B-spline MPM in 2D and 3D

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Introduction

Introduction

- Pile driving

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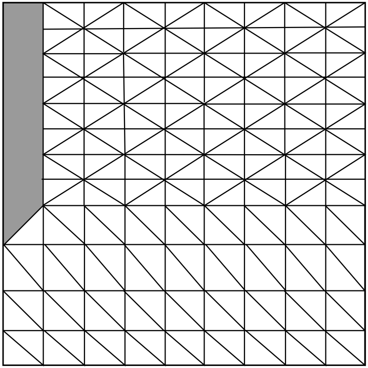
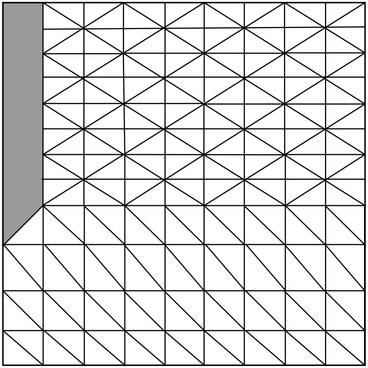
- Pile driving
- Large deformations

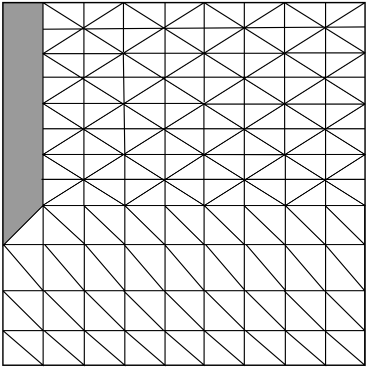


Introduction



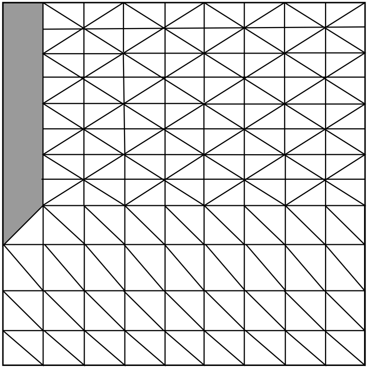
Introduction

- Discretise the domain



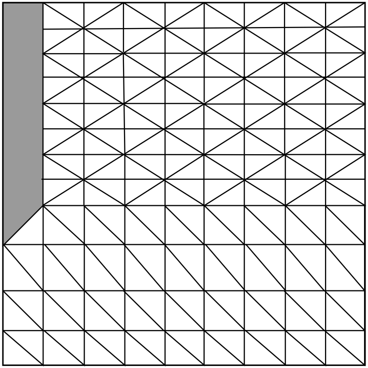
Introduction

- Discretise the domain
- Derive equations of motion

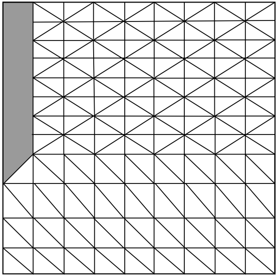


Introduction

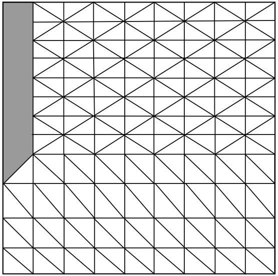
- Discretise the domain
- Derive equations of motion
- Solve using MPM (type of FEM)

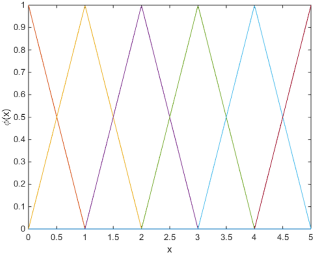


Goal: improve basis functions



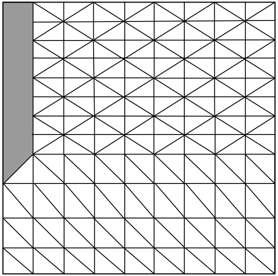
- Current: Piecewise linears

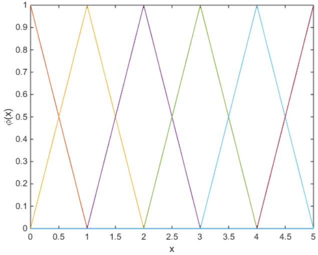




Goal: improve basis functions

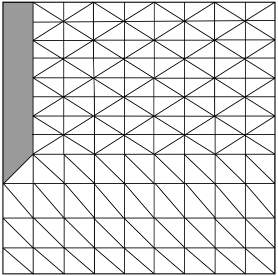
- Current: Piecewise linears
- Wanted: High order, non-negative, smooth

