + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} = 0. \tag{2.14}$$

2.2.2. Momentum balance equation

The stress tensor can be separated into two parts, since we now have a fluid and a solid part. The stress acting on a rectangle domain in Figure 2.1 can be denoted as Equation (2.1) [2] and the stress acting on the fluid part the rectangle domain in Figure 2.1 can be described by the diagonal tensor [2]

$$\bar{\bar{s}} := \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \tag{2.15}$$

where s can be calculated by $s = \alpha P$ [2], [4] with P the fluid pressure and α the Biot constant that depends on the geometry of the medium. Usually $\alpha \approx 1$ can be assumed in Civil Engineering problems [4]. We will assume from now on that $\alpha = 1$. Note the plus-signs in the relation $s = \alpha P$ which describes that scalar s must be positive when the force acting on the fluid is a pressure and the stress tensors σ_x, σ_y and σ_z are negative due to the positive compression convention in the formulation. Note that in [2] a minus-sign is placed before s, since they assume negative pressure when the stresses are positive. The strain tensor in the solid is denoted by Equation (2.2), where ϵ_{ij} for $i, j = x, z$ are described by Equation (2.3) [2], [3]. For a linear solid medium the total stress and effective stress coincided. However, since we now have a saturated medium, there is also the pore pressure in the relationship between total and effective stress which is in tensor form [2], [3]

$$\sigma_{ij} = \sigma'_{ij} + \delta_{ij} P \quad \text{for } i, j = x, z,$$

where σ'_{ij} denotes the effective stress tensor of the solid medium, σ_{ij} the total stress tensor, δ_{ij} the Kronecker delta and P the pore water pressure. Since $\sigma'_{ij} = -2\mu\epsilon_{ij} - \lambda\delta_{ij}\epsilon_{\text{vol}}$, we have that

$$\sigma_{ij} = -2\mu\epsilon_{ij} - \lambda\delta_{ij}\epsilon_{\text{vol}} + \delta_{ij} P \quad \text{for } i, j = x, z, \tag{2.16}$$

where δ_{ij} is the kronecker delta and λ and μ are Lamé's constant.

The equilibrium equations for a fully saturated poroelastic medium is made out of the stresses acting upon the rectangle which is given by [2], [4]

$$\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iz}}{\partial z} = 0 \quad \text{for } i = x, z. \tag{2.17}$$

Then substituting Equation (2.16) into Equation (2.17) gives

$$-2\mu\frac{\partial \epsilon_{ix}}{\partial x} - \lambda\delta_{ix}\frac{\partial \epsilon_{\text{vol}}}{\partial x} + \delta_{ix}\frac{\partial P}{\partial x} - 2\mu\frac{\partial \epsilon_{iz}}{\partial z} - \lambda\delta_{iz}\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \delta_{iz}\frac{\partial P}{\partial z} = 0 \quad \text{for } i = x, z. \tag{2.18}$$

We can rewrite this as [3]

$$\begin{cases} -(\lambda+\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial x} - \mu\nabla^2 u_x + \frac{\partial P}{\partial x} = 0 \\ -(\lambda+\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} - \mu\nabla^2 u_z + \frac{\partial P}{\partial z} = 0 \end{cases},$$

where $\nabla^2 \boldsymbol{v} = \frac{\partial_x^v}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}$ for all vectors $\boldsymbol{v} = \begin{bmatrix} v_x \\ v_z \end{bmatrix}$. We can also rewrite Equation (2.18) as

$$\begin{cases} -\mu\frac{\partial \omega}{\partial z} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial x} + \frac{\partial P}{\partial x} = 0 \\ \mu\frac{\partial \omega}{\partial x} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\partial P}{\partial z} = 0 \end{cases}, \tag{2.19}$$

where $\omega := \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$. which is used in [6]. Equation (2.19) will be used further in this chapter.

According to [2], we can uncouple Equation (2.19) by taking the curl and the divergence to obtain a formula for the vorticity and the volumetric strain, respectively. We will do this in the next sections.

2.2.3. Vorticity equation

After applying the curl on the first equation of Equation (2.19), we get that

$$\begin{aligned} 0 =& \frac{\partial}{\partial z}\left(-\mu\frac{\partial \omega}{\partial z} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial x} + \frac{\partial P}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\mu\frac{\partial \omega}{\partial x} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\partial P}{\partial z}\right) \\ =& -\mu\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial z^2}\right], \end{aligned} \tag{2.20}$$

where ω is now the only unknown variable in the formula.

2.2.4. Volumetric strain equation

After applying the divergence on the first equation of Equation (2.19), we get that

$$
\begin{aligned}
0 =& \frac{\partial}{\partial x}\left(-\mu\frac{\partial \omega}{\partial z} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial x} + \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(-\mu\frac{\partial \omega}{\partial x} - (\lambda+2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\partial P}{\partial z}\right)\\
=& -(\lambda+2\mu)\left[\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right] + \left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2}\right]. \qquad (2.21)
\end{aligned}
$$

After substituting the mass conservation equation given by Equation (2.14) into Equation (2.21), we get

$$
\frac{K_s}{\gamma_w}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda+2\mu)\left[\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right] + p\beta\frac{K_s}{\gamma_w}\frac{\partial P}{\partial t} = 0, \qquad (2.22)
$$

which is a formula depending on the volumetric strain and the pore water pressure. Note that if $\beta = 0$, then the there is only one unknown, namely the volumetric strain. Therefore, the formula written in this way is preferable. Because we now have two equations with two unknowns, namely Equation (2.14) and (2.22) with the volumetric strain and water pressure, this system of equations can be solved for the unknowns volumetric strain and the pore water pressure.

2.2.5. Displacement

Lastly, we need also to describe the relations between the horizontal and vertical displacements and the other unknowns. Since $\omega = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$ and $\epsilon_{\text{vol}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}$, these can be defined as [6]

$$
-\frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} = -\frac{\partial \omega}{\partial z} - \frac{\partial \epsilon_{\text{vol}}}{\partial x}, \qquad (2.23)
$$

$$
-\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} = \frac{\partial \omega}{\partial x} - \frac{\partial \epsilon_{\text{vol}}}{\partial z}. \qquad (2.24)
$$

2.2.6. Boundary conditions

When using Biot's model it is common to take $\sigma_{zz} = \sigma'_{zz} + P$ set equal to a function only depending on time F_{zz} and suppose that $\sigma'_{zz} = 0$ and $P = F_{zz}$. However, according to [6], the assumption of $\sigma'_{zz} + P = F_{zz}$, where F_{zz} a function depending on time, gives a pressure at the surface that is much higher than the pressure of the waves on the surface caused by water running over the porous medium. Since $\sigma_{zz} = F_{zz} = 0 + F_{zz} = \sigma'_{zz} + P$, Terzaghi's principle is met. Another condition at boundary $z = 0$ is that $\sigma_{xz} = F_{xz}$ [3], where F_{xz} is a function only depending on time. Recall that the formula of shear stress is given by $\sigma_{xz} = -2\mu\epsilon_{xz} = -\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$.

At $z = -n_z$ we assume that the displacement for the soil and pore water in z-direction is zero for a deep enough seabed. This means that $u_z = 0$ and that there is no gradient for the pore water pressure which is defined as $\frac{\partial P}{\partial z} = 0$ [5], [7] at $z = -n_z$, respectively. Since the displacement of soil is negligible at the bottom, we find that there is no gradient for the displacements: $\frac{\partial u_x}{\partial z} = 0$ and $\frac{\partial u_z}{\partial x} = 0$ [5]. Hence, there is also no gradient for the volumetric strain $\frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0$ [5]. Furthermore, since $\frac{\partial u_x}{\partial z} = 0$ and $\frac{\partial u_z}{\partial x} = 0$, we have that $\omega = 0$ at $z = -n_z$.

Similarly, at $x = 0$ and $x = -n_x$ it is assumed that the displacements will smoothen out according to [8]. Then we get that $\frac{\partial u_z}{\partial x} = 0$ and $\frac{\partial u_x}{\partial z} = 0$ at $x = 0, L$ [5]. Assuming that the displacements on the sides of the domain are negligible, we have that $u_x = 0$ at $x = 0, n_x$ [5]. Therefore, we also have that the volumetric strain does not have a gradient $\frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0$ at $x = 0, n_x$. Furthermore, we have that the pore water has no gradient at $x = 0$ and $x = n_x$, which is defined by $\frac{\partial P}{\partial x} = 0$, since the water displacements are also assumed to be negligible [5]. Furthermore, since $\frac{\partial u_x}{\partial z} = 0$ and $\frac{\partial u_z}{\partial x} = 0$, we have that $\omega = 0$ at $x = 0$ and $x = n_x$.

In conclusion, we have the following boundary conditions

$$\text{for } z = 0 : \begin{cases} \mu\omega - 2\mu\frac{\partial u_x}{\partial z} = F_{xz} \\ P = F_{zz} \\ -\lambda\epsilon_{\text{vol}} - 2\mu\frac{\partial u_z}{\partial z} = 0 \end{cases}, \tag{2.25}$$

$$\text{for } z = -n_z : \left\{ \omega = u_z = \frac{\partial u_x}{\partial z} = \frac{\partial P}{\partial z} = \frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0 \right. , \tag{2.26}$$

$$\text{and for } x = 0 \text{ and } x = n_x : \left\{ \omega = u_x = \frac{\partial u_z}{\partial x} = \frac{\partial P}{\partial x} = \frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0 \right. , \tag{2.27}$$

where F_{xz} and F_{zz} are a functions only depending on time. Its value and its gradient equal zero.

2.2.7. Initial conditions

We assume that at the beginning, $t = 0$, everything is at rest. Therefore, it is assumed that no stresses act on the surface in the beginning, so there are no stresses and displacements at time $t = 0$ [6]. Since we have no displacement and stresses, the volumetric strain and pressure must be zero too. Then we have that [6]

$$\omega|_{t=0} = u_x|_{t=0} = u_z|_{t=0} = \epsilon_{\text{vol}}|_{t=0} = P|_{t=0} = 0.$$

2.3. Discretisation in space

In the following sections we will discretise the equations of conservation of mass, vorticity, volumetric strain and displacement which was found in Section 2.2. First we will discretise these four equations with respect to space using the finite-element method (FEM) in order to derive the Galerkin equations. We do this per equation. In this numerical approach we assume that $\Omega = (0, n_x) \times (-n_z, 0) \subseteq \mathbb{R}^2$ is the space domain and that $\mathbb{T} = (0, t_{\text{end}})$ is the time domain, with $n_x, n_z, t_{\text{end}} > 0$.

The two-dimensional domain and its boundaries are given as in Figure 2.2, each with their own color. The normal unit vectors with respect to this boundaries are also given in Figure 2.2, These have corresponding colors to their boundary. The normal unit vectors in two dimensions are given by

$$\eta_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Note that the first entry represents the x-direction (horizontal) and the second entry represents the z-direction (vertical).

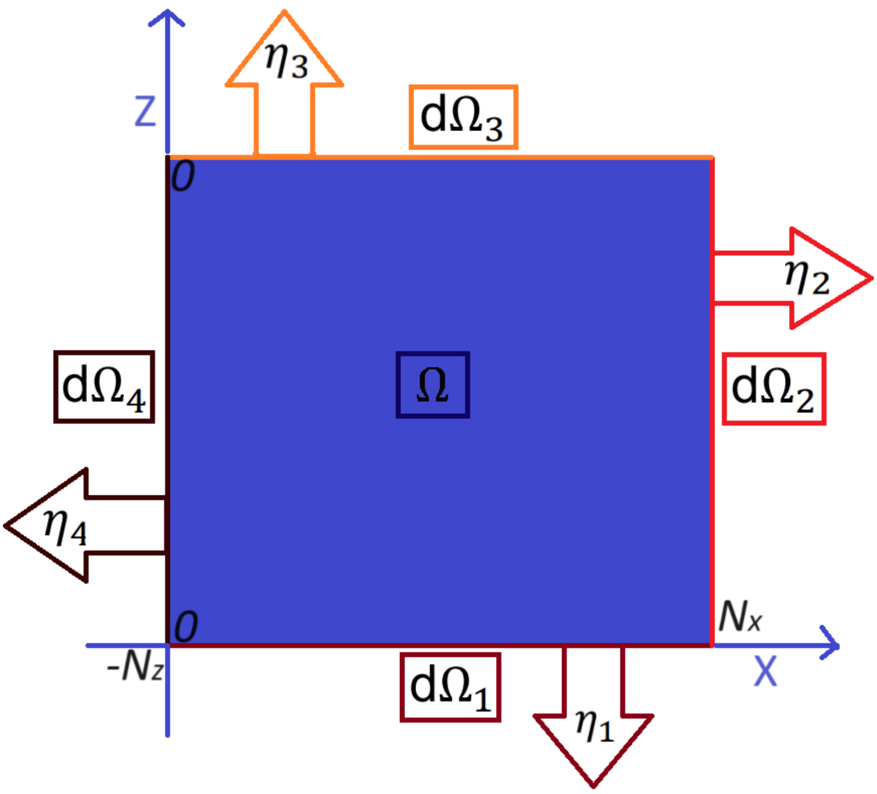

Figure 2.2: Rectangle domain with boundaries and their normal unit vectors. The domain is given by the color blue. The subdomains $\mathrm{d}\Omega_1, \mathrm{d}\Omega_2, \mathrm{d}\Omega_3$ and $\mathrm{d}\Omega_4$ with their normal unit vectors are given by the colors bordeaux red, light red, orange and dark red, respectively.

We define n to be the dimension of the space and N_i are the basis-functions for $i = 1, \ldots, n$ that form a basis for the space. Note that in the next few sections N_i has a superscript which can be the symbol of the volumetric strain, pore water pressure or displacement in horizontal or vertical direction. In this case, N_i for $i = 1, \dots, n$ are defined for the space of that unknown variable specifically. n is assumed to be the same for all spaces.

2.3.1. Conservation of mass equation

We will derive the weak form of the pressure equation and its matrix-vector form. Suppose that the test and trial functions are given by

$$v^P(x,z,t) = \sum_{i=1}^{n} N_i^P(x,z)\bar{v}_i^P(t), \tag{2.28}$$

$$\epsilon_{\text{vol}}(x,z,t) = \sum_{j=1}^{n} N_j^\epsilon(x,z)\bar{\epsilon}_j(t), \tag{2.29}$$

$$P(x,z,t) = \sum_{l=1}^{n} N_l^P(x,z)\bar{P}_l(t). \tag{2.30}$$

After multiplying Equation (2.14) by test function v^ϵ and integrating over the domain Ω, we have that

$$\int_\Omega v^P \left[\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t}\right] \mathrm{d}\Omega = 0. \tag{2.31}$$

Since $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = \nabla \cdot (\nabla P)$, we can apply integration by parts on the $-\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2}$ part followed by the divergence theorem. Then Equation (2.31) becomes

$$-\int_{\mathrm{d}\Omega} v^P\,(\nabla P \cdot \boldsymbol{\eta})\,\mathrm{d}\Gamma + \int_\Omega v^P \frac{\gamma_w}{K_s}\left[p\beta\frac{\partial P}{\partial t} + \frac{\partial \epsilon_{\text{vol}}}{\partial t}\right] + \left(\nabla v^P \cdot \nabla P\right)\mathrm{d}\Omega = 0. \tag{2.32}$$

After setting $v^P(x,z,t) = 0$ for $z = 0$ because of the boundary condition given by Equation (2.25) and using the other boundary conditions given by Equations (2.26) and (2.27), we get that

$$\begin{aligned}\int_{\mathrm{d}\Omega} v^P\,(\nabla P\cdot\boldsymbol{\eta})\,\mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^P\,(\nabla P\cdot\boldsymbol{\eta}_1)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^P\,(\nabla P\cdot\boldsymbol{\eta}_2)\,\mathrm{d}\Gamma \\ &\quad + \int_{\mathrm{d}\Omega_3} v^P\,(\nabla P\cdot\boldsymbol{\eta}_3)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^P\,(\nabla P\cdot\boldsymbol{\eta}_4)\,\mathrm{d}\Gamma \\ &= 0.\end{aligned} \tag{2.33}$$

When substituting the test and trial functions given by Equations (2.28), (2.29) and (2.30) and boundary integral given by Equation (2.33) into Equation (2.32), we get the following Galerkin equations

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^{n} N_i^P \bar{v}_i^P \frac{\gamma_w}{K_s}\left[p\beta\frac{\partial}{\partial t}\left(\sum_{l=1}^{n} N_l^P\bar{P}_l\right) + \frac{\partial}{\partial t}\left(\sum_{j=1}^{n} N_j^\epsilon\bar{\epsilon}_j\right)\right] + \left[\nabla\left(\sum_{i=1}^{n} N_i^P\bar{v}_i^P\right)\cdot\nabla\left(\sum_{l=1}^{n} N_l^P\bar{P}_l\right)\right]\mathrm{d}\Omega \\ &= \sum_{i=1}^{n}\bar{v}_i^P\int_\Omega N_i^P\frac{\gamma_w}{K_s}\left[p\beta\sum_{l=1}^{n} N_l^P\frac{\partial\bar{P}_l}{\partial t} + \sum_{j=1}^{n} N_j^\epsilon\frac{\partial\bar{\epsilon}_j}{\partial t}\right] + \left[\nabla N_i^P\cdot\left(\sum_{l=1}^{n}\bar{P}_l\nabla N_l^P\right)\right]\mathrm{d}\Omega. \end{aligned} \tag{2.34}$$

Since it must hold for arbitrary $\bar{v}_i^P$ with $i = 1, \dots, n$, we have that Equation (2.34) still holds as

$$0 = \int_\Omega N_i^P\frac{\gamma_w}{K_s}\left[p\beta\sum_{l=1}^{n} N_l^P\frac{\partial\bar{P}_l}{\partial t} + \sum_{j=1}^{n} N_j^\epsilon\frac{\partial\bar{\epsilon}_j}{\partial t}\right] + \left[\nabla N_i^P\cdot\left(\sum_{l=1}^{n}\bar{P}_l\nabla N_l^P\right)\right]\mathrm{d}\Omega. \tag{2.35}$$

We can write Equation (2.35) as matrix-vector multiplication

$$A^{PP}\bar{\boldsymbol{P}}_t + A^{P\epsilon}\bar{\boldsymbol{\epsilon}}_t + B^P\bar{\boldsymbol{P}} = 0, \tag{2.36}$$

where

$$A_{ij}^{PP} = \int_\Omega \frac{\gamma_w}{K_s} p\beta N_i^P N_j^P \mathrm{d}\Omega, \quad A_{ij}^{P\epsilon} = \int_\Omega \frac{\gamma_w}{K_s} N_i^P N_j^\epsilon \mathrm{d}\Omega, \quad B_{ij}^P = \int_\Omega \nabla N_i^P \cdot \nabla N_j^P \mathrm{d}\Omega,$$

$$\bar{\boldsymbol{P}} = \begin{bmatrix} \bar{P}_1 \\ \vdots \\ \bar{P}_n \end{bmatrix}, \quad \bar{\boldsymbol{P}}_t = \begin{bmatrix} \frac{\partial \bar{P}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{P}_n}{\partial t} \end{bmatrix}, \quad \bar{\boldsymbol{\epsilon}}_t = \begin{bmatrix} \frac{\partial \bar{\epsilon}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{\epsilon}_n}{\partial t} \end{bmatrix},$$

for $i, j = 1, \ldots, n$. Equation (2.36) is our third matrix problem to solve.

2.3.2. Vorticity equation

We will derive the weak form of the vorticity equation and its matrix-vector form. Suppose that the test and trial functions are given by

$$v^\omega(x,z,t) = \sum_{i=1}^n N_i^\omega(x,z)\bar{v}_i^\omega(t), \tag{2.37}$$

$$\omega(x,z,t) = \sum_{j=1}^n N_j^\omega(x,z)\bar{\omega}_j(t). \tag{2.38}$$

Multiplying Equation (2.20) by test function v^ω and integrating over the domain Ω gives

$$\int_\Omega v^\omega \mu \left[\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial x^2}\right] \mathrm{d}\Omega = 0. \tag{2.39}$$

Since $\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial x^2} = \nabla \cdot (\nabla \omega)$ with ∇ the gradient operator, we can apply integration by parts on Equation (2.39) followed by the divergence theorem. Then Equation (2.39) becomes the weak equation

$$\int_{\mathrm{d}\Omega} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}\right) \mathrm{d}\Gamma - \int_\Omega \nabla v^\omega \cdot \mu \nabla \omega \mathrm{d}\Omega = 0, \tag{2.40}$$

where $\mathrm{d}\Omega$ contains the boundaries of domain Ω, and $\boldsymbol{\eta}$ is the unit normal vector pointing outward to the surface $\mathrm{d}\Omega$. Because of the boundary conditions given by Equations (2.25), (2.26) and (2.27), we set $v^\omega(x,z,t) = 0$ for $z = -n_z$, $x = 0$ and $x = n_x$. Then we get that

$$\begin{aligned} \int_{\mathrm{d}\Omega} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}\right) \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}_1\right) \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}_2\right) \mathrm{d}\Gamma \\ &\quad + \int_{\mathrm{d}\Omega_3} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}_3\right) \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^\omega \mu \left(\nabla\omega \cdot \boldsymbol{\eta}_4\right) \mathrm{d}\Gamma \\ &= \int_{\mathrm{d}\Omega_3} v^\omega \mu \frac{\partial \omega}{\partial z} \mathrm{d}\Gamma. \end{aligned} \tag{2.41}$$

After substituting the test and trial functions given by Equations (2.37) and (2.38), respectively, and boundary integral given by Equation (2.41) into Equation (2.40) we get the following Galerkin equations

$$\begin{aligned} 0 &= \int_{\mathrm{d}\Omega_3} \sum_{i=1}^n N_i^\omega \bar{v}_i^\omega \mu \frac{\partial}{\partial z}\left(\sum_{j=1}^n N_j^\omega \bar{\omega}_j\right) \mathrm{d}\Gamma - \int_\Omega \nabla\left(\sum_{i=1}^n N_i^\omega \bar{v}_i^\omega\right) \cdot \mu \nabla \left(\sum_{i=1}^n N_j^\omega \bar{\omega}_j\right) \mathrm{d}\Omega \\ &= \sum_{i=1}^n \bar{v}_i^\omega \int_{\mathrm{d}\Omega_3} N_i^\omega \mu \sum_{j=1}^n \bar{\omega}_j \frac{\partial N_j^\omega}{\partial z} \mathrm{d}\Gamma - \sum_{i=1}^n \bar{v}_i^\omega \int_\Omega \nabla N_i^\omega \cdot \mu \sum_{j=1}^n \bar{\omega}_j \nabla N_j^\omega \mathrm{d}\Omega. \end{aligned} \tag{2.42}$$

Since it must hold for arbitrary $\bar{v}_i^\omega$ with $i = 1, \ldots, n$, we have that Equation (2.42) still holds as

$$0 = \int_{\mathrm{d}\Omega_3} N_i^\omega \mu \sum_{j=1}^n \bar{\omega}_j \frac{\partial N_j^\omega}{\partial z} \mathrm{d}\Gamma - \int_\Omega \nabla N_i^\omega \cdot \mu \sum_{j=1}^n \bar{\omega}_j \nabla N_j^\omega \mathrm{d}\Omega \quad \text{for } i = 1, \ldots, n. \tag{2.43}$$

We can write Equation (2.43) as matrix-vector multiplication,

$$(B^{\omega} - SD^{\omega})\,\bar{\boldsymbol{w}} = 0, \tag{2.44}$$

where

$$B^{\omega}_{ij} = \int_{\Omega} \mu \left(\nabla N^{\omega}_i \cdot \nabla N^{\omega}_j\right) \mathrm{d}\Omega, \quad SD^{\omega}_{ij} = \int_{\mathrm{d}\Omega_3} \mu N^{\omega}_i \frac{\partial N^{\omega}_j}{\partial z} \mathrm{d}\Gamma, \quad \bar{\boldsymbol{w}} = \begin{bmatrix} \bar{\omega}_1 \\ \vdots \\ \bar{\omega}_n \end{bmatrix},$$

for $i, j = 1, \dots, n$. Equation (2.44) is our first matrix problem to solve.