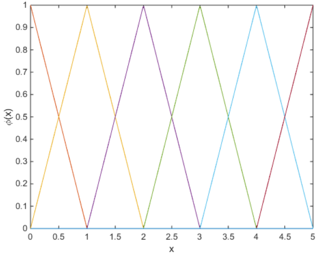


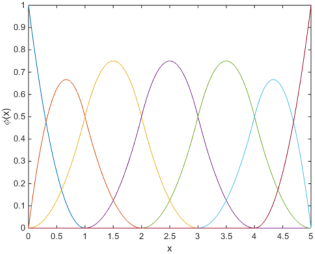


Goal: improve basis functions

- Current: Piecewise linears
- Wanted: High order, non-negative, smooth
- B-splines







Outline

Outline

1. Mathematical model
2. Material Point Method
3. Higher order basis functions
   - Lagrange basis functions
   - B-spline basis functions
4. Preliminary results
5. Conclusion

Mathematical model

Mathematical model

- Conservation of momentum

Mathematical model

- Conservation of momentum

$$\underbrace{\rho\frac{\partial \boldsymbol{v}}{\partial t}}_{m\cdot\boldsymbol{a}} = \underbrace{\nabla\cdot\boldsymbol{\sigma}}_{\boldsymbol{F_{int}}} + \underbrace{\rho\boldsymbol{g}}_{\boldsymbol{F}_{ext}}$$

Mathematical model

- Conservation of momentum

$$\underbrace{\rho \frac{\partial \boldsymbol{v}}{\partial t}}_{m \cdot \boldsymbol{a}} = \underbrace{\nabla \cdot \boldsymbol{\sigma}}_{\boldsymbol{F_{int}}} + \underbrace{\rho \boldsymbol{g}}_{\boldsymbol{F}_{ext}}$$

- Displacement → Stress → Force → Displacement

Outline

Material Point Method

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- Particle in grid method: particles store information, equations solved on grid

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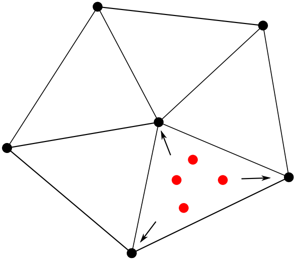
- Particles properties are projected onto the grid



Material Point Method

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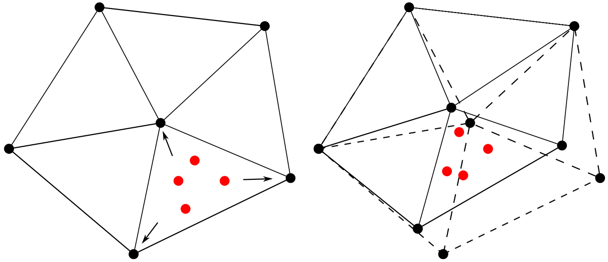
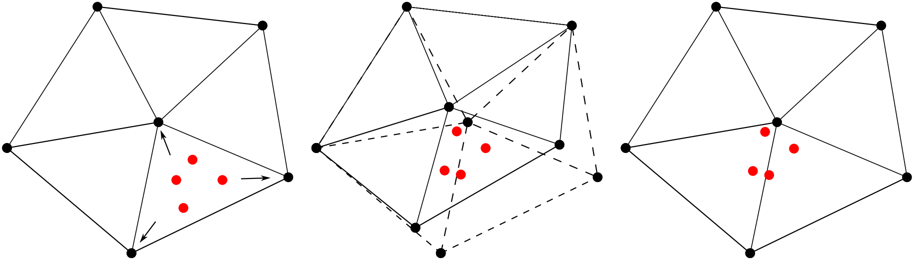


Material Point Method

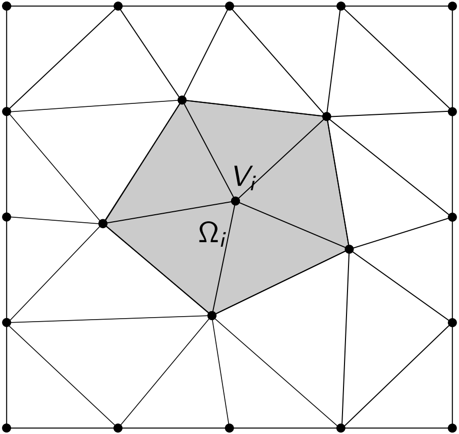
- Particle in grid method: particles store information, equations solved on grid
- Particles properties are projected onto the grid
- Equations are solved on the grid
- Update particles and reset the grid



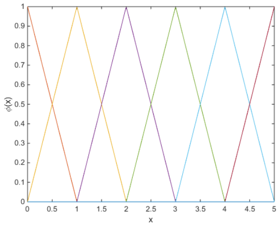
Outline

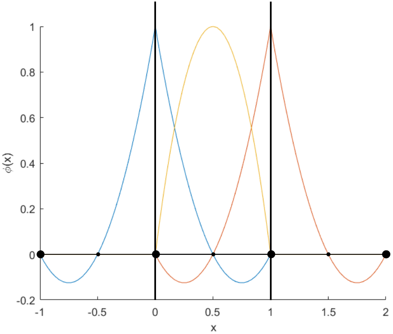
Triangulations

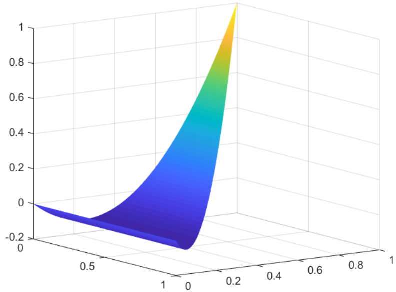
- Easy refinement, good geometry description
- Basis functions: local support, non-negative, smooth



Lagrange basis functions

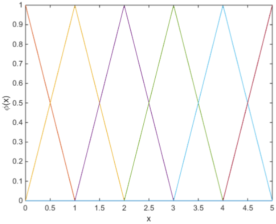


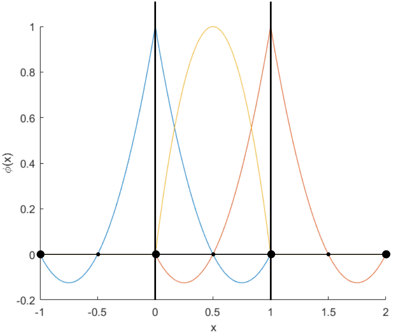


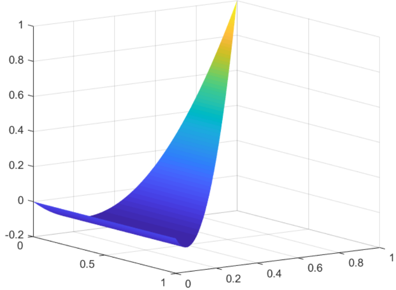


Lagrange basis functions

- Polynomial over each element

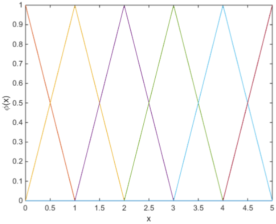


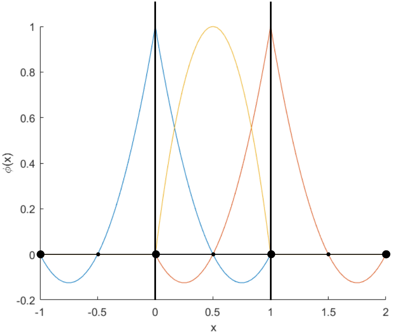


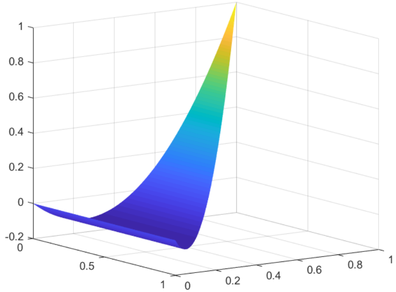


Lagrange basis functions

- Polynomial over each element
- Interpolatory property: δ_{ij}

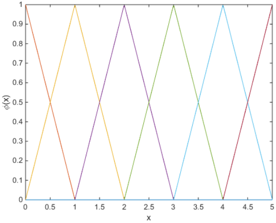


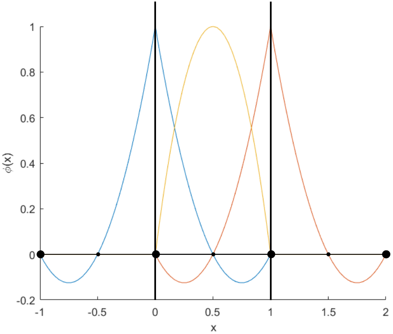


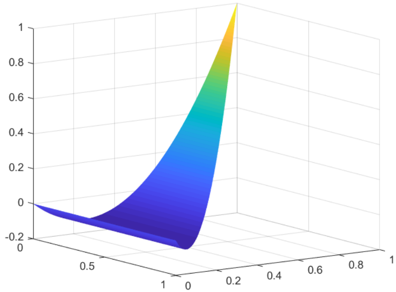


Lagrange basis functions

- Polynomial over each element
- Interpolatory property: δ_{ij}
- Discontinuous derivatives over edges, negative parts