2.3.3. Volumetric strain equation

We will derive the weak form of the volumetric strain equation and its matrix-vector form. Suppose that the test function for ϵ_{vol} is given by

$$v^{\epsilon}(x, z, t) = \sum_{i=1}^{n} N^{\epsilon}_i(x, z)\bar{v}^{\epsilon}_i(t). \tag{2.45}$$

Recall the following test and trial functions

$$\epsilon_{\text{vol}}(x, z, t) = \sum_{j=1}^{n} N^{\epsilon}_j(x, z)\bar{\epsilon}_j(t),$$

$$P(x, z, t) = \sum_{l=1}^{n} N^{P}_l(x, z)\bar{P}_l(t).$$

Note that ϵ_{vol} and P have a first derivative with respect to time which means that their test and trial functions have to depend on time. Multiplying Equation (2.22) by test function v^{ϵ} and integrating over the domain Ω gives

$$\int_{\Omega} v^{\epsilon} \left[\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda + 2\mu)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) + \frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t}\right] \mathrm{d}\Omega = 0. \tag{2.46}$$

Since $\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} = \nabla \cdot (\nabla \epsilon_{\text{vol}})$, we can apply integration by parts on the $\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}$ part and then the divergence theorem. Then Equation (2.46) becomes

$$-\int_{\mathrm{d}\Omega} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta})\,\mathrm{d}\Gamma + \int_{\Omega} v^{\epsilon} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta \frac{\partial P}{\partial t}\right] + (\lambda + 2\mu)\,(\nabla v^{\epsilon} \cdot \nabla \epsilon_{\text{vol}})\,\mathrm{d}\Omega = 0. \tag{2.47}$$

Using the boundary conditions given by Equations (2.25), (2.26) and (2.27), we get that

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta})\,\mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta}_1)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta}_2)\,\mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta}_3)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\text{vol}} \cdot \boldsymbol{\eta}_4)\,\mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{\epsilon} \frac{\partial \epsilon_{\text{vol}}}{\partial z} \mathrm{d}\Gamma.
\end{aligned} \tag{2.48}$$

Substituting the test and trial functions given by Equations (2.45), (2.29), (2.30) and boundary integral given by Equation (2.48) into Equation (2.47) gives the following Galerkin equations

$$
\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} \sum_{i=1}^{n} N_i^\epsilon \bar{v}_i^\epsilon \left(\sum_{j=1}^{n} \bar{\epsilon}_j \frac{\partial N_j^\epsilon}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \int_\Omega \sum_{i=1}^{n} N_i^\epsilon \bar{v}_i^\epsilon \frac{\gamma_w}{K_s} \left[\frac{\partial}{\partial t} \left(\sum_{j=1}^{n} N_j^\epsilon \bar{\epsilon}_j \right) + p\beta \frac{\partial}{\partial t} \left(\sum_{l=1}^{n} N_l^P \bar{P}_l \right) \right] + (\lambda + 2\mu) \left[\nabla \left(\sum_{i=1}^{n} N_i^\epsilon \bar{v}_i^\epsilon \right) \cdot \nabla \left(\sum_{j=1}^{n} N_j^\epsilon \bar{\epsilon}_j \right) \right] \mathrm{d}\Omega \\
= &-\sum_{i=1}^{n} \bar{v}_i^\epsilon \int_{\mathrm{d}\Omega} N_i^\epsilon \left(\mu \sum_{j=1}^{n} \bar{\omega}_j \frac{\partial N_j^\omega}{\partial x} + \sum_{l=1}^{n} \bar{P}_l \frac{\partial N_l^P}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \sum_{i=1}^{n} \bar{v}_i^\epsilon \int_\Omega \frac{\gamma_w}{K_s} N_i^\epsilon \left[\sum_{j=1}^{n} N_j^\epsilon \frac{\partial \bar{\epsilon}_j}{\partial t} + p\beta \sum_{l=1}^{n} N_l^P \frac{\partial \bar{P}_l}{\partial t} \right] + (\lambda + 2\mu) \left[\nabla N_i^\epsilon \cdot \left(\sum_{j=1}^{n} \bar{\epsilon}_j \nabla N_j^\epsilon \right) \right] \mathrm{d}\Omega. \qquad (2.49)
\end{aligned}
$$

Since it must hold for arbitrary $\bar{v}_i^\epsilon$ with $i = 1, \dots, n$, we have that Equation (2.49) still holds as

$$
\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} N_i^\epsilon \left(\sum_{j=1}^{n} \bar{\epsilon}_j \frac{\partial N_j^\epsilon}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \int_\Omega \frac{\gamma_w}{K_s} N_i^\epsilon \left[\sum_{j=1}^{n} N_j^\epsilon \frac{\partial \bar{\epsilon}_j}{\partial t} + p\beta \sum_{l=1}^{n} N_l^P \frac{\partial \bar{P}_l}{\partial t} \right] + (\lambda + 2\mu) \left[\nabla N_i^\epsilon \cdot \left(\sum_{j=1}^{n} \bar{\epsilon}_j \nabla N_j^\epsilon \right) \right] \mathrm{d}\Omega. \qquad (2.50)
\end{aligned}
$$

We can write Equation (2.50) as matrix-vector multiplication,

$$
A^{\epsilon\epsilon} \bar{\boldsymbol{\epsilon}}_t + A^{\epsilon P} \bar{\boldsymbol{P}}_t + (B^\epsilon - SD^\epsilon) \bar{\boldsymbol{\epsilon}} = 0, \qquad (2.51)
$$

where

$$
A_{ij}^{\epsilon\epsilon} = \int_\Omega \frac{\gamma_w}{K_s} N_i^\epsilon N_j^\epsilon \mathrm{d}\Omega, \quad A_{ij}^{\epsilon P} = \int_\Omega \frac{\gamma_w}{K_s} p\beta N_i^\epsilon N_j^P \mathrm{d}\Omega, \quad B_{ij}^\epsilon = \int_\Omega (\lambda + 2\mu) \left[\nabla N_i^\epsilon \cdot \nabla N_j^\epsilon \right] \mathrm{d}\Omega,
$$

$$
SD_i^\epsilon = \int_{\mathrm{d}\Omega_3} N_i^\epsilon \frac{\partial N_j^\epsilon}{\partial z} \mathrm{d}\Gamma, \quad \bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \bar{\epsilon}_1 \\ \vdots \\ \bar{\epsilon}_n \end{bmatrix}, \quad \bar{\boldsymbol{\epsilon}}_t = \begin{bmatrix} \frac{\partial \bar{\epsilon}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{\epsilon}_n}{\partial t} \end{bmatrix}, \quad \bar{\boldsymbol{P}}_t = \begin{bmatrix} \frac{\partial \bar{P}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{P}_n}{\partial t} \end{bmatrix},
$$

for $i, j = 1, \dots, n$.

2.3.4. Displacement equations

We will derive the weak form of the displacement equations and its matrix-vector form. Suppose that the test function is given by

$$
v^{u_x}(x, z, t) = \sum_{i=1}^{n} N_i^{u_x}(x, z) \bar{v}_i^{u_x}(t), \qquad (2.52)
$$

$$
v^{u_z}(x, z, t) = \sum_{i=1}^{n} N_i^{u_z}(x, z) \bar{v}_i^{u_z}(t), \qquad (2.53)
$$

$$
u_x(x, z, t) = \sum_{j=1}^{n} N_j^{u_x}(x, z) \bar{u}_j^x(t), \qquad (2.54)
$$

$$
u_z(x, z, t) = \sum_{j=1}^{n} N_j^{u_z}(x, z) \bar{u}_j^z(t). \qquad (2.55)
$$

Recall the following test and trial functions

$$\epsilon_{\text{vol}}(x,z,t) = \sum_{k=1}^{n} N_k^{\epsilon}(x,z)\bar{\epsilon}_k(t), \quad \omega(x,z,t) = \sum_{l=1}^{n} N_l^{\omega}(x,z)\bar{\omega}_l(t).$$

After multiplying Equations (2.23) and (2.24) by test functions v^{u_x} and v^{u_x}, respectively, and integrating over the domain Ω we get

$$0 = \int_{\Omega} v^{u_x}\left[-\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x}\right] \mathrm{d}\Omega, \tag{2.56}$$

$$0 = \int_{\Omega} v^{u_z}\left[-\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z}\right] \mathrm{d}\Omega. \tag{2.57}$$

Since $\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial z^2} = \nabla \cdot (\nabla u_i)$ for $i = x, z$, applying integration by parts and divergence theorem to Equations (2.56) and (2.57) gives

$$0 = -\int_{\mathrm{d}\Omega} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}\right] \mathrm{d}\Gamma + \int_{\Omega} v^{u_x}\left(\frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x}\right) + \left[\nabla v^{u_x} \cdot \nabla u_x\right] \mathrm{d}\Omega, \tag{2.58}$$

$$0 = -\int_{\mathrm{d}\Omega} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}\right] \mathrm{d}\Gamma + \int_{\Omega} v^{u_z}\left(-\frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z}\right) + \left[\nabla v^{u_z} \cdot \nabla u_z\right] \mathrm{d}\Omega. \tag{2.59}$$

After substituting the boundary conditions given by Equations (2.25), (2.26) and (2.27) into Equations (2.58) and (2.59), we get that

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}\right] \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}_1\right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}_2\right] \mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}_3\right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{u_x}\left[\nabla u_x \cdot \boldsymbol{\eta}_4\right] \mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{u_x}\frac{1}{2}\left(\omega - \frac{1}{\mu}F_{xz}\right) \mathrm{d}\Gamma,
\end{aligned} \tag{2.60}$$

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}\right] \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}_1\right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}_2\right] \mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}_3\right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{u_z}\left[\nabla u_z \cdot \boldsymbol{\eta}_4\right] \mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{u_z}\frac{\lambda}{2\mu}\epsilon_{\text{vol}} \mathrm{d}\Gamma.
\end{aligned} \tag{2.61}$$

Substituting the test and trial functions given by Equations (2.52), (2.53), (2.54), (2.55), (2.29), (2.38) and boundary integrals given by Equations (2.60) and (2.61) into Equations (2.58) and (2.59) gives the following Galerkin equations

$$\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} \sum_{i=1}^{n} N_i^{u_x}\bar{v}_i^{u_x}\frac{1}{2}\left[\left(\sum_{j=1}^{n} N_j^{\omega}\bar{\omega}_j\right) - \frac{1}{\mu}F_{xz}\right] \mathrm{d}\Gamma \\
&+ \int_{\Omega} \sum_{i=1}^{n} N_i^{u_x}\bar{v}_i^{u_x}\left(\frac{\partial}{\partial z}\left(\sum_{j=1}^{n} N_j^{\omega}\bar{\omega}_j\right) + \frac{\partial}{\partial x}\left(\sum_{l=1}^{n} N_l^{\epsilon}\bar{\epsilon}_l\right)\right) + \left[\nabla\left(\sum_{i=1}^{n} N_i^{u_x}\bar{v}_i^{u_x}\right) \cdot \nabla\left(\sum_{j=1}^{n} N_j^{u_x}\bar{u}_j^{x}\right)\right] \mathrm{d}\Omega,
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} \sum_{i=1}^{n} N_i^{u_z}\bar{v}_i^{u_z}\left(\frac{\lambda}{2\mu}\sum_{j=1}^{n} N_j^{\epsilon}\bar{\epsilon}_j\right) \mathrm{d}\Gamma \\
&+ \int_{\Omega} \sum_{i=1}^{n} N_i^{u_z}\bar{v}_i^{u_z}\left(-\frac{\partial}{\partial x}\left(\sum_{j=1}^{n} N_j^{\omega}\bar{\omega}_j\right) + \frac{\partial}{\partial z}\left(\sum_{l=1}^{n} N_l^{\epsilon}\bar{\epsilon}_l\right)\right) + \left[\nabla\left(\sum_{i=1}^{n} N_i^{u_z}\bar{v}_i^{u_z}\right) \cdot \nabla\left(\sum_{j=1}^{n} N_j^{u_z}\bar{u}_j^{z}\right)\right] \mathrm{d}\Omega.
\end{aligned} \tag{2.63}$$

Equations (2.62) and (2.63) can be written as

$$
\begin{aligned}
0 = &-\sum_{i=1}^{n} \bar{v}_i^{u_x} \int_{\mathrm{d}\Omega} N_i^{u_x} \frac{1}{2}\left[\left(\sum_{j=1}^{n} N_j^{\omega}\bar{\omega}_j\right) - \frac{1}{\mu}F_{xz}\right]\mathrm{d}\Gamma \\
&+ \sum_{i=1}^{n} \bar{v}_i^{u_x} \int_{\Omega} N_i^{u_x}\left(\sum_{j=1}^{n}\bar{\omega}_j\frac{\partial N_j^{\omega}}{\partial z} + \sum_{l=1}^{n}\bar{\epsilon}_l\frac{\partial N_l^{\epsilon}}{\partial x}\right) + \left[\nabla N_i^{u_x}\cdot\left(\sum_{j=1}^{n}\bar{u}_j^{x}\nabla N_j^{u_x}\right)\right]\mathrm{d}\Omega, \qquad (2.64)\\
0 = &-\sum_{i=1}^{n} \bar{v}_i^{u_z} \int_{\mathrm{d}\Omega} \frac{\lambda}{2\mu} N_i^{u_z} \sum_{j=1}^{n}\bar{\epsilon}_j N_j^{\epsilon}\mathrm{d}\Gamma \\
&+ \sum_{i=1}^{n} \bar{v}_i^{u_z} \int_{\Omega} N_i^{u_z}\left(-\sum_{j=1}^{n}\bar{\omega}_j\frac{\partial N_j^{\omega}}{\partial x} + \sum_{l=1}^{n}\bar{\epsilon}_l\frac{\partial N_l^{\epsilon}}{\partial z}\right) + \left[\nabla N_i^{u_z}\cdot\left(\sum_{j=1}^{n}\bar{u}_j^{z}\nabla N_j^{u_z}\right)\right]\mathrm{d}\Omega. \qquad (2.65)
\end{aligned}
$$

Since it must hold for arbitrary $\bar{v}_i^{u_x}$ and $\bar{v}_i^{u_z}$ with $i = 1, \ldots, n$, we have that Equations (2.64) and (2.65) still holds as

$$
\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} N_i^{u_x} \frac{1}{2}\left[\left(\sum_{j=1}^{n} N_j^{\omega}\bar{\omega}_j\right) - \frac{1}{\mu}F_{xz}\right]\mathrm{d}\Gamma \\
&+ \int_{\Omega} N_i^{u_x}\left(\sum_{j=1}^{n}\bar{\omega}_j\frac{\partial N_j^{\omega}}{\partial z} + \sum_{l=1}^{n}\bar{\epsilon}_l\frac{\partial N_l^{\epsilon}}{\partial x}\right) + \left[\nabla N_i^{u_x}\cdot\left(\sum_{j=1}^{n}\bar{u}_j^{x}\nabla N_j^{u_x}\right)\right]\mathrm{d}\Omega, \qquad (2.66)\\
0 = &-\int_{\mathrm{d}\Omega_3} \frac{\lambda}{2\mu} N_i^{u_z} \sum_{j=1}^{n}\bar{\epsilon}_j N_j^{\epsilon}\mathrm{d}\Gamma \\
&+ \int_{\Omega} N_i^{u_z}\left(-\sum_{j=1}^{n}\bar{\omega}_j\frac{\partial N_j^{\omega}}{\partial x} + \sum_{l=1}^{n}\bar{\epsilon}_l\frac{\partial N_l^{\epsilon}}{\partial z}\right) + \left[\nabla N_i^{u_z}\cdot\left(\sum_{j=1}^{n}\bar{u}_j^{z}\nabla N_j^{u_z}\right)\right]\mathrm{d}\Omega. \qquad (2.67)
\end{aligned}
$$

We can write Equations (2.66) and (2.67) as matrix-vector multiplications,

$$
\begin{cases}
B^{u_x}\bar{\boldsymbol{u}}^x + (D^{u_x} - SA^{u_x})\bar{\boldsymbol{\omega}} + C^{u_x}\bar{\boldsymbol{\epsilon}} & = -\mathbf{F}^{xz} \\
(B^{u_z} - SA^{u_z})\,\bar{\boldsymbol{u}}^z - C^{u_z}\bar{\boldsymbol{\omega}} + D^{u_z}\bar{\boldsymbol{\epsilon}} & = 0
\end{cases}, \qquad (2.68)
$$

where

$$
SA_{ij}^{u_x} = \int_{\mathrm{d}\Omega_3} \frac{1}{2} N_i^{u_x} N_j^{\omega}\mathrm{d}\Gamma, \quad SA_{ij}^{u_z} = \int_{\mathrm{d}\Omega_3} \frac{\lambda}{2\mu} N_i^{u_z} N_j^{\epsilon}\mathrm{d}\Gamma, \quad B_{ij}^{u_x} = \int_{\Omega} \nabla N_i^{u_x}\cdot\nabla N_j^{u_x}\mathrm{d}\Omega, \quad B_{ij}^{u_z} = \int_{\Omega} \nabla N_i^{u_z}\cdot\nabla N_j^{u_z}\mathrm{d}\Omega,
$$

$$
C_i^{u_x} = \int_{\mathrm{d}\Omega} N_i^{u_x}\frac{\partial N_j^{\epsilon}}{\partial x}\mathrm{d}\Omega, \quad C_i^{u_z} = \int_{\mathrm{d}\Omega} N_i^{u_z}\frac{\partial N_j^{\omega}}{\partial x}\mathrm{d}\Omega, \quad D_{ij}^{u_x} = \int_{\mathrm{d}\Omega} N_i^{u_x}\frac{\partial N_j^{\omega}}{\partial z}\mathrm{d}\Omega, \quad D_{ij}^{u_z} = \int_{\mathrm{d}\Omega} N_i^{u_z}\frac{\partial N_j^{\epsilon}}{\partial z}\mathrm{d}\Omega,
$$

$$
F_i^{xz} = \int_{\mathrm{d}\Omega_3} N_i^{u_x}\frac{1}{2\mu}F_{xz}\mathrm{d}\Gamma,
$$

for $i, j = 1, \ldots, n$.

2.3.5. Final FEM Model

We assume that $N_i^{\epsilon} = N_i^{\omega} = N_i^{p} = N_i^{u_x} = N_i^{u_z} =: N_i$ for all $i = 1, \ldots, n$. After collecting the governing equations given by Equations (2.36), (2.44), (2.51) and (2.68), we get the following set of Galerkin equations:

$$\begin{cases} \mu\,(B-SD)\,\bar{\boldsymbol{w}} & =0 \\ \frac{\gamma_w}{K_s}A\bar{\boldsymbol{\epsilon}}_t+\frac{\gamma_w}{K_s}p\beta A\bar{\boldsymbol{P}}_t+[(\lambda+2\mu)\,B-SD]\,\bar{\boldsymbol{\epsilon}} & =0 \\ \frac{\gamma_w}{K_s}A\bar{\boldsymbol{\epsilon}}_t+\frac{\gamma_w}{K_s}p\beta A\bar{\boldsymbol{P}}_t+B\bar{\boldsymbol{P}} & =0 \\ B\bar{\boldsymbol{u}}^x+(D-\frac{1}{2}SA)\bar{\boldsymbol{\omega}}+C\bar{\boldsymbol{\epsilon}} & =-\boldsymbol{F}^{xz} \\ \left(B-\frac{\lambda}{2\mu}SA\right)\bar{\boldsymbol{u}}^z-C\bar{\boldsymbol{\omega}}+D\bar{\boldsymbol{\epsilon}} & =0 \end{cases}, \tag{2.69}$$

where $A_{ij}=\int_\Omega N_iN_j\mathrm{d}\Omega$, $\quad SA_{ij}=\int_{\mathrm{d}\Omega_3}N_iN_j\mathrm{d}\Gamma$, $\quad B_{ij}=\int_\Omega \nabla N_i\cdot\nabla N_j\mathrm{d}\Omega$,
$C_{ij}=\int_\Omega N_i\frac{\partial N_j}{\partial x}\mathrm{d}\Omega$, $\quad SC_{ij}=\int_{\mathrm{d}\Omega_3}N_i\frac{\partial N_j}{\partial x}\mathrm{d}\Gamma$, $\quad D_{ij}=\int_\Omega N_i\frac{\partial N_j}{\partial z}\mathrm{d}\Omega$, $\quad SD_{ij}=\int_{\mathrm{d}\Omega_3}N_i\frac{\partial N_j}{\partial z}\mathrm{d}\Gamma$
for all $i,j=1,\dots,n$.

We can write Equation (2.69) as one system of matrix-vector multiplication

$$M^t\boldsymbol{\theta}_t+M\boldsymbol{\theta}=\boldsymbol{f}, \tag{2.70}$$

where

$$M^t=\begin{bmatrix} \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A & \emptyset & \emptyset \\ \emptyset & \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \end{bmatrix}\in\mathbb{R}^{5n\times5n},\quad M=\begin{bmatrix} \mu\,(B-SD) & \emptyset & \emptyset & \emptyset & \emptyset \\ -\mu SC & (\lambda+2\mu)\,B & -SD & \emptyset & \emptyset \\ \emptyset & \emptyset & B & \emptyset & \emptyset \\ D-\frac{1}{2}SA & C & \emptyset & B & \emptyset \\ -C & D & \emptyset & \emptyset & (B-SA) \end{bmatrix}\in\mathbb{R}^{5n\times5n},$$

$$\boldsymbol{\theta}=\begin{bmatrix} \bar{\boldsymbol{w}} \\ \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{u}}^x \\ \bar{\boldsymbol{u}}^z \end{bmatrix}\in\mathbb{R}^{5n},\boldsymbol{\theta}_t=\begin{bmatrix} \frac{\partial\bar{\boldsymbol{w}}}{\partial t} \\ \frac{\partial\bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial\bar{\boldsymbol{P}}}{\partial t} \\ \frac{\partial\bar{\boldsymbol{u}}^x}{\partial t} \\ \frac{\partial\bar{\boldsymbol{u}}^z}{\partial t} \end{bmatrix}\in\mathbb{R}^{5n},\quad \boldsymbol{f}=\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{F}^{xz} \\ \mathbf{0} \end{bmatrix}\in\mathbb{R}^{5n}. \tag{2.71}$$

Note that the boundary conditions are included.

2.4. Discretisation in time

For discretising the Galerkin equations in matrix-vector form given by Equation (2.70) we use the Backward-Euler method. The Backward-Euler method is given by

$$\boldsymbol{\theta}_t=\frac{1}{\Delta t}\left(\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^k\right)=g\left(t^{k+1},\boldsymbol{\theta}\left(t^{k+1}\right)\right), \tag{2.72}$$

where $\boldsymbol{\theta}^{k+1}=\boldsymbol{\theta}\left(t^{k+1}\right)$, $\Delta t=t^{k+1}-t^k$ is the time step, and $g\left(t^{k+1},\boldsymbol{\theta}\left(t^{k+1}\right)\right)=\boldsymbol{f}^{k+1}-M\boldsymbol{\theta}^{k+1}$ and $\boldsymbol{f}^{k+1}=\boldsymbol{f}\left(t^{k+1}\right)$. Applying Equation (2.72) to Equation (2.70) gives

$$\left(M^t+\Delta tM\right)\boldsymbol{\theta}^{k+1}=M^t\boldsymbol{\theta}^k+\Delta t\boldsymbol{f}^{k+1}. \tag{2.73}$$

Assuming that $\left(M^t+\Delta tM\right)$ is invertible, Equation (2.73) can be written as

$$\boldsymbol{\theta}^{k+1}=\left(M^t+\Delta tM\right)^{-1}\left(M^t\boldsymbol{\theta}^k+\Delta t\boldsymbol{f}^{k+1}\right). \tag{2.74}$$

The two-dimensional results will not be determined in this literature report. However, this is one of the main goals for our further research.