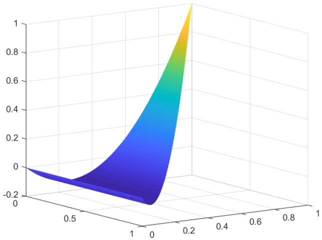






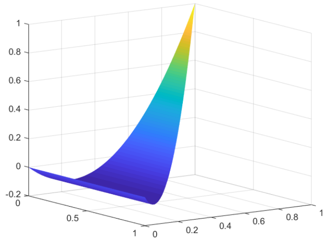
Problems with Lagrange basis

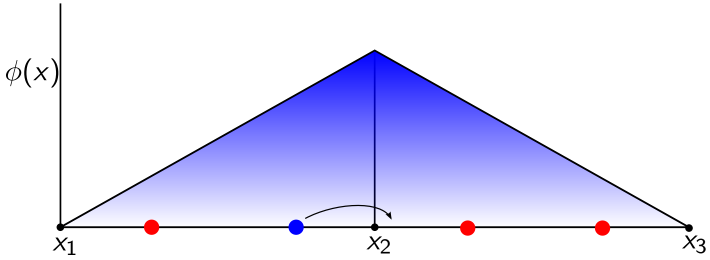
- Discontinuous derivatives



Problems with Lagrange basis

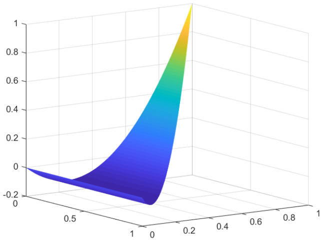
- Discontinuous derivatives
  → Grid crossing error

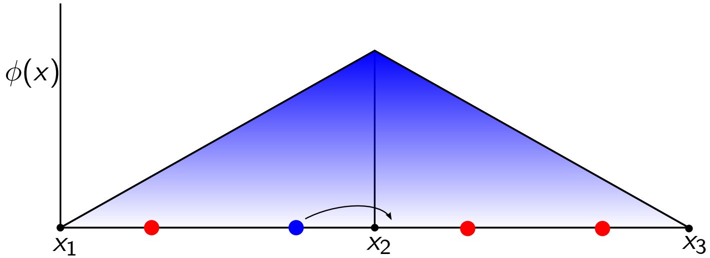




Problems with Lagrange basis

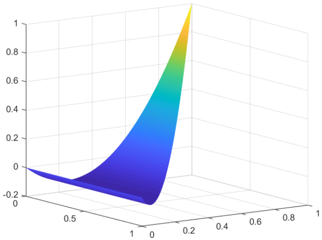
- Discontinuous derivatives
  $\rightarrow$ Grid crossing error
- Negative parts

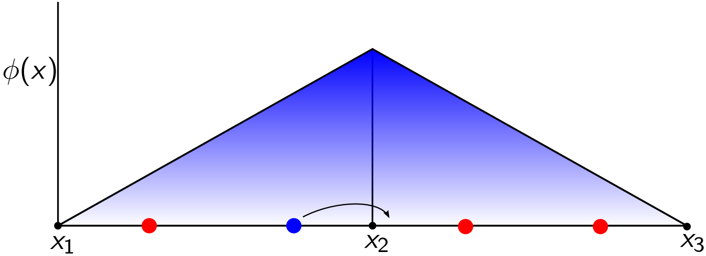




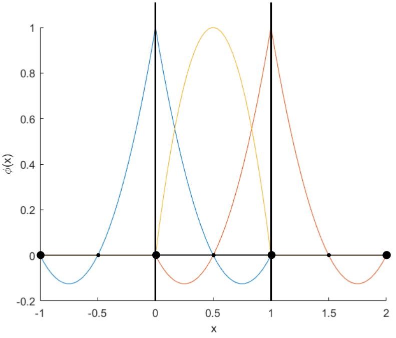
Problems with Lagrange basis

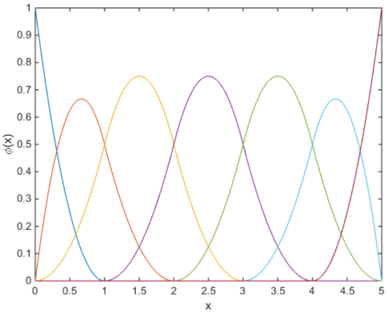
- Discontinuous derivatives
  $\rightarrow$ Grid crossing error
- Negative parts
  $\rightarrow$ Negative masses





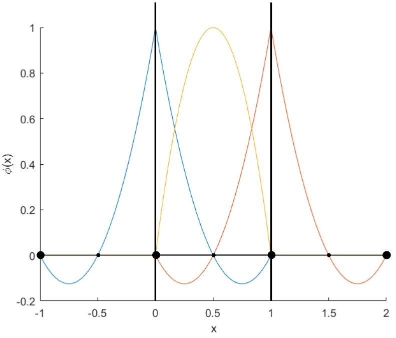
B-spline basis functions

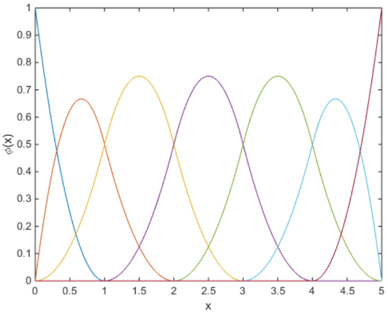




B-spline basis functions

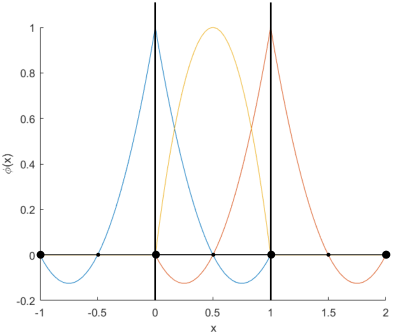
- Piecewise quadratic (or higher order polynomial), smooth, non-negative

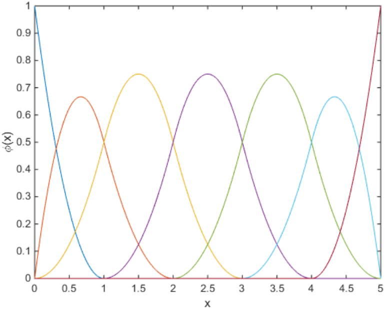




B-spline basis functions

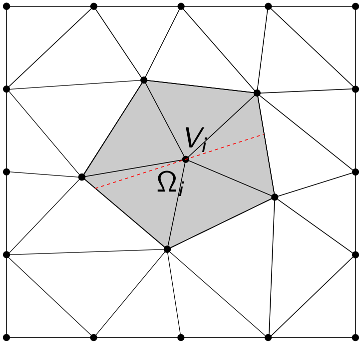
- Piecewise quadratic (or higher order polynomial), smooth, non-negative
- Not interpolatory (δ_{ij})

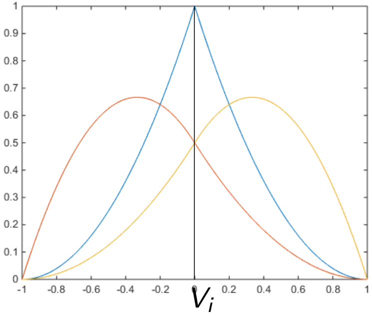




B-spline basis functions in 2D

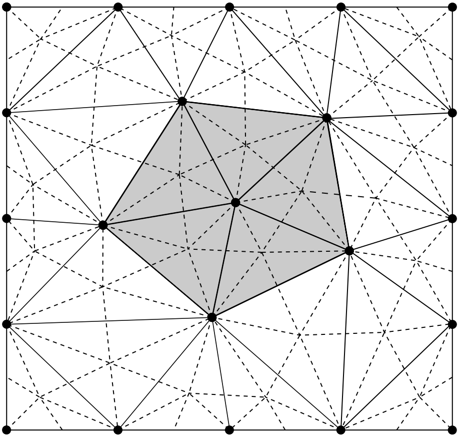
- Basis functions over triangulations
- Smooth, continuous, smooth to zero at edge