3

Biot's Model (1D)

For simplicity, we will look at Biot's model in one dimension and derive its numerical model. In the next sections, the governing equations of Biot's model will be derived in one dimension together with the corresponding boundary conditions. Results are shown for a seabed consisting of one homogeneous layer of soil and for a seabed consisting of two homogeneous layers of two different types of soil. Given that the seabed is n_z metres, the layer of the seabed with one type of soil has thickness n_z and the two layers of the seabed with two types of soil have thickness $\frac{n_z}{2}$.

3.1. One type of soil

We will first give a solution when assuming one layer of fine sand. In this case the porosity, Lamé's constants and hydraulic conductive are constant and their derivatives with respect to z equal zero.

3.1.1. Physical model and numerical model

Noting that in one dimension we have that $\epsilon_{\text{vol}} = \frac{\partial u_z}{\partial z}$, we then have the following constitutive equations in one dimension

$$\begin{cases} \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda + 2\mu)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} & = -\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} \\ \frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial z^2} & = -\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} \\ \frac{\partial u_z}{\partial z} & = \epsilon_{\text{vol}} \end{cases}, \text{ for } z \in \Omega \text{ and } t \in \mathbb{T}, \tag{3.1}$$

with boundary conditions

$$\begin{cases} \epsilon_{\text{vol}} = 0 \\ P(0,t) & = F_{zz}(t) \end{cases}, \quad \begin{cases} u_z(-n_z, t) & = 0 \\ \frac{\partial P(-n_z,t)}{\partial z} & = 0 \\ \frac{\partial \epsilon_{\text{vol}}(-n_z,t)}{\partial z} & = 0 \end{cases}, \text{ for } t \in \mathbb{T}, \tag{3.2}$$

and initial conditions

$$\epsilon_{\text{vol}}(z,0) = P(z,0) = u_z(z,0) = 0, \text{ for } z \in \Omega.$$

Using boundary conditions, the weak equations in one-dimension are given by

$$\begin{cases} \int_{-n_z}^{0} N_i^{\epsilon} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + (\lambda + 2\mu)\left(\frac{\partial v^{\epsilon}}{\partial z} \cdot \frac{\partial \epsilon_{\text{vol}}}{\partial z}\right) \mathrm{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{P} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + \left(\frac{\partial v^{P}}{\partial z} \cdot \frac{\partial P}{\partial z}\right) \mathrm{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{u} \frac{\partial u_z}{\partial z} \mathrm{d}z & = \int_{-n_z}^{0} N_i^{u} \epsilon_{\text{vol}} \mathrm{d}z \end{cases}. \tag{3.3}$$

Substituting one-dimensional trial functions into Equation (3.3) we get the following one-dimensional Galerkin equations in matrix-vector form

$$\begin{cases} \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + (\lambda + 2\mu)\, B\bar{\boldsymbol{\epsilon}} & = \boldsymbol{0} \\ \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + B\bar{\boldsymbol{P}} & = \boldsymbol{0} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{3.4}$$

where $A_{i,j} = \int_{-n_z}^{0} N_i N_j \mathrm{d}z, B_{i,j} = \int_{-n_z}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{d}z, C_{i,j} = \int_{-n_z}^{0} N_i \frac{\partial N_j}{\partial z}\mathrm{d}z.$ and $C_{i,j}^0 = N_i(0)\frac{\partial N_j(0)}{\partial z}$.

We can write Equation (3.4) as two systems of matrix-vector multiplication

$$\begin{cases} M^t \boldsymbol{S}_t + M\boldsymbol{S} & = \boldsymbol{f} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{3.5}$$

where

$$M^t = \begin{bmatrix} \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \\ \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \end{bmatrix}, \quad M = \begin{bmatrix} (\lambda + 2\mu)\, B & \emptyset \\ \emptyset & B \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \end{bmatrix}, \quad \boldsymbol{S}_t = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{P}}}{\partial t} \end{bmatrix}, \quad \boldsymbol{f}(t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}(t) \end{bmatrix}. \tag{3.6}$$

The Dirichlet boundary conditions are included in the numerical model by setting the corresponding rows of matrices M^t and M to zero and then putting pivots in these same rows of M. Furthermore, we set $F_i = 0$ for $i = 1, \dots, n-1$ and $F_n = F_{zz}$. Then we get that $\bar{P}_n = F_n = F_{zz}$ and $\bar{u}_0^z = 0$. Now we apply the time integration given by Equation (2.74) using $M^t, M, \boldsymbol{S}$ and $\boldsymbol{f}$ as described above. Assuming that matrix $\left(M^t + \Delta t M\right)$ and C are invertible, we get that

$$\begin{cases} \boldsymbol{S}^{k+1} & = \left(M^t + \Delta t M\right)^{-1} \left(M^t \boldsymbol{S}^k + \Delta t \boldsymbol{f}^{k+1}\right) \\ \bar{\boldsymbol{u}}_{\boldsymbol{z}}^{k+1} & = C^{-1} A \bar{\boldsymbol{\epsilon}}^{k+1} \end{cases}. \tag{3.7}$$

Finally, we choose that

$$F_{zz}(t) = \begin{cases} (1-\beta)\,(\lambda + 2\mu)\,(1 - \cos(t)), & \text{if } t < \pi \\ 2(1-\beta)\,(\lambda + 2\mu), & \text{if } t \geq \pi \end{cases}. \tag{3.8}$$

Note that F_{zz}, which represents the normal stress, indeed only depends on time and is chosen to be positive for all $t \in \mathbb{T}$. Furthermore, the value of F_{zz} and the value of its derivative with respect to z equals zero at initial time $t = 0$. This way the initial conditions and boundary conditions are still met. Finally, note that the normal stress F_{zz} becomes constant over time.

Then the solution of the one-dimensional model must go to the stationary solution over time. Therefore, will now solve the stationary one-dimensional system in order to find the stationary solution.

3.1.2. Stationary model

The stationary model in one dimension is given by

$$\begin{cases} \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} = 0 \\ \frac{\partial^2 P}{\partial z^2} = 0 \\ \frac{\partial u_z}{\partial z} = \epsilon_{\text{vol}} \end{cases}, \tag{3.9}$$

with boundary conditions given by Equation (3.2) using $F_{zz} = F$ constant $(t \to \infty)$.

The set of equations given by Equation (3.9) gives the following solutions

$$\begin{cases} u_z = \int \epsilon_{\text{vol}} \mathrm{d}z + c_1 = \frac{1}{2}c_2 z^2 + c_3 z + c_1 \\ \epsilon_{\text{vol}} = c_2 z + c_3 \\ P = c_4 z + c_5 \end{cases}. \tag{3.10}$$

Using the boundary conditions, we find that $c_1 = c_2 = c_3 = c_4 = 0$ and $c_5 = F$, i.e.

$$\begin{cases} u_z = 0 \\ \epsilon_{\text{vol}} = 0 \\ P = F \end{cases} . \tag{3.11}$$

Then we find that also the stationary solution for ϵ_{vol} and u_z has a unique solution. Therefore, the solution(s) of Equation (2.69) will converge to a unique solution, namely the stationary solution given by Equation (3.11). However, this (stationary) solution translates to the pore water pressure carrying all load while the effective stress is zero on the whole space $[-n_z, 0]$ which seems physically not possible [6].

3.1.3. Unique solution

When adding the first and second subequations of Equation (3.1), we find

$$\begin{gathered} (\lambda + 2\mu)\,\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} = \frac{\partial^2 P}{\partial z^2} \\ \Leftrightarrow \\ (\lambda + 2\mu)\,\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z} + d_1(t) \\ \Leftrightarrow \\ (\lambda + 2\mu)\,\epsilon_{\text{vol}} = P + d_1 z + d_2, \end{gathered}$$

where $d_1(t)$ and $d_2(t)$ are constant in space. Since we have boundary condition $\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z} = 0$ at $z = 0$, we get that $d_1(t) = 0$. Because we have the boundary conditions $\epsilon_{\text{vol}} = 0$ and $P = F_{zz}$ at $z = -n_z$, we get that $d_2(t) = -F_{zz}$. This implies that

$$(\lambda + 2\mu)\,\epsilon_{\text{vol}} = P - F_{zz}, \quad \text{on } \bar{\Omega} := \Omega \cup \mathrm{d}\Omega = [-n_z, 0].$$

Therefore, we not only have a unique solution for the stationary model given by Equation (3.11), but also for the Galerkin model given by (3.5).

3.1.4. Results

The three variables ϵ_{vol}, P, u_z and their derivatives are plotted five times, namely at $t = 0.0, 1.5, 3.0$ and $4.5, 6.0$, for the case of compressible water and of incompressible water. For the assumption of compressible water we set $\beta = 4.8 \cdot 10^{-10}$ and for the assumption of incompressible water we have by definition $\beta = 0.0$. The effective stress $\sigma'_{zz} = -(\lambda + 2\mu)\,\epsilon_{\text{vol}}$ is also plotted in order to check the relationship $(\lambda + 2\mu)\,\epsilon_{\text{vol}}(z,t) = P(z,t) - F_{zz}$. We use piece-wise linear basis-functions N_i, which is of degree 1 and smoothness 0 and are called hat functions. These are given by

$$N_i = \begin{cases} \frac{z - z_{i-1}}{z_i - z_{i-1}}, & \text{if } z \in [z_{i-1}, z_i] \\ \frac{z_{i+1} - z}{z_{i+1} - z_i}, & \text{if } z \in [z_i, z_{i+1}] \end{cases}, \tag{3.12}$$

for $i = 1, \ldots, n$. Furthermore, for integration of a subdomain we use 1000 integration points, the time step is chosen as $\Delta t = 0.0025$ and the number of subdomains is chosen as $\Delta z = 0.0025$. The porosity, Poisson ratio and the effective size of the grain, d_{10} [m], is given by Table 3.1 and the shear modulus μ and specific weight γ_w are given by Table 3.2. The hydraulic conductivity is $K_s = c \cdot d_{10}$ [m/s] according to Allen Hazen [9]. We will use $c = 1.0$ which is often chosen in civil engineering problems. Recall that λ is given by Equation (2.5). At last, we set $n_z = 2$, i.e. $\bar{\Omega} = [-2, 0]$.

Table 3.2: Parameters shear modulus and specific weight of one layer of soil which is fine sand [6].

Parameters	**Symbols**	**Values**
Shear modulus of 'fine sand' [Pa]	μ	$7.7 \cdot 10^6$
Specific weight of water [Pa]	γ_w	10^4

Table 3.1: Parameters effective size of the grains, porosity and Poisson ratio of one layer of soil which is fine sand [10].

Soil properties	**Symbols**	**Fine sand**
Effective size of the grains [m]	d_{10}	$3.0 \cdot 10^{-5}$
Porosity	p	0.4
Poisson ratio	v_p	0.3

In Figures 3.1 and 3.2 and Figures 3.3 and 3.4, for assuming compressible water and for assuming incompressible water we find that all boundary conditions given by Equation (3.2) are (approximately) satisfied.

When comparing the top figures of Figures 3.1 and 3.3, we find that the effective stress and volumetric strain indeed behave mirrored times a constant. In subfigures of Figures 3.1 and 3.3 for water pressure and volumetric strain, we do not immediately see the relationship $(\lambda + 2\mu)\,\epsilon_{\text{vol}}(z,t) = P(z,t) - F_{zz}$ which is derived in the previous section which is probably due to the large values of the water pressure. In subfigures for the derivative of the water pressure and volumetric strain of Figures 3.2 we do find derivative of the relation, namely $(\lambda + 2\mu)\,\frac{\partial \epsilon_{\text{vol}}(z,t)}{\partial z} = \frac{\partial P(z,t)}{\partial z}$ which indicates that the relationship between the water pressure and the volumetric strain is present for compressible water. We also found, when increasing the time further, that the solutions of the volumetric strain, water pressure and vertical displacement indeed converge to the stationary solution. For incompressible water, it seems that the solutions of the volumetric strain, water pressure and displacement in z-direction (and the effective stress) found is already close to the corresponding steady state solutions given by Equation (3.11). Note that then the relation between the volumetric strain and water pressure still holds: $(\lambda + 2\mu)\,\epsilon_{\text{vol}}(z,t) = P(z,t) - F_{zz} \Longrightarrow 0 = (\lambda + 2\mu)\,0 = F_{zz} - F_{zz} = 0$. This is shown the subfigures of these variables and their derivatives in Figure 3.3. In this case at $t = 0.0$ the water pressure is zero in $\Omega = [-2, 0]$, while at the next time, say $t = 1.5$, the pressure is approximately 25043632.42 Pa over the whole space Ω and the volumetric strain and vertical displacement remain approximately 0 over the whole space. When assuming incompressible water, also [7] noted the pore water pressure is constant over space and thus directly evenly distributed over the whole space. This agrees with [6] that the pore water pressure is directly transferred deeper in the soil for Biot's model with the assumption of incompressible water.

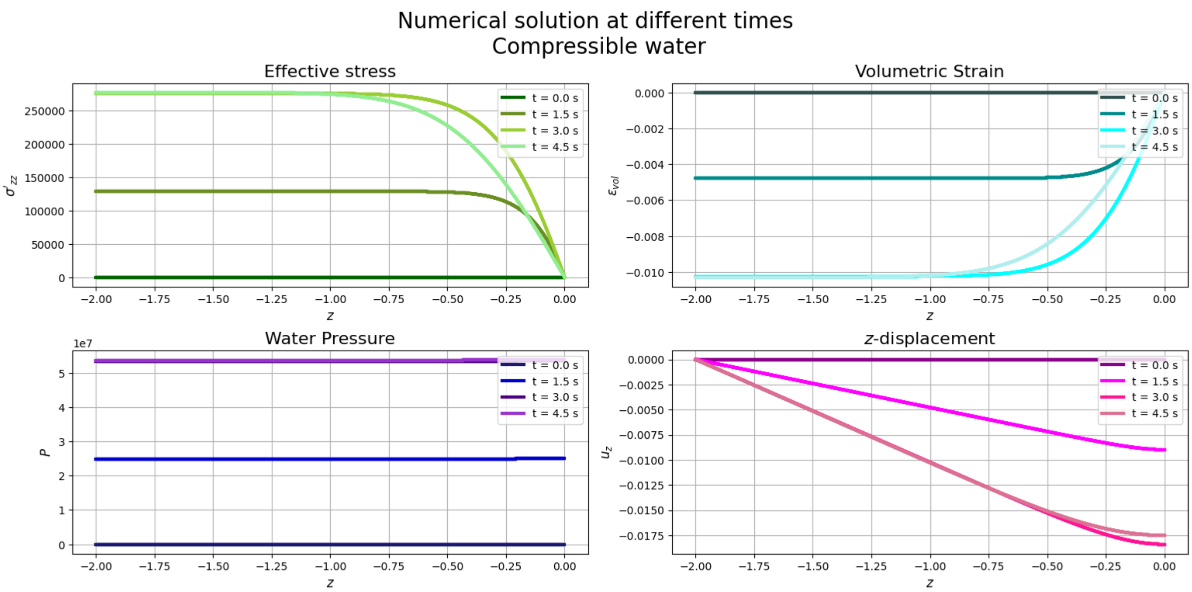

Figure 3.1: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

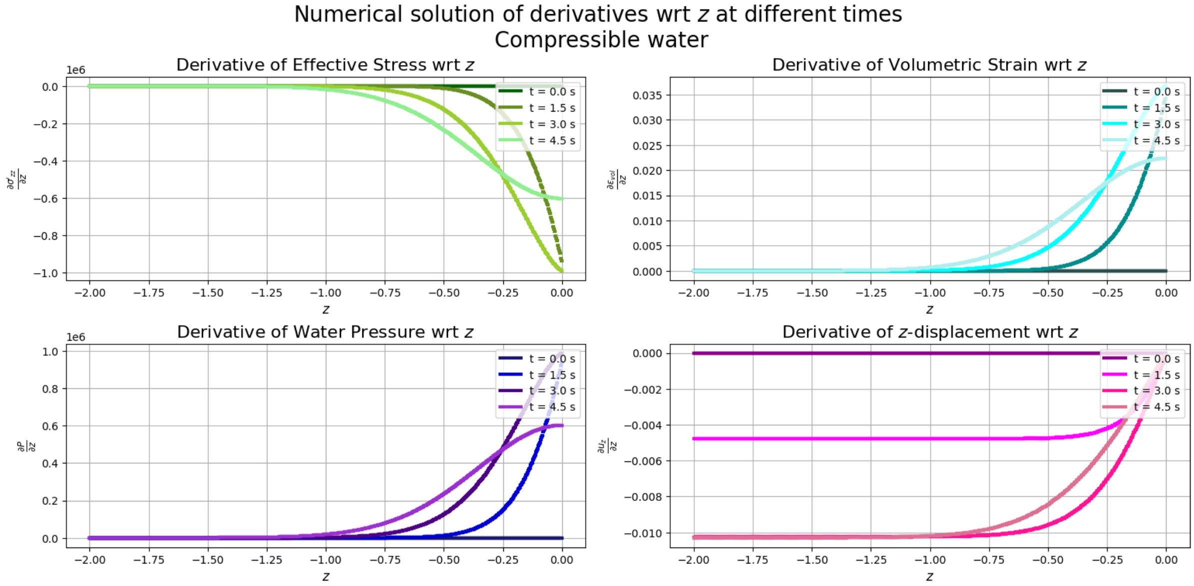

Figure 3.2: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

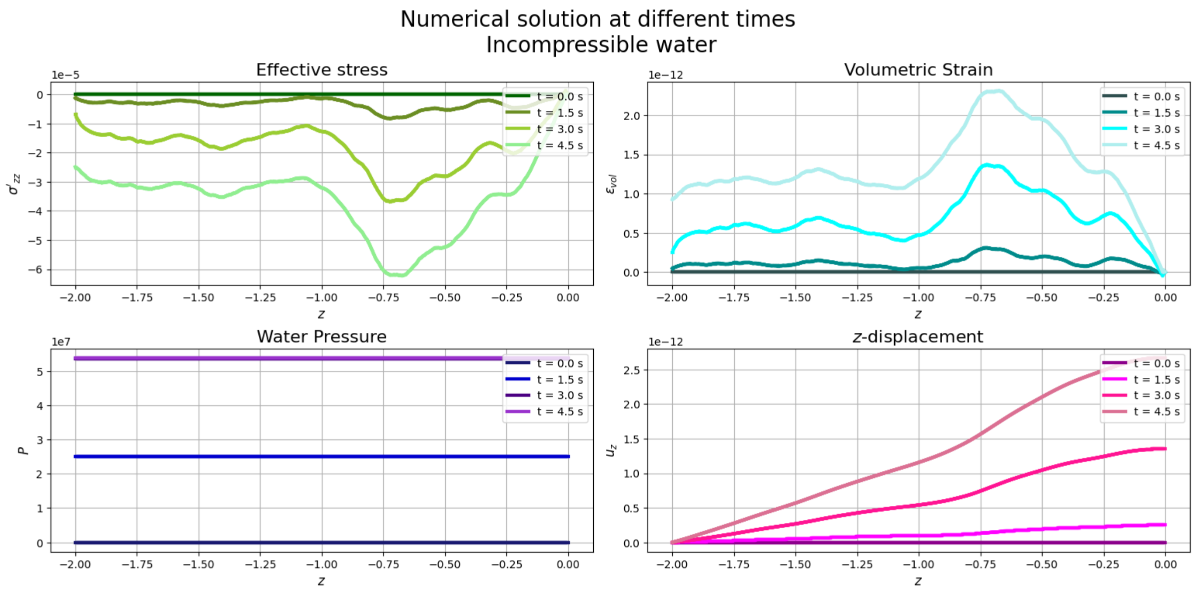

Figure 3.3: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be incompressible ($\beta = 0$).

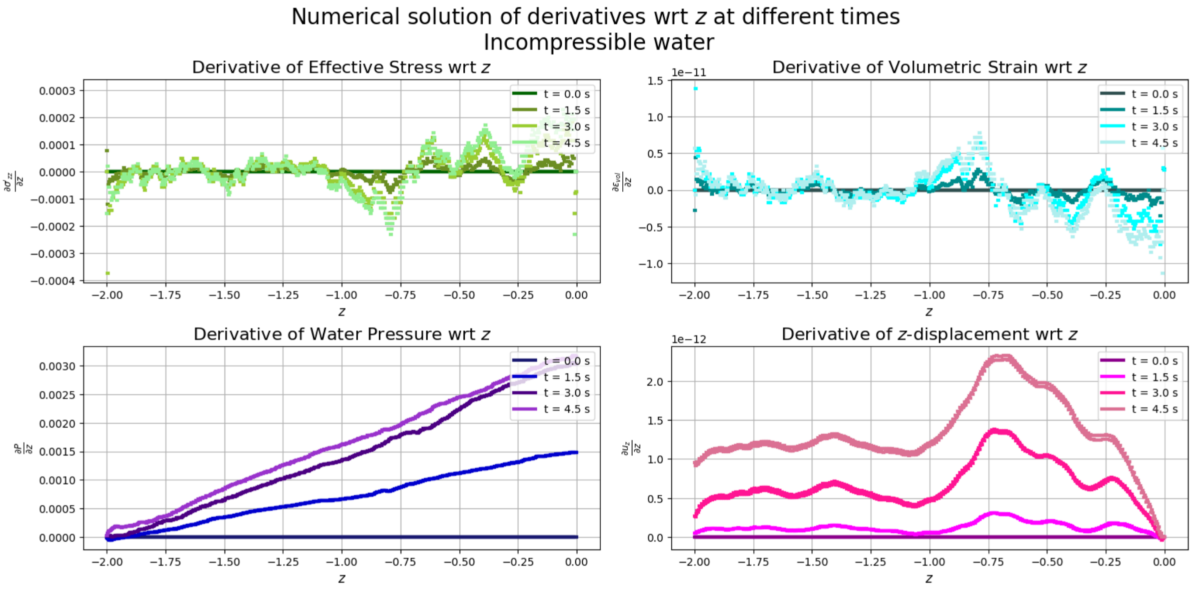

Figure 3.4: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be incompressible ($\beta = 0$).

3.2. Two types of soil

Note that in previous section it was assumed that we had one type of soil. In this section we assume that we have two types of soil and that the transitions happens at -1.0 meter.

3.2.1. Physical model and numerical model

Using the alternative boundary conditions given by (3.2), we get the same one-dimensional weak equations for the two different layers of soil, say fine sand and medium sand, as for one type of soil. However, K_s, p, λ and μ differ per separate layer. We define that the first layer, $z \in [-nz, -\frac{n_z}{2}]$, is medium sand and the second layer, $z \in [-\frac{n_z}{2}, 0]$, is fine sand, i.e. the boundary between the two layer is the middle of the whole domain $\Omega = [-n_z, 0]$. The subdomains are divided such that there is no overlap between layers in one subdomain. However, note that the derivative of the constants that differ per layer does not exist when $z \to -\frac{n_z}{2}$. Since this is only one point and has almost no impact on the numerical model, we ignore this for now.