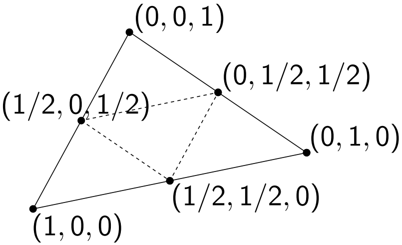
Refine grid

- 6 sub-elements per element

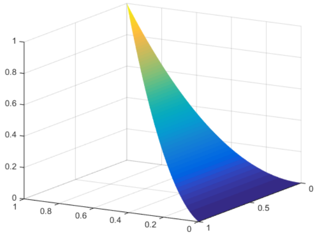


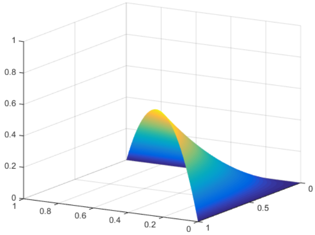
Piecewise parabola

- Define parabola over each subtriangle



$$p(x, y) := b(\zeta) = \sum_{\substack{i+j+k=2,\\ i,j,k\geq 0}} b_{i,j,k} B^2_{i,j,k}(\zeta).$$

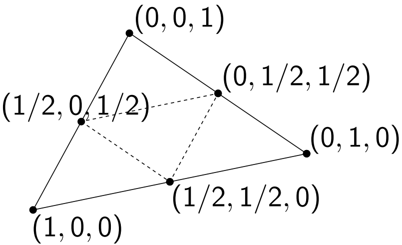


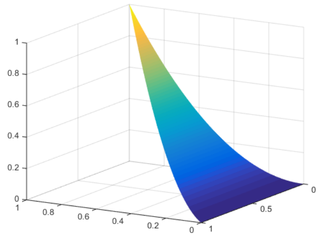


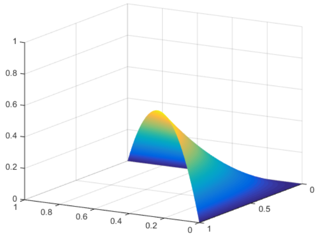
Piecewise parabola

- Define parabola over each subtriangle
- Barycentric coordinates and Bézier ordinates

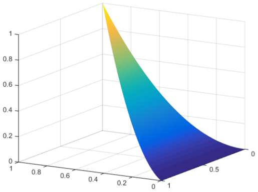
$$p(x,y) := b(\zeta) = \sum_{\substack{i+j+k=2,\\ i,j,k\geq 0}} b_{i,j,k} B^2_{i,j,k}(\zeta).$$







2D B-spline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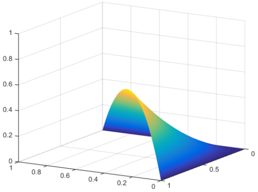
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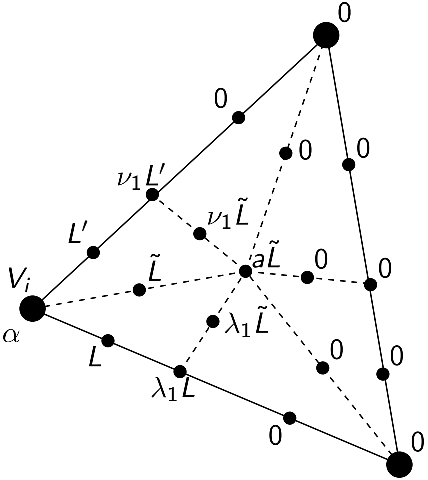
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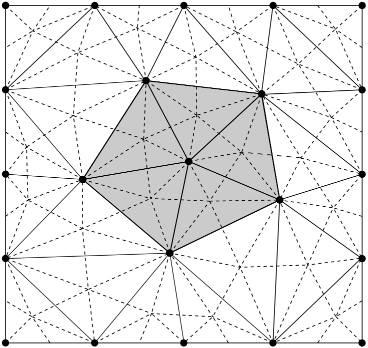
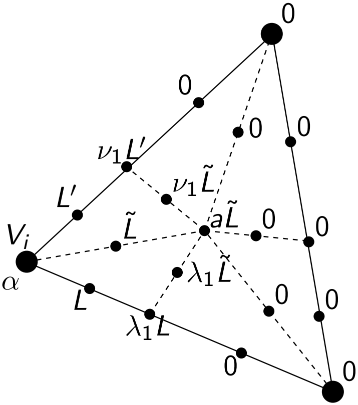
2D B-spline

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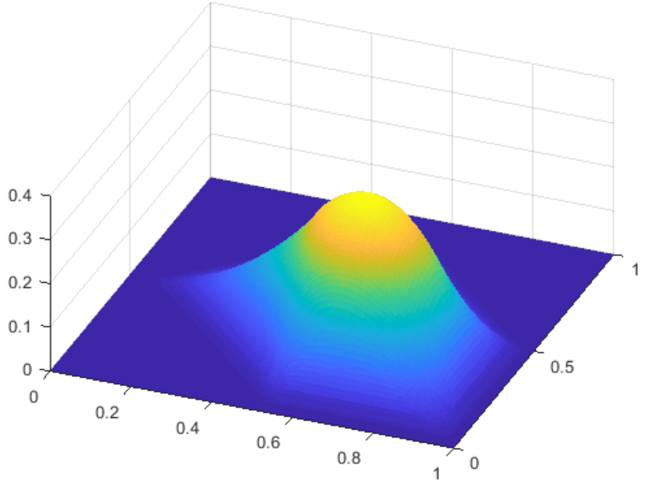
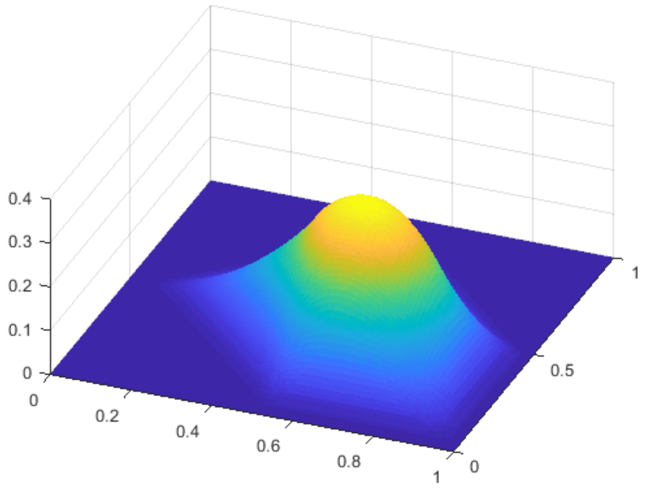
2D B-spline



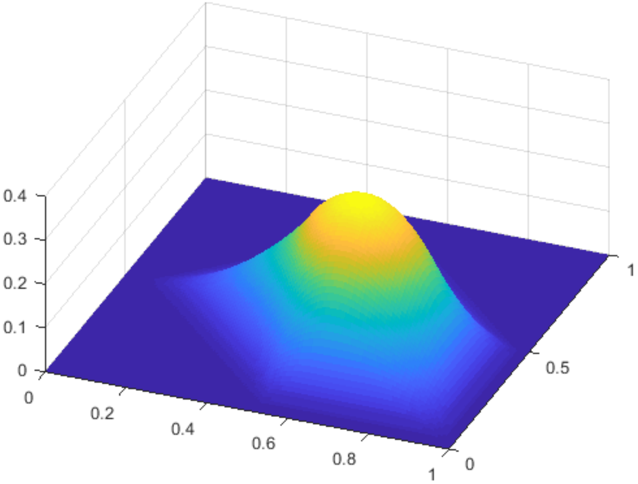
2D B-spline

- Piecewise parabola, smooth, local, non-negative, partition of unity



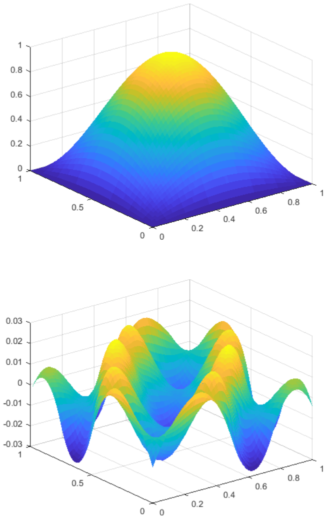
2D B-spline

- Piecewise parabola, smooth, local, non-negative, partition of unity
- 3 basis functions per vertex