Then we get that substituting one-dimensional trial functions into Equation (3.3) gives the following one-dimensional Galerkin equations in matrix-vector form

$$\begin{cases} A^{\epsilon}\bar{\boldsymbol{\epsilon}}_t + A^{P}\bar{\boldsymbol{P}}_t + B^{\epsilon}\bar{\boldsymbol{\epsilon}} & = \mathbf{0} \\ A^{\epsilon}\bar{\boldsymbol{\epsilon}}_t + A^{P}\bar{\boldsymbol{P}}_t + B^{P}\bar{\boldsymbol{P}} & = \mathbf{0}\,, \\ C\bar{\boldsymbol{u}}^{z} = A\bar{\boldsymbol{\epsilon}} \end{cases} \tag{3.13}$$

where

$$\begin{cases} A^{\epsilon}_{i,j} = \frac{\gamma_w^{\text{m}}}{K_S^{\text{m}}} \int_{-n_z}^{-\frac{n_z}{2}} N_i N_j \mathrm{d}\Omega + \frac{\gamma_w^{\text{f}}}{K_S^{\text{f}}} \int_{-\frac{n_z}{2}}^{0} N_i N_j \mathrm{d}\Omega \\ A^{P}_{i,j} = \frac{\gamma_w^{\text{m}}}{K_S^{\text{m}}} p^{\text{m}}\beta \int_{-n_z}^{-\frac{n_z}{2}} N_i N_j \mathrm{d}z + \frac{\gamma_w^{\text{f}}}{K_S^{\text{f}}} p^{\text{f}}\beta \int_{-\frac{n_z}{2}}^{0} N_i N_j \mathrm{d}z \\ B^{\epsilon}_{i,j} = (\lambda^{\text{m}} + 2\mu^{\text{m}}) \int_{-n_z}^{-\frac{n_z}{2}} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{d}z + (\lambda^{\text{f}} + 2\mu^{\text{f}}) \int_{-\frac{n_z}{2}}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{d}z \\ B^{P}_{i,j} = \int_{-n_z}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{d}z \\ C_{i,j} = \int_{-n_z}^{0} N_i \frac{\partial N_j}{\partial z}\mathrm{d}z \end{cases}, \tag{3.14}$$

and the subscripts m and f means the parameter with property of medium and and fine sand, respectively.

We can write Equation (3.13) as two systems of matrix-vector multiplication

$$\begin{cases} M^t \boldsymbol{S}_t + M\boldsymbol{S} & = \boldsymbol{f} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{3.15}$$

where

$$M^t = \begin{bmatrix} A^\epsilon & A^P \\ A^\epsilon & A^P \end{bmatrix}, \quad M = \begin{bmatrix} B^\epsilon & \emptyset \\ \emptyset & B^P \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \end{bmatrix}, \quad \boldsymbol{S}_t = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{P}}}{\partial t} \end{bmatrix}, \quad \boldsymbol{f}(t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}(t) \end{bmatrix}, \tag{3.16}$$

with $A^\epsilon, A^P, B^\epsilon$ and B^P are given by Equation (3.14). The Dirichlet boundary conditions are included in the numerical model by setting the corresponding rows of matrices M^t and M to zero and then putting pivots in these same rows of M. Furthermore, we set again $F_i = 0$ for $i = 1, \ldots, n-1$ and $F_n = F_{zz}$. Then we get that $\bar{P}_n = F_n = F_{zz}$ and $\bar{u}_0^z = 0$.

Now we apply the time integration given by Equation (2.74) using $M^t, M, \boldsymbol{S}$ and $\boldsymbol{f}$ as described in Equation (3.16), where we assume that matrices $(M^t + \Delta t M)$ and C are invertible, we get again the time integration formulas given by Equation (3.7) with F_{zz} chosen as in Equation (3.8), but with M_t, M given by Equation (3.16).

3.2.2. Stationary model and unique solution

Since the only difference between models of having one type or two different layers of soils are some parameters being a different constant for each layer because of the properties of the type of soils, the stationary model and its solution of the one-dimensional model with two different layers of soil remain the same as in model with only one layer of soil. The unique stationary model was given by Equation (3.11). It also holds that this one-dimensional model with two different types of soil still has a unique solution.

3.2.3. Results

The three variables ϵ_{vol}, P, u_z and their derivatives are plotted five times, namely at $t = 0.0, 1.5, 3.0$ and $4.5, 6.0$, for the case of compressible water and of incompressible water. For the assumption of compressible water we take $\beta = 4.8 \cdot 10^{-10}$ and for the assumption of incompressible water we have by definition $\beta = 0.0$. The effective stress $\sigma'_{zz} = -(\lambda + 2\mu)\,\epsilon_{\text{vol}}$ is also plotted. We use the piece-wise linear basis-functions N_i for $i = 1, \ldots, n$, given by Equation (3.12). Furthermore, we use again 1000 integration points, a time step of $\Delta t = 0.0025$ and the number of subdomains is chosen as $\Delta z = 0.0025$. The values of porosity p, Poisson ratio v_p and the effective size of the grains d_{10} [m] per layer are given by Table 3.3 and μ is given by Table 3.4. We use again that $K_s = c \cdot d_{10}$ [m/s] by Allen Hazen [9] with $c = 1.0$. λ is given by Equation (2.5) and recall that we define $\gamma_w = 10^4$. We set again $n_z = 2$, i.e. $\bar{\Omega} = [-2, 0]$.

Table 3.3: Parameters effective size of the grains, porosity and Poisson ratio of two layers of soil which are fine sand and medium sand [10].

Soil properties	**Symbols**	**Fine sand**	**Medium sand**
Effective size of the grains [m]	d_{10}	$3.0 \cdot 10^{-5}$	$2.3 \cdot 10^{-4}$
Porosity	p	0.4	0.44
Poisson ratio	v_p	0.3	0.27

Table 3.4: Lamé's constants of two layers of soil which are fine sand and medium sand [6].

Soil properties	**Symbols**	**Fine sand**	**Medium sand**
Shear modulus [Pa]	μ	$7.7 \cdot 10^6$	$3.9 \cdot 10^6$

Per layer, the results look like the results of the one-dimensional model of Biot with one layer of soil. The values of the effective stress, volumetric strain, water pressure and vertical displacement differ a bit compared to their values for one layer, which makes sense since several parameters depend on the properties of the soil. For example, Lamé constants λ and μ change when the first layer changes into the second. This may cause a jump in value or a kink at the boundary between the layers where $z = -1.0$. Since the effective stress is $\lambda + 2\mu$ times the volumetric strain, we see a bigger jump for the assumption of compressible and incompressible water in the subfigure of the effective stress in Figures 3.5 and 3.7, respectively. When looking at the subfigures for the pore water pressure and volumetric strain of the Figures 3.5 and 3.6 and Figures 3.7 and 3.8, we find again the relation of the volumetric strain and pore water pressure given by $(\lambda + 2\mu)\epsilon_{\text{vol}} = P - F_{zz}$ for both the assumptions of compressible water and incompressible. We expect the water pressure and the z-displacement to be continuous which seems to hold for compressible and incompressible water. This can be seen for compressible and incompressible water when looking at the solutions for the water pressure and vertical displacement in Figures 3.5 and 3.7, respectively. Note that approximately the solutions for incompressible water becomes the stationary solution directly. When increasing the time, we again find that the solutions for compressible water converge to the stationary solution.

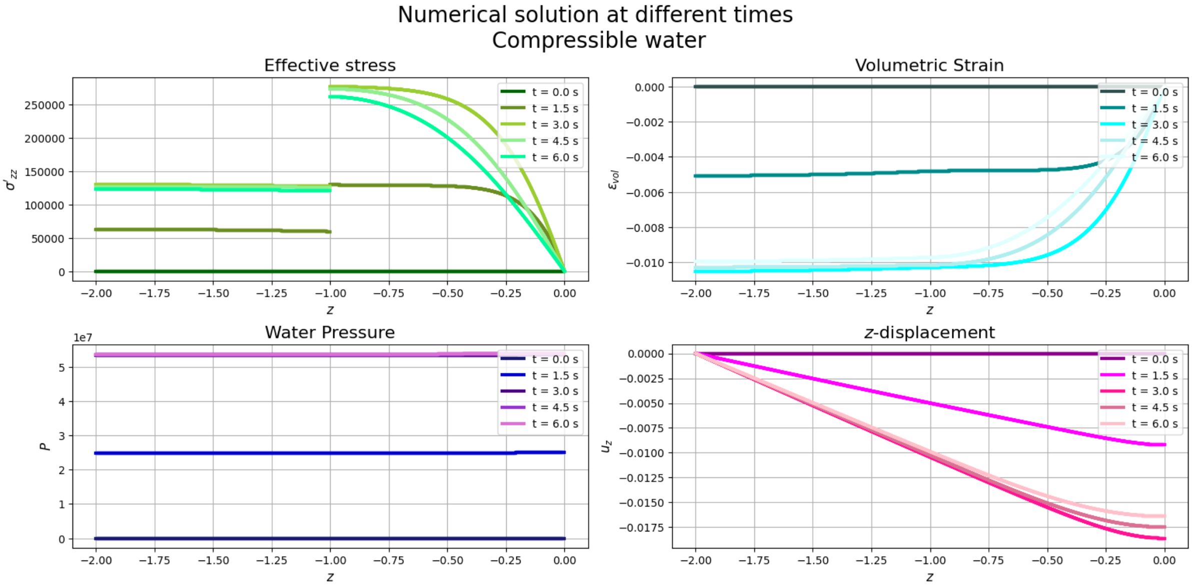

Figure 3.5: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

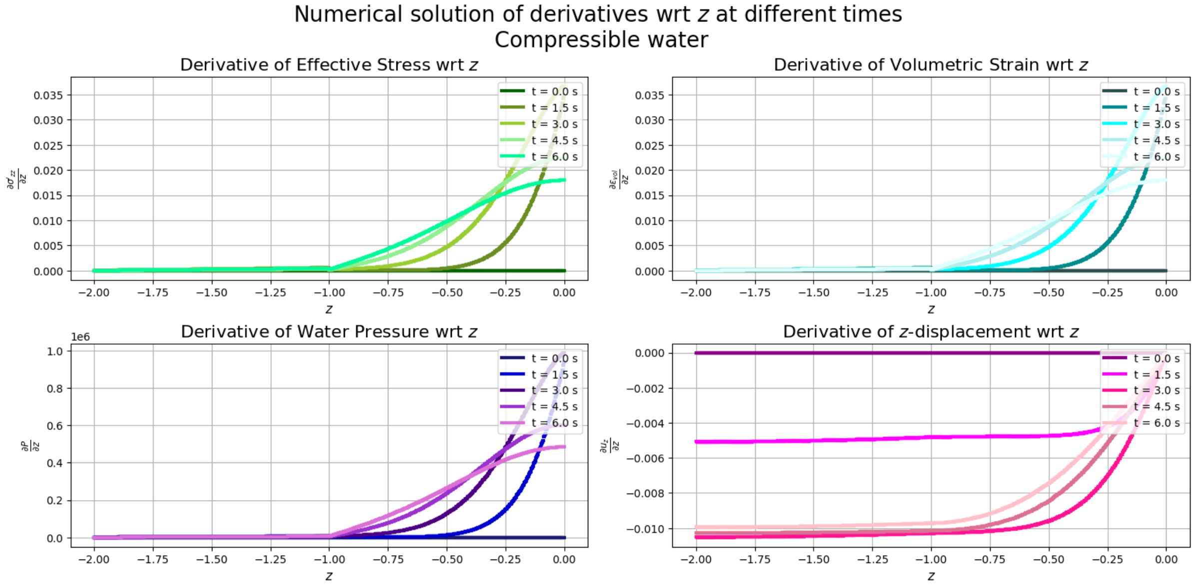

Figure 3.6: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

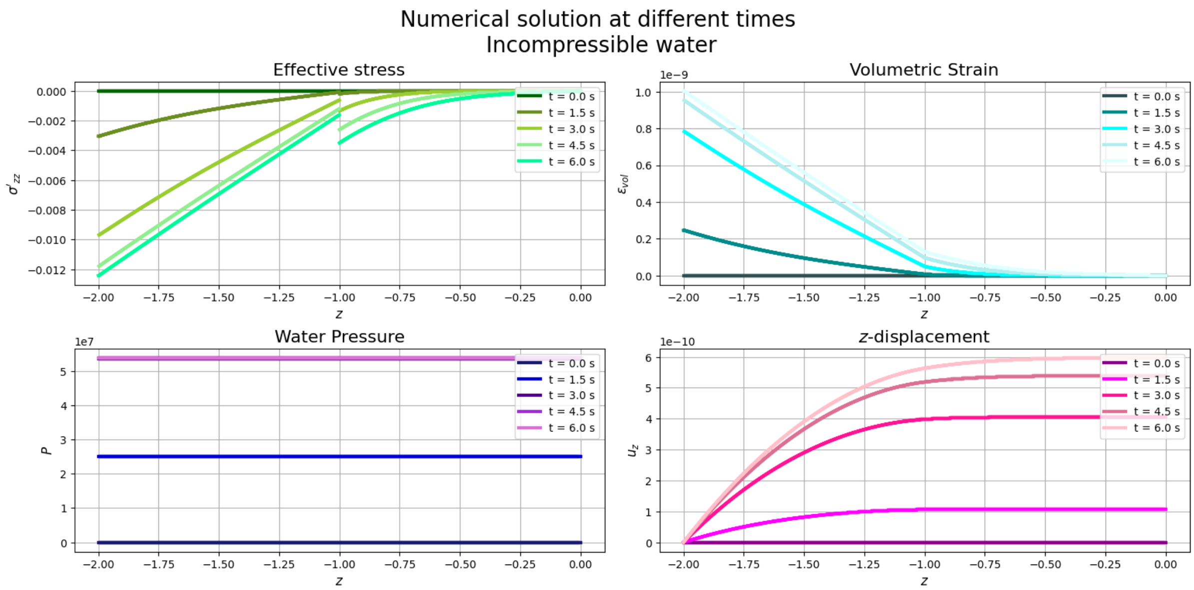

Figure 3.7: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be incompressible ($\beta = 0$).

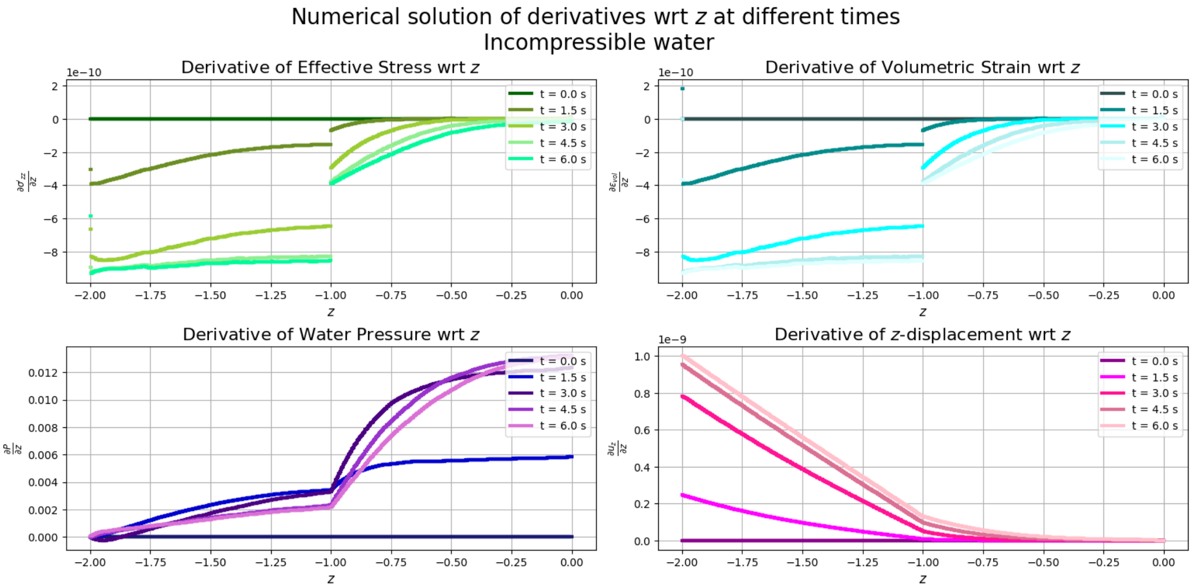

Figure 3.8: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be incompressible ($\beta = 0$).

4

Van Damme and den Ouden-van der Horst Model (2D)

In the next sections we will derive the governing equations for the model of Van Damme and den Ouden-van der Horst in two dimensions and apply numerical methods to discretise in space and time. We will first discretise the new model in space using the finite-element method and in time using the Backward-Euler method.

4.1. Governing equations

We will use Cartesian (x, z)-coordinates. We will follow the derivation of four constitutive equations by [6]. These are given by the vorticity equation, volumetric strain equation, water pressure equation and displacements equation. These follow from the volume balance equation and the momentum balance equations. This approach is based on defining a stress and a stress gradient as boundary conditions which follows from the momentum balance equation [6]. Therefore, it is also in line with the D'Alembert's principle of minimisation of virtual work. However, it does not follow the effective stress principle of Terzaghi [6]. Like we did for Biot's model, we do not take the acceleration terms into account and ignore the body forces in the model of Van Damme and den Ouden-van der Horst.

4.1.1. Volume balance equation

According to [6], the volume balance equation for compressible or incompressible pore water is given by

$$p\beta\frac{\partial P}{\partial t} + \frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(p\frac{\partial w_x}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\frac{\partial w_z}{\partial t}\right) = 0, \tag{4.1}$$

where t denotes time, p denotes the porosity of the medium, β denotes the compressibility of the pore water (if $\beta = 0$ the water is incompressible, and if $\beta \in (0, 1]$ the water is compressible), P denotes the pore water pressure and w_x and w_z denotes the 2D displacement of the pore water in x-direction and z-direction, respectively. According to [6] the volume balance equation of the incompressible particles in a porous medium is given by

$$\frac{\partial(1-p)}{\partial t} + \frac{\partial}{\partial x}\left((1-p)\frac{\partial u_x}{\partial t}\right) + \frac{\partial}{\partial z}\left((1-p)\frac{\partial u_z}{\partial t}\right) = 0, \tag{4.2}$$

where u_x and u_z denotes the 2D displacement of the porous medium in x-direction and z-direction, respectively. Equation (4.2) describes the change in porosity caused by the movement of incompressible particles in a porous medium. Then the volume balance equation for the porous medium is given by adding the volume balance equation of the pore water to the volume balance equation of the particles [6]

$$p\beta\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(p\frac{\partial(w_x - u_x)}{\partial t}\right) + \frac{\partial}{\partial z}\left(p\frac{\partial(w_z - u_z)}{\partial t}\right) + \frac{\partial \epsilon_{\text{vol}}}{\partial t} = 0, \tag{4.3}$$

where $\epsilon_{\text{vol}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}$ is the volumetric strain of the porous medium.

4.1.2. Momentum balance equations

[6] derives the momentum balance equations using D'Alembert's principle of virtual work. This principle states that for the reversible displacements the total virtual work of the imposed forces plus the inertial forces vanish [6]. The momentum balance equations are given by [6]

$$-\mu\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\lambda + 2\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(w_x - u_x)}{\partial t} = 0, \tag{4.4}$$

$$\mu\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\lambda + 2\mu)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(w_z - u_z)}{\partial t} = 0, \tag{4.5}$$

$$\frac{\partial P}{\partial x} + \frac{\gamma_w}{K_s}\frac{\partial p(w_x - u_x)}{\partial t} = 0, \tag{4.6}$$

$$\frac{\partial P}{\partial z} + \frac{\gamma_w}{K_s}\frac{\partial p(w_z - u_z)}{\partial t} = 0, \tag{4.7}$$

where K_s denotes the hydraulic conductivity, γ_w denotes the specific weight. λ and μ are Lamé's constant and are related to the elasticity modulus E and Poisson ratio v_p of the porous medium which is given by [1]

$$\lambda = \frac{v_p E}{(1 + v_p)(1 - 2v_p)}, \tag{4.8}$$

$$\mu = \frac{E}{2(1 + v_p)}. \tag{4.9}$$

4.1.3. Vorticity equation

Applying the curl on the momentum balance equations, we get a constitutive equation for the vorticity [6]. Therefore, the vorticity is defined to be the curl of the displacement field which is given by $\omega = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$. Via substituting Equations (4.6) and (4.7) into Equations (4.4) and (4.5) the Darcy's friction terms are replaced by the pressure gradients. Substituting ϵ_{vol} and ω, this gives the following two equations

$$\frac{\partial P}{\partial x} - \mu\frac{\partial \omega}{\partial z} - (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0, \tag{4.10}$$

$$\frac{\partial P}{\partial z} + \mu\frac{\partial \omega}{\partial x} - (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0. \tag{4.11}$$

Then taking first the curl of Equations (4.10) and (4.11), and second multiplying the resulting equation with -1, gives [6]

$$\mu\left[\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial x^2}\right] = 0. \tag{4.12}$$

Note that Equation (4.12) does only depend on the vorticity ω now and not on the pressure P and volumetric strain ϵ_{vol} anymore. Equation (4.12) forms the first constitutive equation.

4.1.4. Volumetric strain equation

Substituting ϵ_{vol} and ω and then taking the divergence of Equations (4.4) and (4.5), gives

$$-(\lambda + 2\mu)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) - \frac{\gamma_w}{K_s}\left(\frac{\partial}{\partial x}\left[p\frac{\partial (w_x - u_x)}{\partial t}\right] + \frac{\partial}{\partial z}\left[p\frac{\partial (w_z - u_z)}{\partial t}\right]\right) = 0. \tag{4.13}$$

Substituting Equation (4.3) into Equation (4.13), we get

$$\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda + 2\mu)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) = -\frac{\gamma_w}{K_s} p\beta\frac{\partial P}{\partial t}. \tag{4.14}$$

Note that in Equation (4.14) we account for the effects of damping. According to [6], if we would account for the acceleration terms Equation (4.14) would become a wave equation in the case of no damping, because the pore water is not part of this equation. Equation (4.14) forms the second constitutive equation.

4.1.5. Water pressure equation

We also need a constitutive equation describing the relation for the pore water pressures. This is done similar as for the volumetric strain. So now we take the divergence of the momentum balance equations for the pore water which are given by Equations (4.6) and (4.7). This gives a storage equation given by [6]

$$\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = -\frac{\gamma_w}{K_s} \frac{\partial \epsilon_{\text{vol}}}{\partial t}. \tag{4.15}$$

4.1.6. Displacement equations

Beside the relations described above, it is also needed to form some relations between the horizontal and vertical displacements and the vorticity and volumetric strain. These are given by [6]

$$-\frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} = -\frac{\partial \omega}{\partial z} - \frac{\partial \epsilon_{\text{vol}}}{\partial x}, \tag{4.16}$$

$$-\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} = \frac{\partial \omega}{\partial x} - \frac{\partial \epsilon_{\text{vol}}}{\partial z}. \tag{4.17}$$

Equations (4.16) and (4.17) represent the fourth and fifth constitutive equations, respectively.

4.1.7. Boundary conditions

In this new model only the pore water pressure and displacement must be defined at the boundary, while in Biot's model also the effective stress was defined. According to [6], when boundary conditions can be well-determined the use of geotechnical models to such a situation is limited by defining both the pore water pressure and the effective stress.

The boundary condition at $z = 0$ for ω and u_x depend on the boundary conditions for the normal stress and for P, ϵ_{vol} and u_z on the shear stress [6]. Recall that for Biot's model it is common to take $\sigma_{zz} = \sigma'_{zz} + P = -\lambda\epsilon_{\text{vol}} - 2\mu\frac{\partial u_z}{\partial z} + P$. However, according to [6], the assumption of $\sigma'_{zz} + P = F_{zz}$, where F_{zz} is a function depending on time, gives a pressure at the surface that is much higher than the pressure of the waves on the surface caused by water running over the porous medium. $\sigma'_{zz} = 0$ and $P = F_{zz}$ were assumed in Biot's model which means that the water pressure is carrying all the load. The assumption $\sigma'_{zz} = F_{zz}$ gives a discontinuity in the water pressure at $z = 0$ [6]. Assuming that $P = F_{zz}$, a solution is found where the pressure inside the pores is equal to the force of water flow on the surface, but the porous medium can experience a pulling force [6]. Since the two assumptions $\sigma'_{zz} + P = F_{zz}$ and $\sigma'_{zz} = F_{zz}$ give unlikely situations, we choose the assumption $P = F_{zz}$ like [6]. However, then Terzaghi's stress principle is not met in case of hydrodynamic loads, since we do not define the effective stresses at the surface: $\sigma_{zz} = F_{zz} \neq \sigma'_{zz} + F_{zz} = \sigma'_{zz} + P$. Note that Terzaghi's principle is valid in case of statics and linear loads [6]. Another condition at boundary $z = 0$ is that the vertical momentum balance equation must hold. This equation is given by $(\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} - \mu\frac{\partial \omega}{\partial x} - \frac{\partial P}{\partial z} = 0$ [6]. The third condition at $z = 0$ can be given by $\sigma_{xz} = F_{xz}$ [6], where F_{xz} is a function only depending on time. Recall that the formula of shear stress is given by $\sigma_{xz} = -2\mu\epsilon_{xz}$ and that $\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$. Substituting these two definitions into the third condition, we get $-\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = F_{xz}$. Rewriting this latter equation in terms of ω and u_x gives $\mu\omega - 2\mu\frac{\partial u_x}{\partial z} = F_{xz}$ which will be used as the third boundary condition from now on.