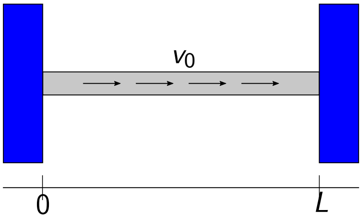


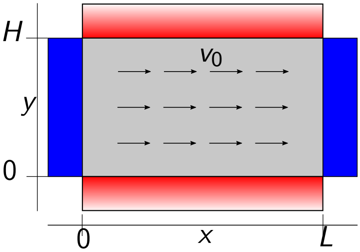
Outline

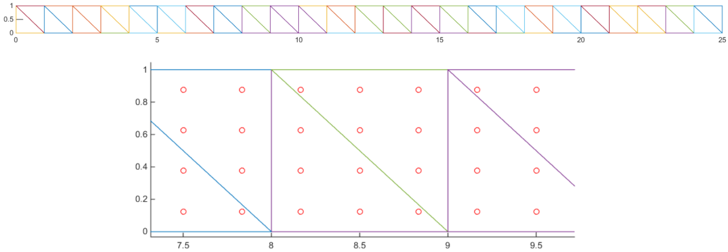
Spatial convergence of basis functions



MPM benchmark: vibrating bar

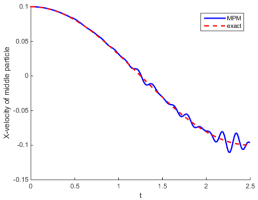




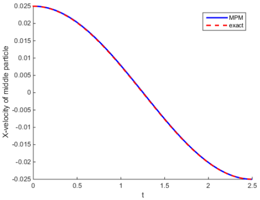


Results: vibrating bar

Lagrange basis

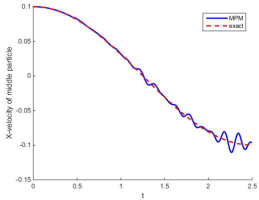
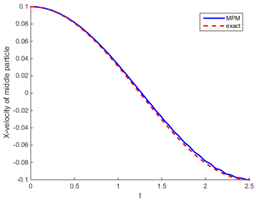
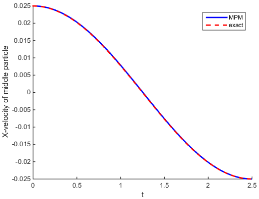
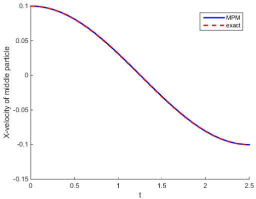


With grid crossing



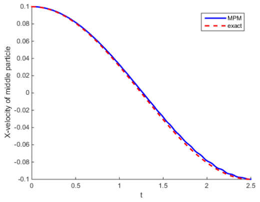
Without grid crossing

Results: vibrating bar

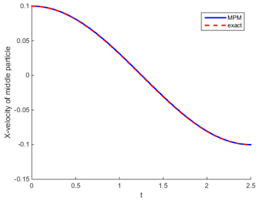
Lagrange basis	B-spline basis
 With grid crossing	 6 particles per element
 Without grid crossing	 96 particles per element

Results: vibrating bar

B-spline basis



6 particles per element

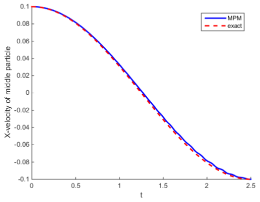


96 particles per element

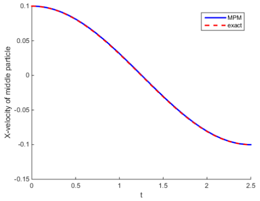
Results: vibrating bar

- No grid crossing error

B-spline basis



6 particles per element

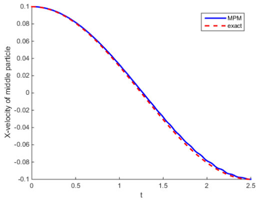


96 particles per element

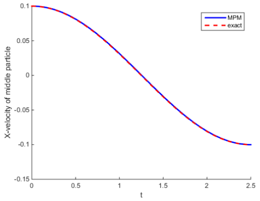
Results: vibrating bar

B-spline basis

- No grid crossing error
- Many integration points necessary



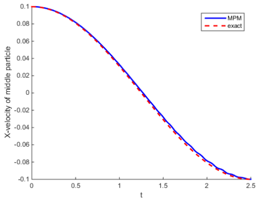
6 particles per element



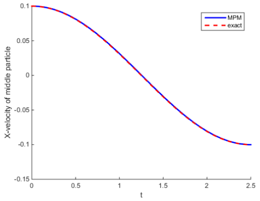
96 particles per element

Results: vibrating bar

B-spline basis



6 particles per element



96 particles per element

- No grid crossing error
- Many integration points necessary
- Non-zero y-velocity

Outline

Conclusion for B-spline basis

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- Disadvantages
  - Cumbersome implementation
  - Hard to extend to higher order polynomials
  - Many particles required for integration

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  - Cumbersome implementation
  - Hard to extend to higher order polynomials
  - Many particles required for integration
- Advantages
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  - Higher order spatial convergence

Summary

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- Goal: implement B-spline basis in MPM

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- Piecewise parabolic basis function

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- Outlook
  - Gauss point for integration
  - Implement B-splines in Deltares code

Questions?