At $z = -n_z$ we define $\omega = u_z = \frac{\partial u_x}{\partial z} = \frac{\partial P}{\partial z} = \frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0$, since the influence of the waves on such a depth is assumed to be nil for these specific variables or their derivative with respect to z [5], [6].

Similarly, at $x = 0$ and $x = -n_x$ we have $\omega = u_x = \frac{\partial u_z}{\partial x} = \frac{\partial P}{\partial x} = \frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0$. These boundary conditions at $x = 0$ and $x = -n_x$ are based on the situation of a standing wave that increases and decreases the load on the soil in horizontal direction [6].

In conclusion, we have the following boundary conditions

$$\text{for } z = 0 : \begin{cases} \mu\omega - 2\mu\frac{\partial u_x}{\partial z} & = F_{xz} \\ P & = F_{zz} \\ (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} - \mu\frac{\partial \omega}{\partial x} - \frac{\partial P}{\partial z} & = 0 \end{cases}, \tag{4.18}$$

$$\text{for } z = -n_z : \left\{ \omega = u_z = \frac{\partial u_x}{\partial z} = \frac{\partial P}{\partial z} = \frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0 \right. , \tag{4.19}$$

$$\text{and for } x = 0 \text{ and } x = n_x : \left\{ \omega = u_x = \frac{\partial u_z}{\partial x} = \frac{\partial P}{\partial x} = \frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0 \right. , \tag{4.20}$$

where F_{xz} and F_{zz} are a functions only depending on time. Their value and their gradient equal zero.

4.1.8. Initial conditions

We assume that that there are no waves and everything is initially at rest. This means that we assume that no stresses act on the surface in the beginning, so there are no stresses and displacements at time $t = 0$ [6]. Since we have no displacement and stresses, the volumetric strain and pressure must be zero too. Then we have that [6]

$$u_x|_{t=0} = u_z|_{t=0} = \epsilon_{\text{vol}}|_{t=0} = \omega|_{t=0} = P|_{t=0} = 0. \tag{4.21}$$

4.2. Discretisation in space

In the following sections we will discretise Equations (4.28), (4.37), (4.42), (4.16) and (4.17). First we will discretise these equations with respect to space using the finite-element method (FEM). We do this per equation. When discretising with respect to space is finished, we discretise the resulting equations with respect to time using Backward-Euler since this method is unconditionally stable. In this discretisation we assume that $\Omega = (0, n_x) \times (-n_z, 0) \subseteq \mathbb{R}^2$ is the space domain and that $\mathbb{T} = (0, t_{\text{end}})$ is the time domain, with $n_x, n_z, t_{\text{end}} > 0$. The derivation of the Galerkin equations of the new model is pretty similar to the ones of Biot's model, since all governing equations are the same and most of the boundary conditions too. Therefore, we skip some computations of the Galerkin equations for this new model and refer to the computations of the Galerkin equations done for Biot's model.

We again assume that the two-dimensional domain and its boundaries and the unit vectors normal to these boundaries are given as in Figure 2.2. Recall that the normal unit vectors in two dimensions are given by

$$\eta_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{4.22}$$

where the first entry represents the x-direction and the second entry represents the z-direction.

Recall that n is the dimension of the space and N_i are the basis-functions for $i = 1, \dots, n$ that form a basis for the space. When having the symbol of the volumetric strain, pore water pressure or displacement in horizontal or vertical direction, n and N_i for $i = 1, \dots, n$ are defined for the space of that unknown variable. However, we assume that n is the same for all spaces.

4.2.1. Vorticity equation

We will derive the weak form of the vorticity equation and its matrix-vector form. Suppose that the test and trial functions are given by

$$v^{\omega}(x, z, t) = \sum_{i=1}^{n} N_i^{\omega}(x, z)\bar{v}_i^{\omega}(t), \tag{4.23}$$

$$\omega(x, z, t) = \sum_{j=1}^{n} N_j^{\omega}(x, z)\bar{\omega}_j(t). \tag{4.24}$$

Multiplying Equation (4.12) by test function v^{ω} and integrating over the domain Ω gives

$$\int_{\Omega} v^{\omega}\mu\left[\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial x^2}\right]\mathrm{d}\Omega = 0. \tag{4.25}$$

Since $\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial x^2} = \nabla\cdot(\nabla\omega)$ with ∇ the gradient operator, we can apply integration by parts on Equation (4.25) followed by the divergence theorem. Then Equation (4.25) becomes the weak equation

$$\int_{\mathrm{d}\Omega} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta})\,\mathrm{d}\Gamma - \int_{\Omega}\nabla v^{\omega}\cdot\mu\nabla\omega\mathrm{d}\Omega = 0, \tag{4.26}$$

where $\mathrm{d}\Omega$ contains the boundaries of domain Ω, and $\boldsymbol{\eta}$ is the unit normal vector pointing outward to the surface $\mathrm{d}\Omega$. Because of the boundary conditions given by Equations (4.18), (4.19) and (4.20), we set $v^{\omega}(x, z, t) = 0$ for $z = -n_z$, $x = 0$ and $x = n_x$. Then we get that

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta})\,\mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta}_1)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta}_2)\,\mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta}_3)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{\omega}\mu\,(\nabla\omega\cdot\boldsymbol{\eta}_4)\,\mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{\omega}\mu\frac{\partial\omega}{\partial z}\mathrm{d}\Gamma.
\end{aligned} \tag{4.27}$$

Substituting the test and trial functions given by Equations (4.23) and (4.24), respectively, into Equation (4.26) and using the computations as for Biot's model in Section 2.3.2 gives the following Galerkin equations in matrix-vector multiplication,

$$(B^{\omega} - SD^{\omega})\,\bar{\boldsymbol{w}} = 0, \tag{4.28}$$

where

$$B_{ij}^{\omega} = \int_{\Omega} \mu \left(\nabla N_i^{\omega} \cdot \nabla N_j^{\omega}\right) \mathrm{d}\Omega, \quad SD_{ij}^{\omega} = \int_{\mathrm{d}\Omega_3} \mu N_i^{\omega} \frac{\partial N_j^{\omega}}{\partial z} \mathrm{d}\Gamma, \quad \bar{\boldsymbol{w}} = \begin{bmatrix} \bar{\omega}_1 \\ \vdots \\ \bar{\omega}_n \end{bmatrix},$$

for $i, j = 1, \ldots, n$. Equation (4.28) is our first matrix problem to solve.

4.2.2. Volumetric strain equation

We will derive the weak form of the volumetric strain equation and its matrix-vector form. Suppose that the test and trial functions are given by

$$v^{\epsilon}(x,z,t) = \sum_{i=1}^{n} N_i^{\epsilon}(x,z)\bar{v}_i^{\epsilon}(t), \tag{4.29}$$

$$\epsilon_{\mathrm{vol}}(x,z,t) = \sum_{j=1}^{n} N_j^{\epsilon}(x,z)\bar{\epsilon}_j(t), \tag{4.30}$$

$$P(x,z,t) = \sum_{l=1}^{n} N_l^{P}(x,z)\bar{P}_l(t). \tag{4.31}$$

Note that ϵ_{vol} and P have a first derivative with respect to time which means that their test and trial functions have to depend on time. Multiplying Equation (4.14) by test function v^{ϵ} and integrating over the domain Ω gives

$$\int_{\Omega} v^{\epsilon} \left[\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\mathrm{vol}}}{\partial t} - (\lambda + 2\mu)\left(\frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial z^2}\right) + \frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t}\right] \mathrm{d}\Omega = 0. \tag{4.32}$$

Since $\frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial z^2} = \nabla \cdot (\nabla \epsilon_{\mathrm{vol}})$, we can apply integration by parts on the $\frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\mathrm{vol}}}{\partial z^2}$ part and then the divergence theorem. Then Equation (4.32) becomes

$$-\int_{\mathrm{d}\Omega} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta})\,\mathrm{d}\Gamma + \int_{\Omega} v^{\epsilon} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\mathrm{vol}}}{\partial t} + p\beta \frac{\partial P}{\partial t}\right] + (\lambda + 2\mu)\,(\nabla v^{\epsilon} \cdot \nabla \epsilon_{\mathrm{vol}})\,\mathrm{d}\Omega = 0. \tag{4.33}$$

Using the boundary conditions given by Equations (4.18), (4.19) and (4.20), we get that

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta})\,\mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta}_1)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta}_2)\,\mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta}_3)\,\mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{\epsilon}(\lambda + 2\mu)\,(\nabla \epsilon_{\mathrm{vol}} \cdot \boldsymbol{\eta}_4)\,\mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{\epsilon}\left(\mu \frac{\partial \omega}{\partial x} + \frac{\partial P}{\partial z}\right) \mathrm{d}\Gamma.
\end{aligned} \tag{4.34}$$

Note that the integrand of integral for the boundary $\mathrm{d}\Omega_3$ differs than the one for Biot's model, since this particular boundary conditions is different. In this case, we have that $-(\lambda + 2\mu)\epsilon_{\mathrm{vol}} + \mu \frac{\partial \omega}{\partial x} + \frac{\partial P}{\partial z} = 0$ instead of $\epsilon_{\mathrm{vol}} = 0$ at boundary $\mathrm{d}\Omega_3$. Because of this difference, we write out the computations of the Galerking equations.

Substituting the test and trial functions given by Equations (4.29), (4.30) and (4.31) into Equation (4.33) gives the following Galerkin equations

$$
\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} \sum_{i=1}^{n} N_i^{\epsilon} \bar{v}_i^{\epsilon} \left(\mu \sum_{j=1}^{n} \bar{\omega}_j \frac{\partial N_j^{\omega}}{\partial x} + \sum_{l=1}^{n} \bar{P}_l \frac{\partial N_l^{P}}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \int_{\Omega} \sum_{i=1}^{n} N_i^{\epsilon} \bar{v}_i^{\epsilon} \frac{\gamma_w}{K_s} \left[\frac{\partial}{\partial t} \left(\sum_{j=1}^{n} N_j^{\epsilon} \bar{\epsilon}_j \right) + p\beta \frac{\partial}{\partial t} \left(\sum_{l=1}^{n} N_l^{P} \bar{P}_l \right) \right] + (\lambda + 2\mu) \left[\nabla \left(\sum_{i=1}^{n} N_i^{\epsilon} \bar{v}_i^{\epsilon} \right) \cdot \nabla \left(\sum_{j=1}^{n} N_j^{\epsilon} \bar{\epsilon}_j \right) \right] \mathrm{d}\Omega \\
= &-\sum_{i=1}^{n} \bar{v}_i^{\epsilon} \int_{\mathrm{d}\Omega} N_i^{\epsilon} \left(\mu \sum_{j=1}^{n} \bar{\omega}_j \frac{\partial N_j^{\omega}}{\partial x} + \sum_{l=1}^{n} \bar{P}_l \frac{\partial N_l^{P}}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \sum_{i=1}^{n} \bar{v}_i^{\epsilon} \int_{\Omega} \frac{\gamma_w}{K_s} N_i^{\epsilon} \left[\sum_{j=1}^{n} N_j^{\epsilon} \frac{\partial \bar{\epsilon}_j}{\partial t} + p\beta \sum_{l=1}^{n} N_l^{P} \frac{\partial \bar{P}_l}{\partial t} \right] + (\lambda + 2\mu) \left[\nabla N_i^{\epsilon} \cdot \left(\sum_{j=1}^{n} \bar{\epsilon}_j \nabla N_j^{\epsilon} \right) \right] \mathrm{d}\Omega. \qquad (4.35)
\end{aligned}
$$

Since it must hold for arbitrary $\bar{v}_i^{\epsilon}$ with $i = 1, \dots, n$, we have that Equation (4.35) still holds as

$$
\begin{aligned}
0 = &-\int_{\mathrm{d}\Omega_3} N_i^{\epsilon} \left(\mu \sum_{j=1}^{n} \bar{\omega}_j \frac{\partial N_j^{\omega}}{\partial x} + \sum_{l=1}^{n} \bar{P}_l \frac{\partial N_l^{P}}{\partial z} \right) \mathrm{d}\Gamma \\
&+ \int_{\Omega} \frac{\gamma_w}{K_s} N_i^{\epsilon} \left[\sum_{j=1}^{n} N_j^{\epsilon} \frac{\partial \bar{\epsilon}_j}{\partial t} + p\beta \sum_{l=1}^{n} N_l^{P} \frac{\partial \bar{P}_l}{\partial t} \right] + (\lambda + 2\mu) \left[\nabla N_i^{\epsilon} \cdot \left(\sum_{j=1}^{n} \bar{\epsilon}_j \nabla N_j^{\epsilon} \right) \right] \mathrm{d}\Omega. \qquad (4.36)
\end{aligned}
$$

We can write Equation (4.36) as matrix-vector multiplication,

$$
A^{\epsilon\epsilon} \bar{\boldsymbol{\epsilon}}_t + A^{\epsilon P} \bar{\boldsymbol{P}}_t + B^{\epsilon} \bar{\boldsymbol{\epsilon}} - SC^{\epsilon} \bar{\boldsymbol{\omega}} - SD^{\epsilon} \bar{\boldsymbol{P}} = 0, \qquad (4.37)
$$

where

$$
A_{ij}^{\epsilon\epsilon} = \int_{\Omega} \frac{\gamma_w}{K_s} N_i^{\epsilon} N_j^{\epsilon} \mathrm{d}\Omega, \quad A_{ij}^{\epsilon P} = \int_{\Omega} \frac{\gamma_w}{K_s} p\beta N_i^{\epsilon} N_j^{P} \mathrm{d}\Omega, \quad B_{ij}^{\epsilon} = \int_{\Omega} (\lambda + 2\mu) \left[\nabla N_i^{\epsilon} \cdot \nabla N_j^{\epsilon} \right] \mathrm{d}\Omega,
$$

$$
SC_i^{\epsilon} = \int_{\mathrm{d}\Omega_3} \mu N_i^{\epsilon} \frac{\partial N_j^{\omega}}{\partial x} \mathrm{d}\Gamma, \quad SD_i^{\epsilon} = \int_{\mathrm{d}\Omega_3} N_i^{\epsilon} \frac{\partial N_j^{P}}{\partial z} \mathrm{d}\Gamma, \quad \bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \bar{\epsilon}_1 \\ \vdots \\ \bar{\epsilon}_n \end{bmatrix}, \quad \bar{\boldsymbol{\epsilon}}_t = \begin{bmatrix} \frac{\partial \bar{\epsilon}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{\epsilon}_n}{\partial t} \end{bmatrix}, \quad \bar{\boldsymbol{P}}_t = \begin{bmatrix} \frac{\partial \bar{P}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{P}_n}{\partial t} \end{bmatrix},
$$

for $i, j = 1, \dots, n$. Equation (4.37) is our second matrix problem to solve.

4.2.3. Pressure equation

We will derive the weak form of the pressure equation and its matrix-vector form. Suppose that the test function is given by

$$
v^P(x, z, t) = \sum_{i=1}^{n} N_i^P(x, z) \bar{v}_i^P(t). \qquad (4.38)
$$

Recall the following test and trial functions

$$
\epsilon_{\mathrm{vol}}(x, z, t) = \sum_{j=1}^{n} N_j^{\epsilon}(x, z) \bar{\epsilon}_j(t), \quad P(x, z, t) = \sum_{l=1}^{n} N_l^P(x, z) \bar{P}_l(t).
$$

Multiplying Equation (4.15) by test function v^{ϵ} and integrating over the domain Ω gives

$$
\int_{\Omega} v^P \left[\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s} \frac{\partial \epsilon_{\mathrm{vol}}}{\partial t} \right] \mathrm{d}\Omega = 0. \qquad (4.39)
$$

Since $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = \nabla \cdot (\nabla P)$, we can apply integration by parts on the $-\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2}$ part followed by the divergence theorem. Then Equation (4.39) becomes

$$-\int_{\mathrm{d}\Omega} v^P \left(\nabla P \cdot \boldsymbol{\eta}\right) \mathrm{d}\Gamma + \int_{\Omega} v^P \frac{\gamma_w}{K_s}\left[p\beta \frac{\partial P}{\partial t} + \frac{\partial \epsilon_{\text{vol}}}{\partial t}\right] + \left(\nabla v^P \cdot \nabla P\right) \mathrm{d}\Omega = 0. \tag{4.40}$$

Setting $v^P(x,z,t) = 0$ for $z = 0$ because of the boundary condition given by Equation (4.18) and using the other boundary conditions given by Equations (4.19) and (4.20), we get that

$$\begin{aligned}\int_{\mathrm{d}\Omega} v^P \left(\nabla P \cdot \boldsymbol{\eta}\right) \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^P \left(\nabla P \cdot \boldsymbol{\eta}_1\right) \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^P \left(\nabla P \cdot \boldsymbol{\eta}_2\right) \mathrm{d}\Gamma \\ &\quad + \int_{\mathrm{d}\Omega_3} v^P \left(\nabla P \cdot \boldsymbol{\eta}_3\right) \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^P \left(\nabla P \cdot \boldsymbol{\eta}_4\right) \mathrm{d}\Gamma \\ &= 0. \end{aligned} \tag{4.41}$$

Substituting the test and trial functions given by Equations (4.38), (4.30) and (4.31) into Equation (4.40) and using the computations as for Biot's model in Section 2.3.1 gives the following Galerkin equations in matrix-vector multiplication,

$$A^{PP}\bar{\boldsymbol{P}}_t + A^{P\epsilon}\bar{\boldsymbol{\epsilon}}_t + B^P\bar{\boldsymbol{P}} = 0, \tag{4.42}$$

where

$$A^{PP}_{ij} = \int_{\Omega} \frac{\gamma_w}{K_s} p\beta N_i^P N_j^P \mathrm{d}\Omega, \quad A^{P\epsilon}_{ij} = \int_{\Omega} \frac{\gamma_w}{K_s} N_i^P N_j^\epsilon \mathrm{d}\Omega, \quad B^P_{ij} = \int_{\Omega} \nabla N_i^P \cdot \nabla N_j^P \mathrm{d}\Omega,$$
$$\bar{\boldsymbol{P}} = \begin{bmatrix} \bar{P}_1 \\ \vdots \\ \bar{P}_n \end{bmatrix}, \quad \bar{\boldsymbol{P}}_t = \begin{bmatrix} \frac{\partial \bar{P}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{P}_n}{\partial t} \end{bmatrix}, \quad \bar{\boldsymbol{\epsilon}}_t = \begin{bmatrix} \frac{\partial \bar{\epsilon}_1}{\partial t} \\ \vdots \\ \frac{\partial \bar{\epsilon}_n}{\partial t} \end{bmatrix},$$

for $i,j = 1,\dots,n$. Equation (4.42) is our third matrix problem to solve.

4.2.4. Displacement equations

We will derive the weak form of the displacement equations and its matrix-vector form. Suppose that the test function is given by

$$v^{u_x}(x,z,t) = \sum_{i=1}^{n} N_i^{u_x}(x,z)\bar{v}_i^{u_x}(t), \tag{4.43}$$

$$v^{u_z}(x,z,t) = \sum_{i=1}^{n} N_i^{u_z}(x,z)\bar{v}_i^{u_z}(t), \tag{4.44}$$

$$u_x(x,z,t) = \sum_{j=1}^{n} N_j^{u_x}(x,z)\bar{u}_j^{x}(t), \tag{4.45}$$

$$u_z(x,z,t) = \sum_{j=1}^{n} N_j^{u_z}(x,z)\bar{u}_j^{z}(t). \tag{4.46}$$

Recall the following test and trial functions

$$\epsilon_{\text{vol}}(x,z,t) = \sum_{k=1}^{n} N_k^{\epsilon}(x,z)\bar{\epsilon}_k(t), \quad \omega(x,z,t) = \sum_{l=1}^{n} N_l^{\omega}(x,z)\bar{\omega}_l(t).$$

After multiplying Equations (4.16) and (4.17) by test functions v^{u_x} and v^{u_x}, respectively, and integrating over the domain Ω we get

$$0 = \int_\Omega v^{u_x} \left[-\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + \frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x} \right] \mathrm{d}\Omega, \tag{4.47}$$

$$0 = \int_\Omega v^{u_z} \left[-\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right) - \frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z} \right] \mathrm{d}\Omega. \tag{4.48}$$

Since $\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial z^2} = \nabla \cdot (\nabla u_i)$ for $i = x, z$, applying integration by parts and divergence theorem gives

$$0 = -\int_{\mathrm{d}\Omega} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta} \right] \mathrm{d}\Gamma + \int_\Omega v^{u_x} \left(\frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x} \right) + \left[\nabla v^{u_x} \cdot \nabla u_x \right] \mathrm{d}\Omega, \tag{4.49}$$

$$0 = -\int_{\mathrm{d}\Omega} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta} \right] \mathrm{d}\Gamma + \int_\Omega v^{u_z} \left(-\frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z} \right) + \left[\nabla v^{u_z} \cdot \nabla u_z \right] \mathrm{d}\Omega. \tag{4.50}$$

Using the boundary conditions given by Equation (4.18), (4.19) and (4.20), we get that

$$\begin{aligned}
\int_{\mathrm{d}\Omega} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta} \right] \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta}_1 \right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta}_2 \right] \mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta}_3 \right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{u_x} \left[\nabla u_x \cdot \boldsymbol{\eta}_4 \right] \mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{u_x} \frac{1}{2} \left(\omega - \frac{1}{\mu} F_{xz} \right) \mathrm{d}\Gamma, \qquad (4.51) \\
\int_{\mathrm{d}\Omega} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta} \right] \mathrm{d}\Gamma &= \int_{\mathrm{d}\Omega_1} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta}_1 \right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_2} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta}_2 \right] \mathrm{d}\Gamma \\
&\quad + \int_{\mathrm{d}\Omega_3} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta}_3 \right] \mathrm{d}\Gamma + \int_{\mathrm{d}\Omega_4} v^{u_z} \left[\nabla u_z \cdot \boldsymbol{\eta}_4 \right] \mathrm{d}\Gamma \\
&= \int_{\mathrm{d}\Omega_3} v^{u_z} \frac{\partial u_z}{\partial z} \mathrm{d}\Gamma. \qquad (4.52)
\end{aligned}$$

Substituting the test and trial functions given by Equations (4.43), (4.44), (4.45), (4.46), (4.30) and (4.24) into Equations (4.49) and (4.50) and using the computations as for Biot's model in Section 2.3.4 gives the following Galerkin equations in matrix-vector multiplication,

$$\begin{cases} B^{u_x} \bar{\mathbf{u}}^x + (D^{u_x} - SA^{u_x}) \bar{\boldsymbol{\omega}} + C^{u_x} \bar{\boldsymbol{\epsilon}} & = -\mathbf{F}^{xz} \\ (B^{u_z} - SD^{u_z}) \bar{\mathbf{u}}^z - C^{u_z} \bar{\boldsymbol{\omega}} + D^{u_z} \bar{\boldsymbol{\epsilon}} & = 0 \end{cases}, \tag{4.53}$$

where

$$SA_{ij}^{u_x} = \int_{\mathrm{d}\Omega_3} \frac{1}{2} N_i^{u_x} N_j^{\omega} \mathrm{d}\Gamma, \quad B_{ij}^{u_x} = \int_\Omega \nabla N_i^{u_x} \cdot \nabla N_j^{u_x} \mathrm{d}\Omega, \quad B_{ij}^{u_z} = \int_\Omega \nabla N_i^{u_z} \cdot \nabla N_j^{u_z} \mathrm{d}\Omega, \quad C_i^{u_x} = \int_{\mathrm{d}\Omega} N_i^{u_x} \frac{\partial N_j^{\epsilon}}{\partial x} \mathrm{d}\Omega,$$

$$C_i^{u_z} = \int_{\mathrm{d}\Omega} N_i^{u_z} \frac{\partial N_j^{\omega}}{\partial x} \mathrm{d}\Omega, \quad D_{ij}^{u_x} = \int_{\mathrm{d}\Omega} N_i^{u_x} \frac{\partial N_j^{\omega}}{\partial z} \mathrm{d}\Omega, \quad D_{ij}^{u_z} = \int_{\mathrm{d}\Omega} N_i^{u_z} \frac{\partial N_j^{\epsilon}}{\partial z} \mathrm{d}\Omega, \quad SD_{ij}^{u_z} = \int_{\mathrm{d}\Omega_3} N_i^{u_z} \frac{\partial N_j^{u_z}}{\partial z} \mathrm{d}\Gamma,$$

$$F_i^{xz} = \int_{\mathrm{d}\Omega_3} N_i^{u_x} \frac{1}{2\mu} F_{xz} \mathrm{d}\Gamma,$$

for $i, j = 1, \ldots, n$. Equation (4.53) is our fourth matrix problem to solve.

4.2.5. Final FEM Model

We assume that $N_i^{\epsilon} = N_i^{\omega} = N_i^{P} = N_i^{u_x} = N_i^{u_z} =: N_i$ for all $i = 1, \ldots, n$. After collecting the governing equations given by Equations (4.42), (4.28), (4.37) and (4.53), we get the following set of Galerkin equations:

$$\begin{cases} \mu\,(B-SD)\,\bar{\boldsymbol{w}} & = 0 \\ \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + (\lambda+2\mu)\,B\bar{\boldsymbol{\epsilon}} - \mu SC\bar{\boldsymbol{\omega}} - SD\bar{\boldsymbol{P}} & = 0 \\ \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + B\bar{\boldsymbol{P}} & = 0 \\ B\bar{\boldsymbol{u}}^x + (D - \frac{1}{2}SA)\bar{\boldsymbol{\omega}} + C\bar{\boldsymbol{\epsilon}} & = -\boldsymbol{F}^{xz} \\ (B-SD)\,\bar{\boldsymbol{u}}^z - C\bar{\boldsymbol{\omega}} + D\bar{\boldsymbol{\epsilon}} & = 0 \end{cases}, \tag{4.54}$$

where $A_{ij} = \int_\Omega N_i N_j \mathrm{d}\Omega$, $SA_{ij} = \int_{\mathrm{d}\Omega_3} N_i N_j \mathrm{d}\Gamma$, $B_{ij} = \int_\Omega \nabla N_i \cdot \nabla N_j \mathrm{d}\Omega$, $C_{ij} = \int_\Omega N_i \frac{\partial N_j}{\partial x}\mathrm{d}\Omega$, $SC_{ij} = \int_{\mathrm{d}\Omega_3} N_i \frac{\partial N_j}{\partial x}\mathrm{d}\Gamma$, $D_{ij} = \int_\Omega N_i \frac{\partial N_j}{\partial z}\mathrm{d}\Omega$, $SD_{ij} = \int_{\mathrm{d}\Omega_3} N_i \frac{\partial N_j}{\partial z}\mathrm{d}\Gamma$ for all $i, j = 1, \dots, n$.
We can write Equation (4.54) as one system of matrix-vector multiplication

$$M^t\boldsymbol{\theta}_t + M\boldsymbol{\theta} = \boldsymbol{f}, \tag{4.55}$$

where

$$M^t = \begin{bmatrix} \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A & \emptyset & \emptyset \\ \emptyset & \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \end{bmatrix} \in \mathbb{R}^{5n\times 5n}, \quad M = \begin{bmatrix} \mu\,(B-SD) & \emptyset & \emptyset & \emptyset & \emptyset \\ -\mu SC & (\lambda+2\mu)\,B & -SD & \emptyset & \emptyset \\ \emptyset & \emptyset & B & \emptyset & \emptyset \\ D - \frac{1}{2}SA & C & \emptyset & B & \emptyset \\ -C & D & \emptyset & \emptyset & (B-SD) \end{bmatrix} \in \mathbb{R}^{5n\times 5n},$$

$$\boldsymbol{\theta} = \begin{bmatrix} \bar{\boldsymbol{w}} \\ \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{u}}^x \\ \bar{\boldsymbol{u}}^z \end{bmatrix} \in \mathbb{R}^{5n}, \boldsymbol{\theta}_t = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{w}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{P}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{u}}^x}{\partial t} \\ \frac{\partial \bar{\boldsymbol{u}}^z}{\partial t} \end{bmatrix} \in \mathbb{R}^{5n}, \quad \boldsymbol{f} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{F}^{xz} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{5n}. \tag{4.56}$$

Note that the boundary conditions are included.

4.3. Discretisation in time

For discretising the final FEM model given by Equation (4.55) we use the Backward-Euler method again. Recall that after applying the Backward-Euler method given by Equation (2.72) to Equation (4.55), we get that

$$\boldsymbol{\theta}^{k+1} = \left(M^t + \Delta t M\right)^{-1}\left(M^t\boldsymbol{\theta}^k + \Delta t\boldsymbol{f}^{k+1}\right), \tag{4.57}$$

when assuming $\left(M^t + \Delta t M\right)$ is invertible.

For simplicity, we will look at the Van Damme and den Ouden-van der Horst Model in one dimension which will be described in the next section. The two-dimensional results will not be determined in this literature report. However, this is one of the main goals for our further research.

5

Van Damme and den Ouden-van der Horst Model (1D)

For our first analysis we reduce the two-dimensional case to one-dimensional. This makes the numerical analysis more simple, however, it gives a good understanding of the behaviour in z-direction. The model in one-dimension can be determined by setting the shear stress to zero and by letting the normal stress due to the wave be a function which only depends on time. In this case, there will be no alterations in x-direction. This means that all displacements in x-direction, (u^x), and all derivatives with respect to x, $\left(\frac{\partial\cdot}{\partial x}\right)$, are equal to zero. Note that in this model the entire domain is of one type of soil and is a simplification of the two-dimensional model. Therefore, the main goal in this section is to represent the relations between the volumetric strain, water pressure and z-displacement. At last, recall that in one-dimension we have $z \in \Omega := [-n_z, 0]$.

5.1. One type of soil

We will first give a solution when assuming one layer of fine sand. In this case the porosity, Lamé constants, and hydraulic conductive are constant and their derivatives with respect to z equal zero.

5.1.1. Model

This gives the following constitutive equations in one-dimension,

$$\begin{cases} \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda + 2\mu)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} & = -\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} \\ \frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial z^2} & = -\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} \\ \frac{\partial u_z}{\partial z} & = \epsilon_{\text{vol}} \end{cases}, \text{ for } z \in \Omega \text{ and } t \in \mathbb{T} \tag{5.1}$$

with boundary conditions

$$\begin{cases} P(0,t) & = F_{zz}(t) \\ (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}(0,t)}{\partial z} & = \frac{\partial P(0,t)}{\partial z} \end{cases}, \quad \begin{cases} u_z(-n_z, t) & = 0 \\ \frac{\partial P(-n_z,t)}{\partial z} & = 0 \\ \frac{\partial \epsilon_{\text{vol}}(-n_z,t)}{\partial z} & = 0 \end{cases}, \text{ for } t \in \mathbb{T} \tag{5.2}$$

and initial conditions

$$\epsilon_{\text{vol}}(z,0) = P(z,0) = u_z(z,0) = 0, \text{ for } z \in \Omega. \tag{5.3}$$

Using boundary conditions, the weak equations in one-dimension are given by

$$\begin{cases} -N_i^{\epsilon}(0)\frac{\partial P(0,t)}{\partial z} + \int_{-n_z}^{0} N_i^{\epsilon}\frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + (\lambda + 2\mu)\left(\frac{\partial v^{\epsilon}}{\partial z}\cdot\frac{\partial \epsilon_{\text{vol}}}{\partial z}\right)\text{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{P}\frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + \left(\frac{\partial v^{P}}{\partial z}\cdot\frac{\partial P}{\partial z}\right)\text{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{u}\frac{\partial u_z}{\partial z}\text{d}z & = \int_{-n_z}^{0} N_i^{u}\epsilon_{\text{vol}}\text{d}z \end{cases}, \tag{5.4}$$

Substituting one-dimensional trial functions into Equation (5.4) we get the following one-dimensional Galerkin equations in matrix-vector form

$$\begin{cases} \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + (\lambda + 2\mu)\, B\bar{\boldsymbol{\epsilon}} - C^0\bar{\boldsymbol{P}} & = \mathbf{0} \\ \frac{\gamma_w}{K_s} A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s} p\beta A\bar{\boldsymbol{P}}_t + B\bar{\boldsymbol{P}} & = \mathbf{0} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{5.5}$$

where $A_{i,j} = \int_{-n_z}^{0} N_i N_j \mathrm{d}z, B_{i,j} = \int_{-n_z}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{d}z, C_{i,j} = \int_{-n_z}^{0} N_i \frac{\partial N_j}{\partial z}\mathrm{d}z.$ and $C_{i,j}^0 = N_i(0)\frac{\partial N_j(0)}{\partial z}$.

We can write Equation (5.5) as two systems of matrix-vector multiplication

$$\begin{cases} M^t\boldsymbol{S}_t + M\boldsymbol{S} & = \boldsymbol{f} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{5.6}$$

where

$$M^t = \begin{bmatrix} \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \\ \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \end{bmatrix}, \quad M = \begin{bmatrix} (\lambda+2\mu)\,B & -C^0 \\ \emptyset & B \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \end{bmatrix}, \quad \boldsymbol{S}_t = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{P}}}{\partial t} \end{bmatrix}, \quad \boldsymbol{f}(t) = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{F}(t) \end{bmatrix}. \tag{5.7}$$

The Dirichlet boundary conditions are included in the numerical model by setting the corresponding rows of matrices M^t and M to zero and then putting pivots in these same rows of M. Furthermore, we set $F_i = 0$ for $i = 1, \dots, n-1$ and $F_n = F_{zz}$. Then we get that $\bar{P}_n = F_n = F_{zz}$ and $\bar{u}_0^z = 0$. After applying the time integration given by Equation (4.57) using $M^t, M, \boldsymbol{S}$ and $\boldsymbol{f}$ as described above and assuming that matrix $(M^t + \Delta t M)$ and C are invertible, we get that

$$\begin{cases} \boldsymbol{S}^{k+1} & = \left(M^t + \Delta t M\right)^{-1}\left(M^t\boldsymbol{S}^k + \Delta t \boldsymbol{f}^{k+1}\right) \\ \bar{\boldsymbol{u}}_{\boldsymbol{z}}^{k+1} & = C^{-1}A\bar{\boldsymbol{\epsilon}}^{k+1} \end{cases}. \tag{5.8}$$

Finally, we choose again that

$$F_{zz}(t) = \begin{cases} (1-\beta)\,(\lambda+2\mu)\,(1-\cos(t)), & \text{if } t < \pi \\ 2(1-\beta)\,(\lambda+2\mu), & \text{if } t \geq \pi \end{cases}. \tag{5.9}$$

Recall that F_{zz} represents the normal stress and only depends on time and is chosen to be positive for all $t \in \mathbb{T}$. Furthermore, F_{zz} converges to a constant over time.

Therefore, the solution of the one-dimensional model must go to the stationary solution over time. In the next section we will solve the stationary one-dimensional system in order to find the stationary solution.

5.1.2. Stationary model

The stationary model in one dimension is given by

$$\begin{cases} \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} = 0 \\ \frac{\partial^2 P}{\partial z^2} = 0 \\ \frac{\partial u_z}{\partial z} = \epsilon_{\text{vol}} \end{cases}, \tag{5.10}$$

with boundary conditions given by Equation (5.2) using $F_{zz} = F$ constant ($t \to \infty$). The set of equations given by Equation (5.10) gives the following solutions

$$\begin{cases} u_z = \int \epsilon_{\text{vol}} \mathrm{d}z + c_1 = \frac{1}{2}c_2 z^2 + c_3 z + c_1 \\ \epsilon_{\text{vol}} = c_2 z + c_3 \\ P = c_4 z + c_5 \end{cases}. \tag{5.11}$$

Using the boundary conditions, we find that $c_2 = c_4 = 0$, $c_1 = c_3 n_z$ and $c_5 = F$, i.e.

$$\begin{cases} u_z = c_3 z + c_3 n_z \\ \epsilon_{\text{vol}} = c_3 \\ P = F \end{cases} . \tag{5.12}$$

Then we find that also the stationary solution for ϵ_{vol} and u_z has infinitely many stationary solutions. Therefore, its likely that we also have infinitely many solutions of the one-dimensional model [5].

5.1.3. Infinitely many solutions

When adding the first and second subequations of Equation (5.1), we find

$$\begin{aligned} (\lambda + 2\mu)\,\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} &= \frac{\partial^2 P}{\partial z^2} \\ &\Leftrightarrow \\ (\lambda + 2\mu)\,\frac{\partial \epsilon_{\text{vol}}}{\partial z} &= \frac{\partial P}{\partial z} + d_1(t) \\ &\Leftrightarrow \\ (\lambda + 2\mu)\,\epsilon_{\text{vol}} &= P + d_1 z + d_2, \end{aligned}$$

where $d_1(t)$ and $d_2(t)$ are constant in space. Since we have boundary condition $\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z} = 0$ at $z = 0$, we get that $d_1(t) = 0$. This implies that

$$(\lambda + 2\mu)\,\epsilon_{\text{vol}} = P + d_2, \quad \text{on } \bar{\Omega} := \Omega \cup \mathrm{d}\Omega = [-n_z, 0]. \tag{5.13}$$

Note that d_2 stays undefined using the set of boundary conditions given by Equation (5.2). Therefore, the values of ϵ_{vol} and P are expected to have the same behaviour but their values can be of different signs.

5.1.4. Results

The three variables ϵ_{vol}, P, u_z and their derivatives are plotted five times, namely at $t = 0.0, 1.5, 3.0$ and $4.5, 6.0$, for the case of compressible water and of incompressible water. For the assumption of compressible water we set $\beta = 4.8 \cdot 10^{-10}$ and for the assumption of incompressible water we have by definition $\beta = 0.0$. The effective stress $\sigma'_{zz} = -(\lambda + 2\mu)\,\epsilon_{\text{vol}}$ is also plotted in order to check the boundary condition $(\lambda + 2\mu)\,\frac{\partial \epsilon_{\text{vol}}(0,t)}{\partial z} = \frac{\partial P(0,t)}{\partial z}$. We use piece-wise linear basis-functions N_i, which is of degree 1 and smoothness 0 and are called hat functions. These are given by

$$N_i = \begin{cases} \frac{z - z_{i-1}}{z_i - z_{i-1}}, & \text{if } z \in [z_{i-1}, z_i] \\ \frac{z_{i+1} - z}{z_{i+1} - z_i}, & \text{if } z \in [z_i, z_{i+1}] \end{cases}, \tag{5.14}$$

for $i = 1, \ldots, n$. Furthermore, for integration of a subdomain we use 1000 integration points, the time step is chosen as $\Delta t = 0.0025$ and the number of subdomains is chosen as $\Delta z = 0.0025$. The porosity, Poisson ratio and the effective size of the grain, d_{10} [m], is given by Table 3.1 and the shear modulus μ and specific weight γ_w are given by Table 3.2. The hydraulic conductivity is $K_s = c \cdot d_{10}$ [m/s] according to Allen Hazen [9]. We will use $c = 1.0$. We recall that λ is given by Equation (4.8). At last, we set $n_z = 2$, i.e. $\bar{\Omega} = [-2, 0]$.

Then we find that ϵ and P behave the same which can be seen in upper right and lower left subplots of Figures 5.1 and 5.3. In upper right and lower left subplots of Figures 5.2 and 5.4, one can see that the derivative of volumetric strain and water pressure with respect to z are equal, i.e. $(\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z}$ holds indeed. We also find that $\sigma_{zz} + P$ is constant in the z-direction, which can be noticed in the upper left and lower left subplots of Figures 5.1 and 5.3. However, note that this constant may differ in time like in these figures. Furthermore, in Figure 5.3 we find for increasing t the water pressure solution becomes more of a constant line, namely $P(z,t) = F_{zz}(t) = 2(1-\beta)(\lambda + 2\mu)$ as $t \to \infty$. This is because F_{zz} is defined to be a constant for $t \geq \pi$. In Figure 5.1 we find that this convergence is not shown (yet) for the assumption of compressible water. However, when further increasing time t, a very slow convergence is shown.

At last, we noticed when varying the space step by changing the number of subdomains and/or quadrature points, the behaviour and/or values of the variables changed. This is also the case when changing the time step. One reason could be that $d_2(t)$ is not defined and thus can be any constant in space depending on time and another is simply rounding errors in calculations.

Since there are infinitely many solutions, this behaviour was expected. Nevertheless, this model represents the relation between the variables ϵ_{vol}, P and u_z (and σ'_{zz}) well which was the main goal in this section.

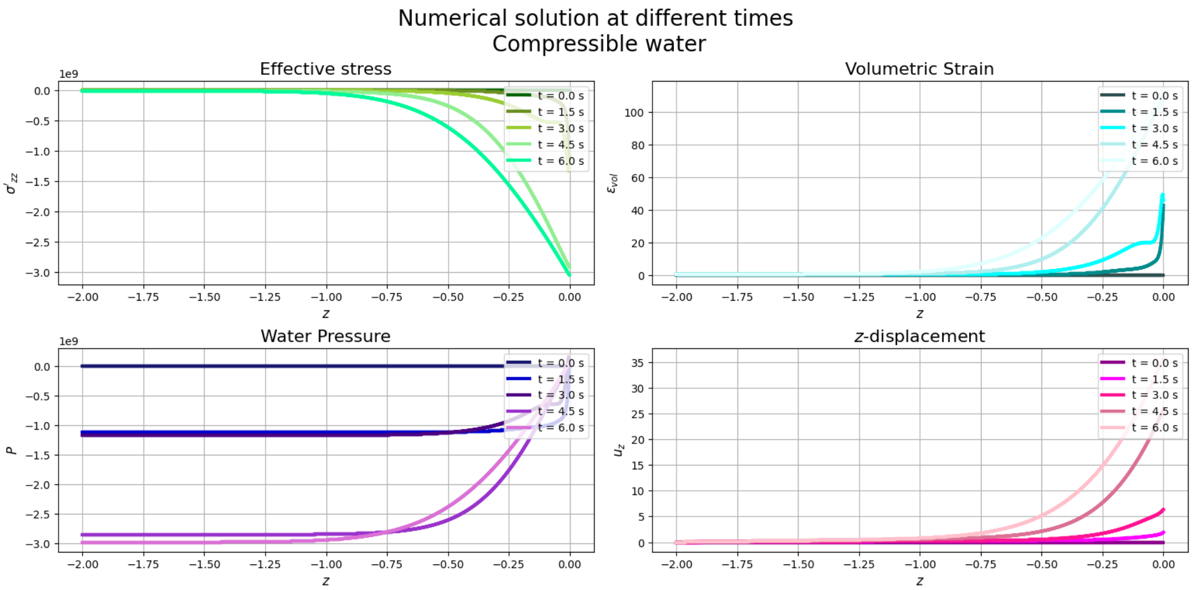

Figure 5.1: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

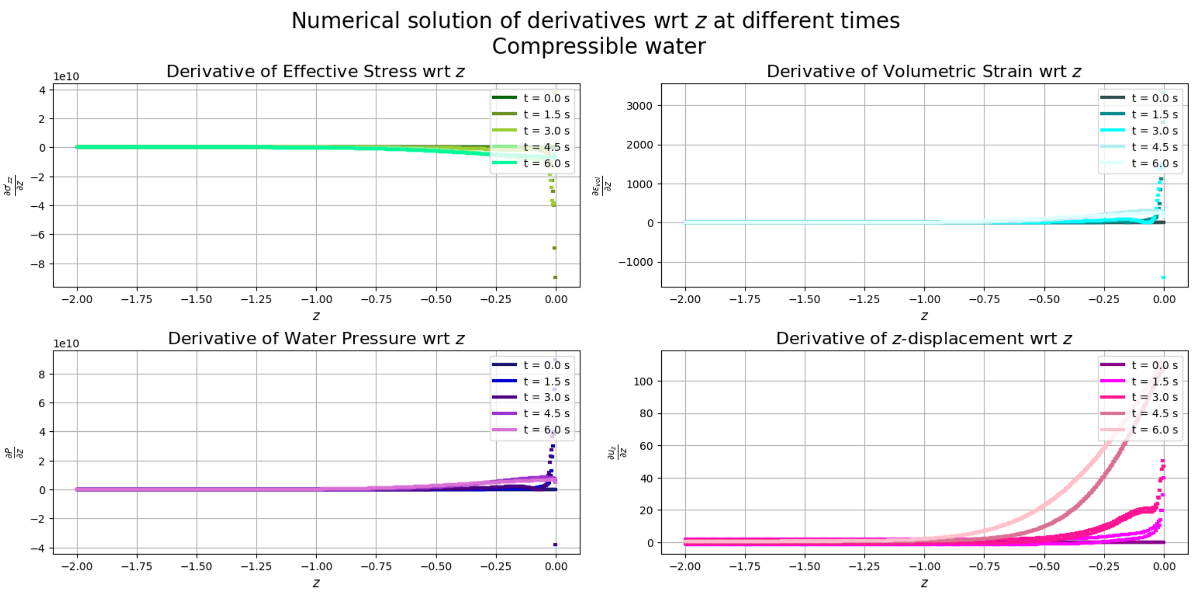

Figure 5.2: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

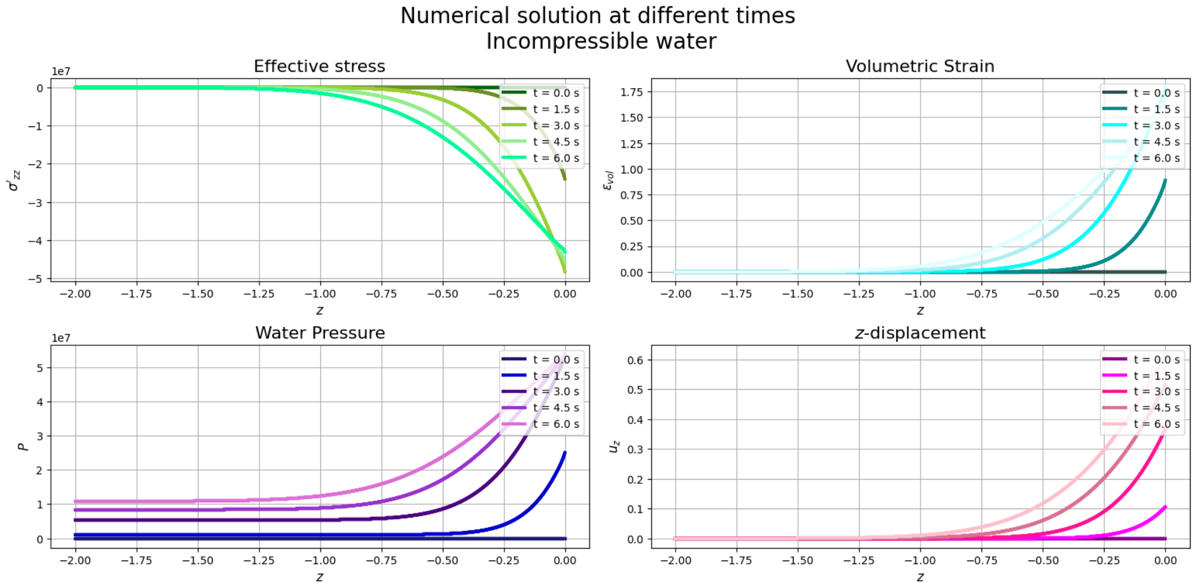

Figure 5.3: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be incompressible ($\beta = 0$).

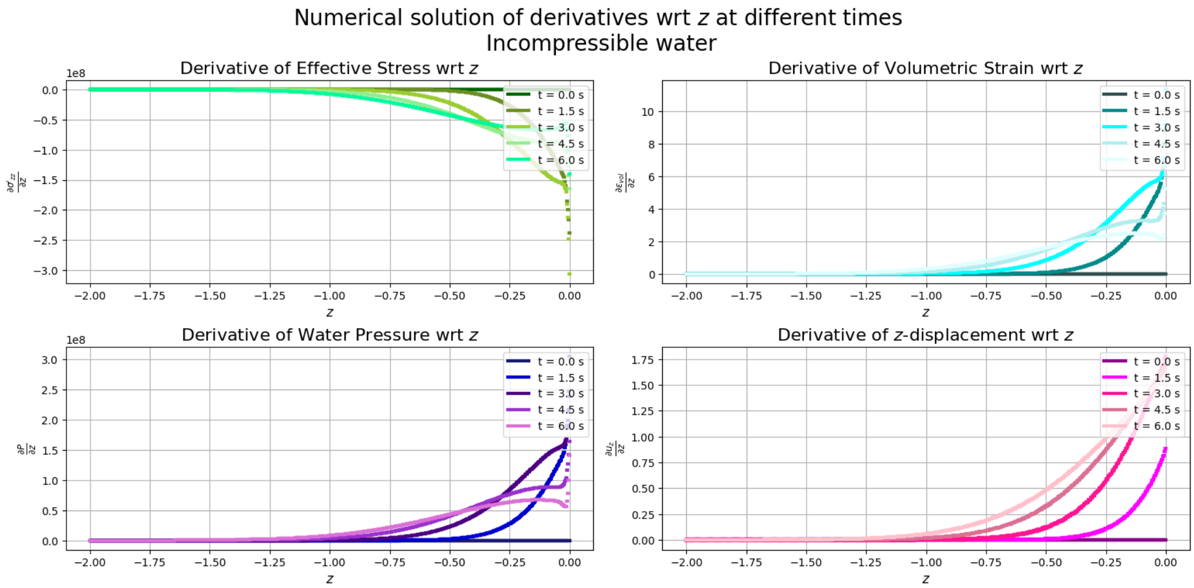

Figure 5.4: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be incompressible ($\beta = 0$).

5.2. One type of soil with alternative boundary condition

To be able to compare results with different parameters correctly, it is necessarily to have a unique solution. To find a unique solution, we explicitly set $d_2(t)$ equal to some constant in z-direction, say zero. In this case, we can define $(\lambda + 2\mu)\epsilon_{\text{vol}} = P$ at the surface instead of $(\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z}$. This more strict boundary condition comes naturally in two dimensions, since in two dimensions it follows from the momentum balance equations when deriving the analytical solutions [6]. However, for one dimension this can not be shown.

5.2.1. Physical model and numerical model

Then the vertical momentum balance equation still holds at the surface, but the variation in value is limited to one value. In other words, we now have the following boundary conditions,

$$\begin{cases} P(0,t) & = F_{zz}(t) \\ (\lambda + 2\mu)\,\epsilon_{\text{vol}}(0,t) & = P(0,t) \end{cases}, \quad \begin{cases} u_z(-n_z,t) & = 0 \\ \frac{\partial P(-n_z,t)}{\partial z} & = 0 \\ \frac{\partial \epsilon_{\text{vol}}(-n_z,t)}{\partial z} & = 0 \end{cases}, \text{ for } t \in \mathbb{T} \tag{5.15}$$

and initial conditions given by Equation (5.3).

Using these alternative boundary conditions, the weak equations in one-dimension are given by

$$\begin{cases} \int_{-n_z}^{0} N_i^{\epsilon} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + (\lambda + 2\mu)\left(\frac{\partial v^{\epsilon}}{\partial z} \cdot \frac{\partial \epsilon_{\text{vol}}}{\partial z}\right)\text{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{P} \frac{\gamma_w}{K_s}\left[\frac{\partial \epsilon_{\text{vol}}}{\partial t} + p\beta\frac{\partial P}{\partial t}\right] + \left(\frac{\partial v^{P}}{\partial z} \cdot \frac{\partial P}{\partial z}\right)\text{d}z & = 0 \\ \int_{-n_z}^{0} N_i^{u} \frac{\partial u_z}{\partial z}\text{d}z & = \int_{-n_z}^{0} N_i^{u}\epsilon_{\text{vol}}\text{d}z \end{cases}. \tag{5.16}$$

Substituting one-dimensional trial functions into Equation (5.16) we get the following one-dimensional Galerkin equations in matrix-vector form

$$\begin{cases} \frac{\gamma_w}{K_s}A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s}p\beta A\bar{\boldsymbol{P}}_t + (\lambda + 2\mu)\,B\bar{\boldsymbol{\epsilon}} & = \boldsymbol{0} \\ \frac{\gamma_w}{K_s}A\bar{\boldsymbol{\epsilon}}_t + \frac{\gamma_w}{K_s}p\beta A\bar{\boldsymbol{P}}_t + B\bar{\boldsymbol{P}} & = \boldsymbol{0} \\ C\bar{\boldsymbol{u}}^z = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{5.17}$$

where $A_{i,j} = \int_{-n_z}^{0} N_i N_j \text{d}z$, $B_{i,j} = \int_{-n_z}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\text{d}z$ and $C_{i,j} = \int_{-n_z}^{0} N_i \frac{\partial N_j}{\partial z}\text{d}z$. We can write Equation (5.17) as two systems of matrix-vector multiplication as in Equation (5.7) where

$$M^t = \begin{bmatrix} \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \\ \frac{\gamma_w}{K_s}A & \frac{\gamma_w}{K_s}p\beta A \end{bmatrix}, \quad M = \begin{bmatrix} (\lambda + 2\mu)\,B & \emptyset \\ \emptyset & B \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \end{bmatrix}, \quad \boldsymbol{S}_t = \begin{bmatrix} \frac{\partial \bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial \bar{\boldsymbol{P}}}{\partial t} \end{bmatrix}, \quad \boldsymbol{f}(t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}(t) \end{bmatrix}. \tag{5.18}$$

The Dirichlet boundary conditions are included in the numerical model by setting the corresponding rows of matrices M^t and M to zero and then putting pivots in these same rows of M. Furthermore, we set $F_i = 0$ for $i = 1, \dots, n-1$ and $F_n = F_{zz}$. Then we get that $\bar{P}_n = F_n = F_{zz}, \bar{\epsilon}_n = \frac{1}{\lambda+2\mu}\bar{P}_n = \frac{F_{zz}}{\lambda+2\mu}$ and $\bar{u}_0^z = 0$.

Now we apply the Backward-Euler method given by Equation (4.57) using $M^t, M, \boldsymbol{S}$ and $\boldsymbol{f}$ as described in Equation (5.18), where we assume that matrices $(M^t + \Delta t M)$ and C are invertible. Then we get again the time integration formulas given by Equation (5.8) with F_{zz} chosen as in Equation (5.9), but with M_t, M given by Equation (5.18). To check whether we have convergence to a unique stationary solution, we take a look again at the stationary model but now with the alternative boundary conditions given by Equation (5.15) in the next two sections.

5.2.2. Stationary model

The stationary model is given by Equation (5.10) but instead of the boundary conditions given by (5.2) its comes along with the boundary conditions given by Equation (5.15). Then we find that the coefficients

C_i for $i = 1, \dots, 5$ of Equation (5.11) are given by $c_2 = c_4 = 0$, $c_3 = \frac{1}{\lambda+2\mu} c_5$ and $c_5 = F$, i.e.

$$\begin{cases} u_z = \frac{F}{\lambda+2\mu} + \frac{F}{\lambda+2\mu} n_z \\ \epsilon_{\text{vol}} = \frac{F}{\lambda+2\mu} \\ P = F \end{cases} . \tag{5.19}$$

Note that we now have only one stationary solution to the one-dimensional stationary model.

5.2.3. Unique solution

Assuming $d_2(t)$ to be constant in Equation (5.13), here chosen $d_2(t) = 0$, we get a unique solution to the one-dimensional model. We expect that by explicitly setting $d_2(t)$ to a constant in space, the variables ϵ_{vol} and P (and σ'_{zz}) will not change drastically anymore when for example varying the step in space and when assuming that the water is compressible (with $\beta = 4.8 \cdot 10^{-10}$) or incompressible ($\beta = 0.0$). Therefore, we now have a model that has a unique solution and tends to a unique solution over time.

5.2.4. Results

The three variables ϵ_{vol}, P, u_z and their derivatives are plotted five times, namely at $t = 0.0, 1.5, 3.0$ and $4.5, 6.0$, for the case of compressible water and of incompressible water. For the assumption of compressible water we take $\beta = 4.8 \cdot 10^{-10}$ and for the assumption of incompressible water we have by definition $\beta = 0.0$. The effective stress $\sigma'_{zz} = -(\lambda + 2\mu)\,\epsilon_{\text{vol}}$ is also plotted in order to check the boundary condition $(\lambda + 2\mu)\,\epsilon_{\text{vol}}(0, t) = P(0, t)$. We use the piece-wise linear basis-functions N_i for $i = 1, \dots, n$, given by Equation (5.14). Furthermore, we use again 1000 integration points, a time step of $\Delta t = 0.0025$ and the number of subdomains is chosen as $\Delta z = 0.0025$. The parameters porosity, Poisson ratio and effective size of the grains are given by Table 3.1 and the shear modulus and specific weight are given by Table 3.2. Recall that we use for the hydraulic conductivity the formula by Allen Hazen [9] again with choosing $c = 1.0$ and that λ is given by Equation (4.8). We set again $n_z = 2$, i.e. $\bar{\Omega} = [-2, 0]$.

Then we find that variables σ'_{zz} and P behave the same but mirrored and when we add both solutions we get the constant zero which was imposed as condition on the upper boundary. This can be seen in upper left and lower left subplots of Figures 5.5 and 5.7. The derivatives of ϵ_{vol} and P with respect to z are the same like they were with the original boundary condition $(\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z}$ at $z = 0$. We expect this to happen, since the equation $(\lambda + 2\mu)\epsilon_{\text{vol}} = P$ on $\bar{\Omega} = [-n_z, 0]$ derived in Equation (5.13) with $d_2(t) = 0$, guarantees that the derivative of this formula with respect to z also holds on $\bar{\Omega}$. This can be seen in upper left and lower left subplots of Figures 5.5 and 5.7.

Furthermore, in Figure 5.7 we find the convergence to the stationary solution is not recognizable yet. However, when increasing time t even more the solutions indicate that unique solutions reach the unique stationary solutions.

At last, for $d_2(t)$ not defined we noticed when varying some parameters the relations between the variables ϵ_{vol}, P and u_z kept the same but their behaviour and values not necessarily, since there were infinitely many solutions. Using the alternative boundary conditions, we set $d_2(t) = 0$ and leads to a unique solution. This way not only the relations between the variables stay the same but, in general, also their behaviour and values as well. Their are some differences in value but this is minimal with respect to the order of the values and are probably due to computation errors.

In conclusion, this model represents the relation between the variables ϵ_{vol}, P and u_z (and σ'_{zz}) well and also gives a unique solution that tends to a unique stationary solution over time.

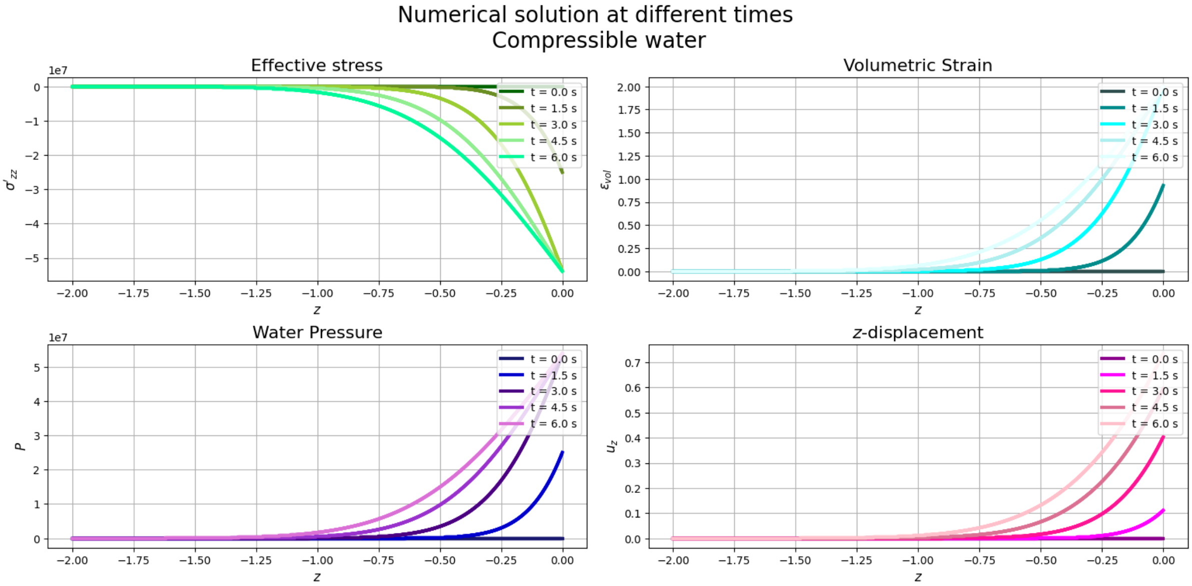

Figure 5.5: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when $d_2(t) = 0$ is constant is space and when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

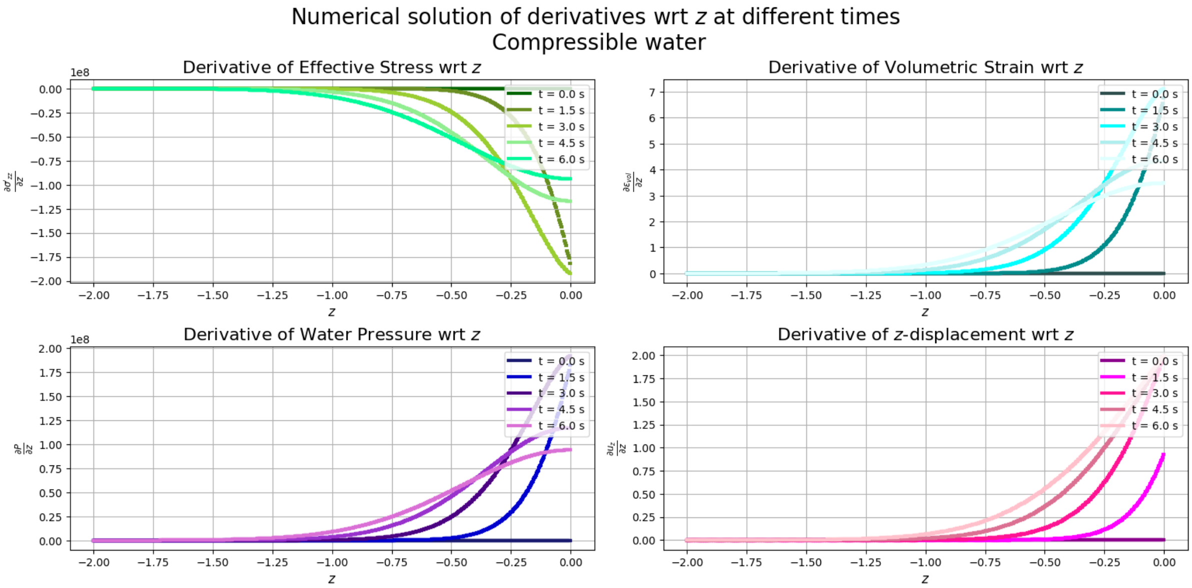

Figure 5.6: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when $d_2(t) = 0$ is constant is space and when water is assumed to be incompressible with $\beta = 4.8 \cdot 10^{-10}$.

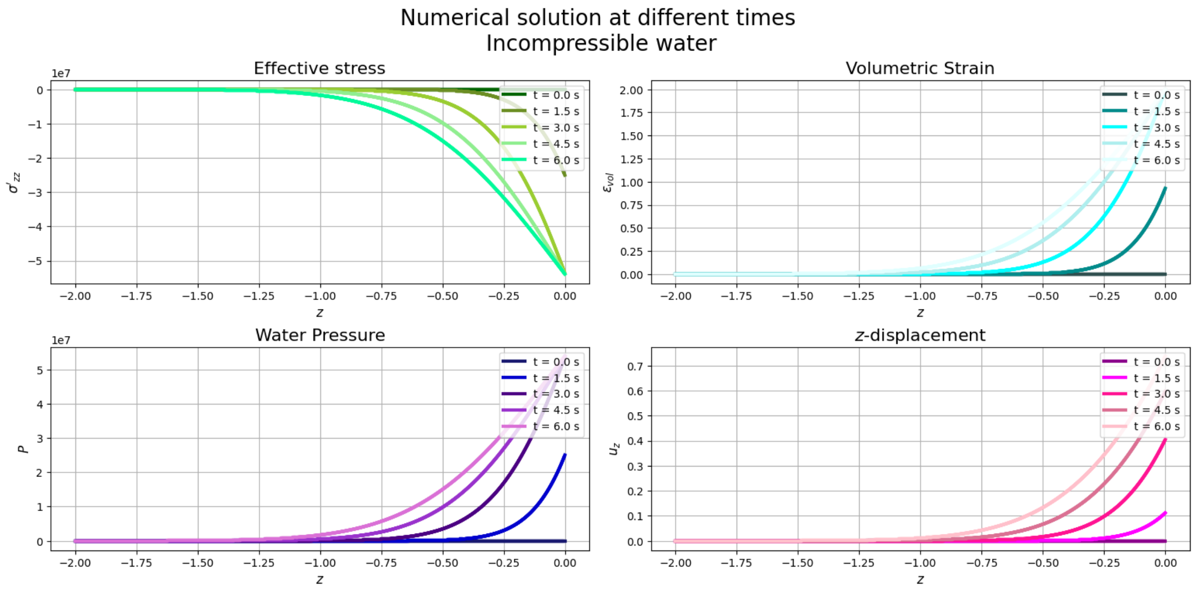

Figure 5.7: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when $d_2(t) = 0$ is constant is space and when water is assumed to be incompressible ($\beta = 0$).

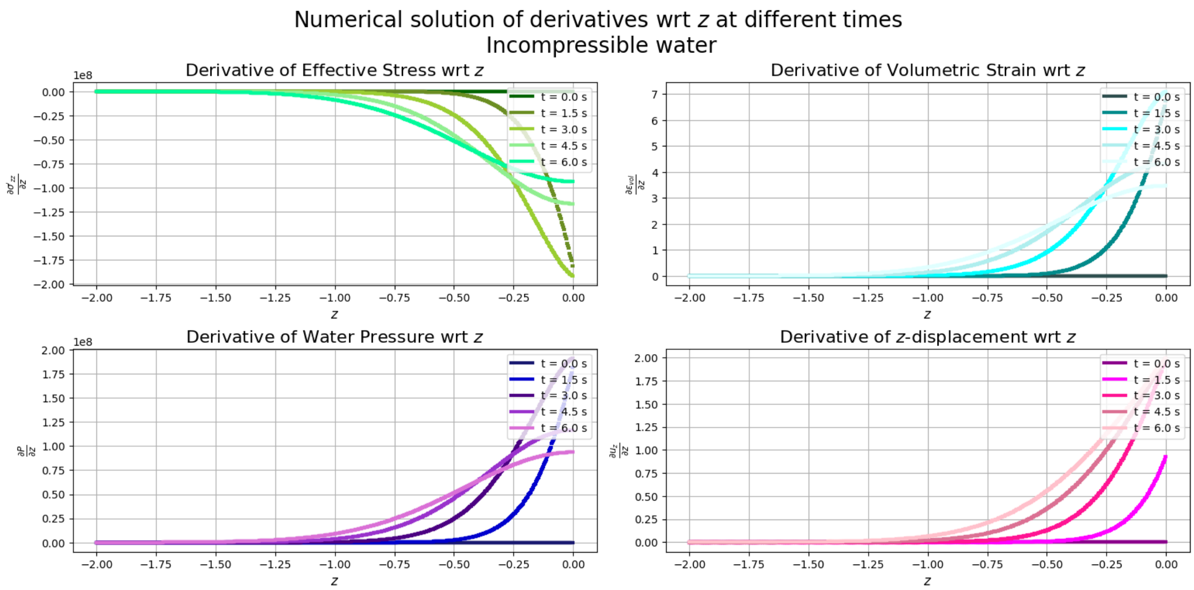

Figure 5.8: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when $d_2(t) = 0$ is constant in space and when water is assumed to be incompressible ($\beta = 0$).

5.3. Two types of soil with alternative boundary condition

Note that in previous section it was assumed that we had one type of soil. In this section we assume that we have two types of soil and that the transitions happens at -1.0 meter. In this section we assume that $d_2(t) = 0$ again, since then we can compare the results of two types of soil with one type without the variation of d_2 over time.

5.3.1. Physical model and numerical model

Using the alternative boundary conditions given by Equation (5.15) and the initial conditions given by Equation (5.3), we get the same one-dimensional weak equations for the two different layers of soil, say fine sand and medium sand, as for one type of soil. However, K_s, p, λ and μ differ per separate layer. We define that the first layer, $z \in [-nz, -\frac{n_z}{2}]$, is medium sand and the second layer, $z \in [-\frac{n_z}{2}, 0]$, is fine sand, i.e. the boundary between the two layer is the middle of the whole domain $\Omega = [-n_z, 0]$. The subdomains are divided such that there is no overlap between layers in one subdomain. However, note that the derivative of the constants that differ per layer does not exist when $z \to -\frac{n_z}{2}$. Since this is only one point and has almost no impact on the numerical model, we ignore this for now.

Then we get that substituting one-dimensional trial functions into Equation (5.16) gives the following one-dimensional Galerkin equations in matrix-vector form

$$\begin{cases} A^{\epsilon}\bar{\boldsymbol{\epsilon}}_t + A^{P}\bar{\boldsymbol{P}}_t + B^{\epsilon}\bar{\boldsymbol{\epsilon}} & = \boldsymbol{0} \\ A^{\epsilon}\bar{\boldsymbol{\epsilon}}_t + A^{P}\bar{\boldsymbol{P}}_t + B^{P}\bar{\boldsymbol{P}} & = \boldsymbol{0} \\ C\bar{\boldsymbol{u}}^z = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{5.20}$$

where

$$\begin{cases} A^{\epsilon}_{i,j} = \frac{\gamma_w^{\mathrm{m}}}{K_s^{\mathrm{m}}}\int_{-n_z}^{-\frac{n_z}{2}} N_i N_j \mathrm{d}\Omega + \frac{\gamma_w^{\mathrm{f}}}{K_s^{\mathrm{f}}}\int_{-\frac{n_z}{2}}^{0} N_i N_j \mathrm{d}\Omega \\ A^{P}_{i,j} = \frac{\gamma_w^{\mathrm{m}}}{K_s^{\mathrm{m}}} p^{\mathrm{m}}\beta \int_{-n_z}^{-\frac{n_z}{2}} N_i N_j \mathrm{dz} + \frac{\gamma_w^{\mathrm{f}}}{K_s^{\mathrm{f}}} p^{\mathrm{f}}\beta \int_{-\frac{n_z}{2}}^{0} N_i N_j \mathrm{dz} \\ B^{\epsilon}_{i,j} = (\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}})\int_{-n_z}^{-\frac{n_z}{2}} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{dz} + (\lambda^{\mathrm{f}} + 2\mu^{\mathrm{f}})\int_{-\frac{n_z}{2}}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{dz} \\ B^{P}_{i,j} = \int_{-n_z}^{0} \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\mathrm{dz} \\ C_{i,j} = \int_{-n_z}^{0} N_i \frac{\partial N_j}{\partial z}\mathrm{dz} \end{cases}, \tag{5.21}$$

and the subscripts m and f means the parameter with property of medium and and fine sand, respectively. We can write Equation (5.20) as two systems of matrix-vector multiplication

$$\begin{cases} M^t\boldsymbol{S}_t + M\boldsymbol{S} & = \boldsymbol{f} \\ C\bar{\boldsymbol{u}}^z & = A\bar{\boldsymbol{\epsilon}} \end{cases}, \tag{5.22}$$

where

$$M^t = \begin{bmatrix} A^{\epsilon} & A^{P} \\ A^{\epsilon} & A^{P} \end{bmatrix}, \quad M = \begin{bmatrix} B^{\epsilon} & \emptyset \\ \emptyset & B^{P} \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \bar{\boldsymbol{\epsilon}} \\ \bar{\boldsymbol{P}} \end{bmatrix}, \quad \boldsymbol{S}_t = \begin{bmatrix} \frac{\partial\bar{\boldsymbol{\epsilon}}}{\partial t} \\ \frac{\partial\bar{\boldsymbol{P}}}{\partial t} \end{bmatrix}, \quad \boldsymbol{f}(t) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}(t) \end{bmatrix}, \tag{5.23}$$

where $A^{\epsilon}, A^{P}, B^{\epsilon}$ and B^{P} are given by Equation (5.21).

The Dirichlet boundary conditions are included in the numerical model by setting the corresponding rows of matrices M^t and M to zero and then putting pivots in these same rows of M. Furthermore, we set again $F_i = 0$ for $i = 1, \ldots, n-1$ and $F_n = F_{zz}$. Then we get that $\bar{P}_n = F_n = F_{zz}$ and $\bar{u}^z_0 = 0$.

After applying the Backward-Euler method given by Equation (4.57) using $M^t, M, \boldsymbol{S}$ and $\boldsymbol{f}$ as described in Equation (5.23) and assuming that matrix $(M^t + \Delta t M)$ and C are invertible, we get again the time integration formulas given by Equation (5.8) with F_{zz} chosen as in Equation (5.9), but with M_t, M given by Equation (5.23).

5.3.2. Stationary model and unique solution

Since the only difference between models of having one type or two different layers of soils are some parameters depending on space now, the stationary model and its solution of the one-dimensional model with two different layers of soil remains the same as in model with only one layer of soil. The unique stationary model was given by Equation (5.19). It also still holds that this one-dimensional model with alternative solution and two different types of soil has still a unique solution.

5.3.3. Results

The three variables ϵ_{vol}, P, u_z and their derivatives are plotted five times, namely at $t = 0.0, 1.5, 3.0$ and $4.5, 6.0$, for the case of compressible water and of incompressible water. For the assumption of compressible water we take $\beta = 4.8 \cdot 10^{-10}$ and for the assumption of incompressible water we have by definition $\beta = 0.0$. The effective stress $\sigma'_{zz} = -(\lambda + 2\mu)\,\epsilon_{\text{vol}}$ is also plotted in order to check the boundary condition $(\lambda + 2\mu)\,\epsilon_{\text{vol}}(0,t) = P(0,t)$. We use the piece-wise linear basis-functions N_i for $i = 1, \dots, n$, given by Equation (5.14). Furthermore, we use again 1000 integration points, a time step of $\Delta t = 0.0025$ and the number of subdomains is chosen as $\Delta z = 0.0025$. The values of porosity p, Poisson ratio v_p and the effective size of the grains d_{10} [m] per layer are given by Table 3.3 and μ is given by Table 3.4. We use again the formula of Allen Hazen [9] with $c = 1.0$, λ is given by Equation (4.8) and recall that we define $\gamma_w = 10^4$. We set again $n_z = 2$, i.e. $\bar{\Omega} = [-2, 0]$.

The results look like the results of the model with one layer of soil and using alternative boundary conditions. However, since some parameters change, their is a difference in values for $z \in [-2, -1]$ and there is a jump or kink in the solutions and their derivatives at $z = -1.0$ [m] which can be seen in Figures 5.9, 5.10 and 5.12. The volumetric strain and effective stress may have a jump in space, since the Lamé's constants are involved in the equation for volumetric strain (and thus for effective stress, since $\sigma'_{zz} = (\lambda + 2\mu)\epsilon_{\text{vol}}$ and these vary in space now. However, we expect the water pressure and the z-displacement to be continuous. These expectations seems to hold for compressible and incompressible.

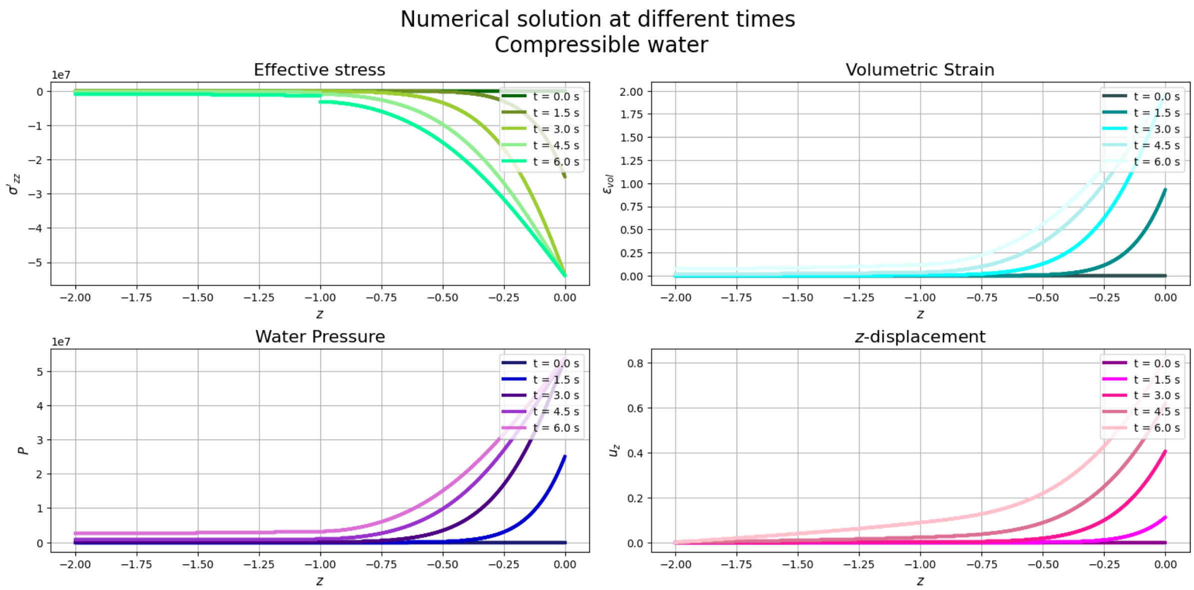

Figure 5.9: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

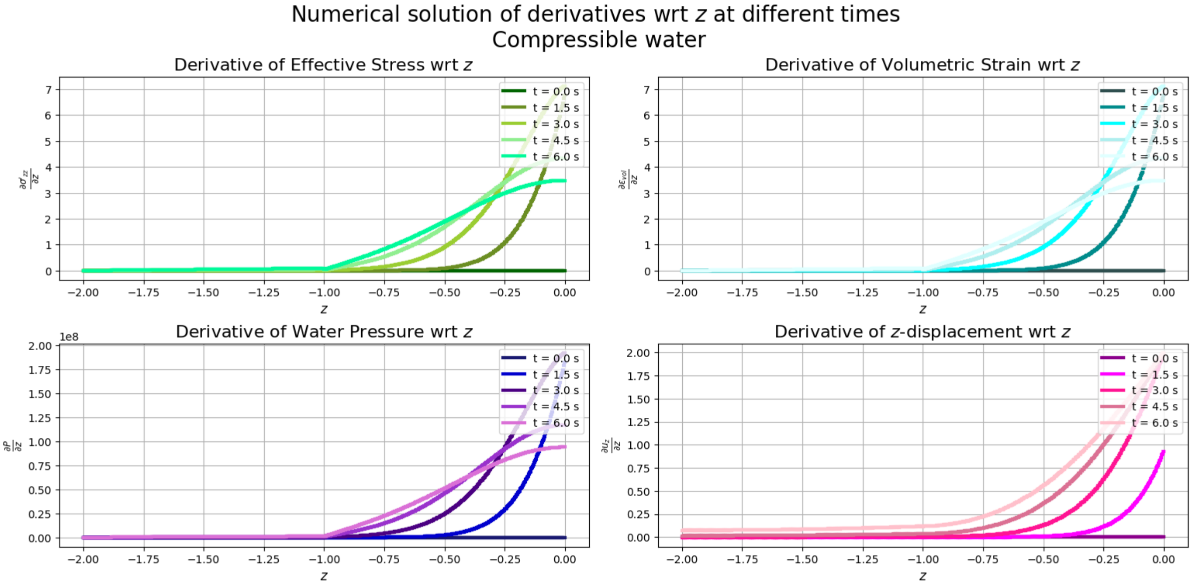

Figure 5.10: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be compressible with $\beta = 4.8 \cdot 10^{-10}$.

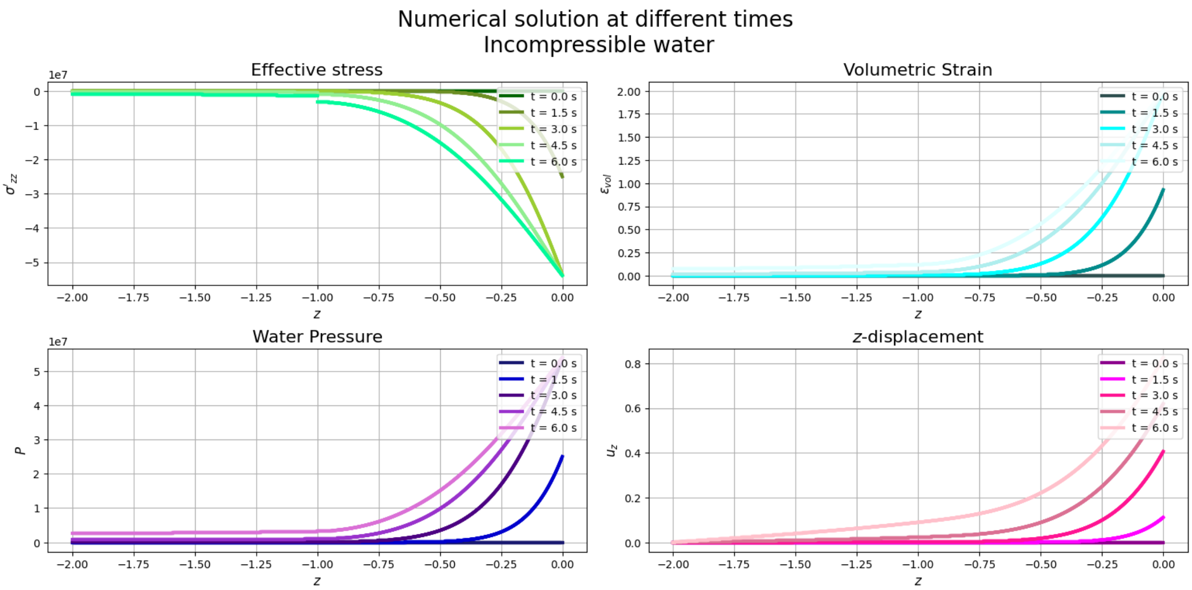

Figure 5.11: $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times, when water is assumed to be incompressible ($\beta = 0$).

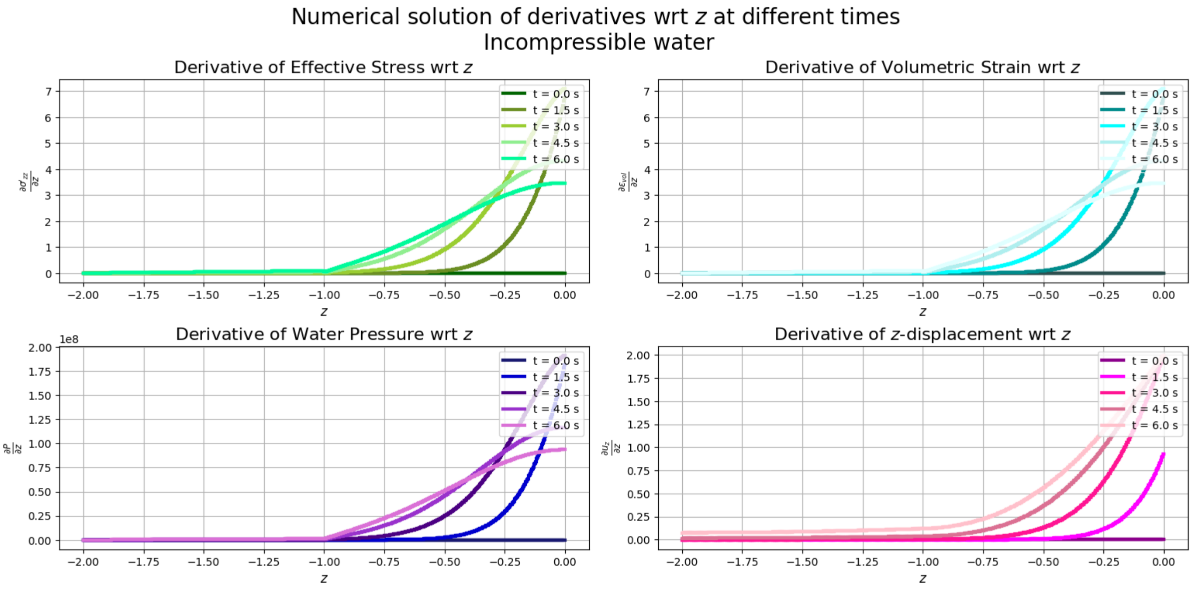

Figure 5.12: Derivatives of $\sigma'_{zz}, \epsilon_{\text{vol}}, P, u_z$ at different times with respect to z, when water is assumed to be incompressible ($\beta = 0$).

6 Conclusion and discussion

The goal of this literature report is to describe the behaviour of a seabed subjected to waves using numerical methods like the finite-element method and show the one-dimensional results. The pressure in levees is currently described by Biot's model. In Biot's model it is assumed that the pore water is compressible, the effective stresses of the porous soil are zero at the surface and the load due to the waves is completely carried by the pressure. A new model of Van Damme and Den Ouden-Van der Horst assumes that the stresses are absorbed by the pore water particles and the soil particles. Note that this way all momentum balance equations are satisfied everywhere on the computational domain and its boundaries, but the effective stress principle of Terzaghi is not in case of hydrodynamic load. On the other hand Biot's model does agree with Terzaghi.

In Chapter 2 Biot's model in two dimensions is derived to which a numerical approach is applied in order to solve Biot's model. For discretising in space the finite-element method is used and for discretising in time the Backward-Euler method is used. The finite-element method is used to solve the model, because the method is flexible in terms of computation domain and different types of boundary conditions can be included. In applying the finite-element method to the two-dimensional model, we approximate the unknown variables volumetric strain, pore water pressure and deformation in z-direction by using a linear combination of basis functions. This way the weak equations become the Galerkin equations. Then the Galerkin equations are solved by discretising in time using the Backward-Euler method. For discretising in time we us this first-order method, since the Backward-Euler method is an implicit method and thus is unconditionally stable and the accuracy is less of importance because we want to find whether the numerical model gives a solution first.

In Chapter 3 we simplified Biot's model in two dimensions to a one-dimensional model for our first analysis and discretised again in space using the finite-element method and in time using the Backward-Euler method. For deriving the numerical results of Biot's model, we set a compressibility of $4.8 \cdot 10^{-10}$ when assuming that the water is compressible. When assuming that the water is incompressible, the compressibility is 0.0 by definition. In Section 3.1, we found that the one-dimensional model of Biot with the assumption of one homogeneous layer of soil gives a unique solution for the volumetric strain, water pressure and vertical displacement. When assuming compressible water we get a solution for the volumetric strain, water pressure and vertical displacement (and effective stress) that satisfies all boundary conditions and the relationship between the volumetric strain and water pressure holds, while when assuming incompressible water we get the stationary solution back almost immediately which is physically unlikely. In Section 3.2, we found that the one-dimensional model of Biot with the assumption of two layers of two types of soil gives again a solution for the volumetric strain, water pressure and vertical displacement that converges to its unique stationary solution. We also found that there may be a jump or kink in the solution and its derivative at the boundary between the two layers of soil, since some parameters are a different constant on each layer. Indeed, a jump could be seen for the effective stress. In each of the sections of Chapter 3, we find that the one-dimensional results of the volumetric strain, pore water pressure and deformation in vertical direction (and effective stress) are different for the assumption of compressible water and the assumption of incompressible water. The solutions for incompressible water converge almost directly to the stationary solution, while for compressible water they take some more time.

In Chapter 4 the two-dimensional model of Van Damme and Den Ouden-Van der Horst is described and a numerical approach is given to solve this new model. For discretising in space the finite-element method is used and for discretising in time the Backward-Euler method is used. Like done for Biot's model, we find the Galerkin equations for the equations of the volumetric strain, pore water pressure and deformation in z-direction by using a linear combination of basis functions, when applying the finite-element method to the two-dimensional model. Then by using the Backward-Euler method again for the discretising in time, the Galerkin equations are solved and the solutions for the volumetric strain, pore water pressure and deformation are found.

In Chapter 5 we simplified the two-dimensional model to a one-dimensional model for our first analysis and discretised again in space using the finite-element method and in time using the Backward-Euler method. For deriving the numerical results of the model of Van Damme and Den Ouden-Van der Horst, we set again the compressibility of pore water equal to $4.8 \cdot 10^{-10}$ when assuming that the water is compressible en is 0.0 when assuming that the water is incompressible. In Section 5.1 we found that the one-dimensional model of Van Damme and Den Ouden-Van der Horst when assuming one layer of one type of soil gives infinitely many solutions when assuming that the pressure at the surface matches the load acting on the surface and that the vertical momentum balance equation must still hold at the surface. However, when we make the boundary condition at the surface about the vertical momentum balance equation more strict and say that the pore water pressure must be equal to the volumetric strain times a constant instead of their derivatives, we get a unique solution which is described in Section 5.2. Setting this more strict boundary condition feels natural, since it follows from the momentum balance equations in two dimensions. However, for one dimension this can not be proven. In Section 5.3 we again use this more strict boundary condition, but now we assume that seabed exists of two layers of different types of soil. We get again a unique solution which is similar as for one layer of one type of soil, but the increase or decrease goes faster or slower depending on the properties of the soils. Furthermore, we noticed again a small jump for the effective stress in value due to some different constant parameters like the Lamé's constants. In each of the sections of Chapter 5, we find that the one-dimensional results are different for the assumption of compressible water versus incompressible water when looking at the values of the variables volumetric strain, pore water pressure and deformation (and effective stress) in vertical direction. However, simply looking at the behaviour the results are similar. A small change in value and the behaviour of the variables being the same is expected, since the value of compressibility is very small when assuming compressible water.

Comparing Chapter 2 and 4, one can find that the governing equations of the model of Van Damme and den Ouden-van der Horst are the same and thus the steps taken for derivation of the governing equations are similar to the derivation of the governing equations of Biot's model. However, the thinking steps are different. Therefore, the derivation of the governing equations of this new model is still written down. The main difference between Biot's model and the model of Van Damme and den Ouden-van der Horst will be the imposed boundary equations at the surface.

When we compare the results of Chapters 3 en 5, Biot's model and Van Damme and the model of Den Ouden-Van der Horst in one dimension, we find that both models seem to describe the behaviour of the seabed subjected to waves similarly for compressible water. However, the results have different values. By changing the compressibility we could derive the same solutions for Biot's model as for the new model. For incompressible water, the stationary solution is almost immediately found by Biot's model while the results of the new model converge over time to its stationary solution. However, as said in the introduction the solution of Biot's model for incompressible water does not match real observations. Therefore, it is common to assume that the water is compressible. Whether the water is truly compressible depends on the problem we want to solve. Therefore, since both models make different assumptions (boundary conditions and the (in)compressibility of water), it depends on the problem which model makes physically more sense.

For the rest of this master thesis the numerical system in two dimensions will be implemented for one homogeneous layer of soil but also for two layers of two different types of soil. Especially multiple layers of different kinds of soil is of interest to our research, since there are many assumptions about the intersections of the layers to be consider. Another extension is to compare the (numerical) solutions of Biot's model and the model of Van Damme and Den Ouden-Van der Horst based on physics, mathematics and data of experiments more extensively, since this research question remained unanswered in this literature report. A different extension could be extending the two models to a three-dimensional setting. However, this probably will be too time-consuming and will not yield any new information about the behaviour of the seabed subjected to waves. We could also extend the one-dimensional and two-dimensional model by adding the acceleration terms as an extension to this literature report. This way we can determine whether our assumption of the acceleration terms being negligible is valid. Another extension could be trying to derive the amount of energy at the surface and look whether the same amount of energy goes in as out during a wave in two dimensions. This could validate the different chosen boundary conditions at the surface in the model of Van Damme and Den Ouden-Van der Horst. Lastly, as a extension we could also make the computational domain more general. In this literature report, we assumed a square grid in two dimensions and a vertical line in one-dimension. However, in reality the layers of the seabed are not necessarily rectangular but can be diagonal or wavy for example.

Bibliography

[1] A. Verruijt, *Soil Mechanics*. 2012.

[2] M. Biot, "Theory of propagation of elastic waves in a fluid-saturated porous solid i. low-frequency range," *The Journal of the Acoustical Society of America*, vol. 28, no. 2, pp. 168–178, 1956.

[3] A. Verruijt, *Offshore soil mechanics*. 2006.

[4] Y. Cai and H. Sun, *Solutions for Biot's Poroelastic Theory in Key Engineering Fields*. Butterworth-Heinemann, 2017, pp. 1–17.

[5] M. Mersie, "Overtopping failure in levees," M.S. thesis, TU Delft and Rijkswaterstaat, 2021.

[6] M. van Damme and D. den Ouden-van der Horst, "An alternative process-based approach to predicting the response of water saturated porous media to hydrodynamic loads," *Journal of Porous Media*, 2023. DOI: `10.1615/JPorMedia.2023045106`.

[7] J. Ye and D.-S. Jeng, "Effects of bottom shear stresses on the wave-induced dynamic response in a porous seabed: Poro–wssi (shear) model," *Acta Mech. Sin.*, vol. 27, no. 6, pp. 898–910, 2011. DOI: `10.1007/s10409-011-0469-1`.

[8] M. van Damme and D. den Ouden - van der Horst, "Methodology for calculating the response of porous media for evaluating grass response to overtopping loads," *Proceedings of 4th International Seminar on Dam Protections against Overtopping*, 2022.

[9] A. Salarashayeri and M. Siosemarde, "Prediction of soil hydraulic conductivity from particle-size distribution," *International Journal of Geological and Environmental Engineering*, vol. 6, no. 1, 2012.

[10] C. F. Li and F. P. Gao, "Characterization of spatio-temporal distributions of wave induced pore pressure in a non-cohesive seabed: Amplitude-attentuation and phase lag," *Ocean Engineering*, vol. 253, 2022.