Outline

The ARRIUS Engine

Combustion

CERFACS

Mathematics

Discretization

Eigenvalue Problems

Solution Methods

Implementation

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Conclusion

Computation of Thermo-Acoustic Instabilities in Combustors

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1. The ARRIUS Engine
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4. Mathematics
   - Discretization
   - Eigenvalue Problems
   - Solution Methods
   - Implementation
5. Results
6. Conclusion

Helicopter

Figure: The AS355 helicopter

Engine

Figure: The Turbomeca ARRIUS 1A1 turbine engine

Engine, mesh of combustion chamber

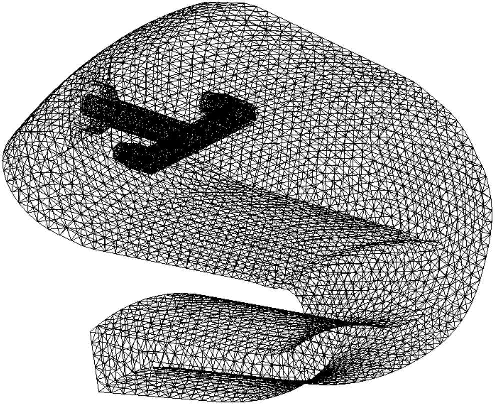


Figure: The numerical grid of a part of the combustion chamber

Combustion

Combustion is a sequence of chemical reactions between a fuel and an oxidant accompanied by the production of heat and (sometimes) light.

Heat causes pressure change (1)

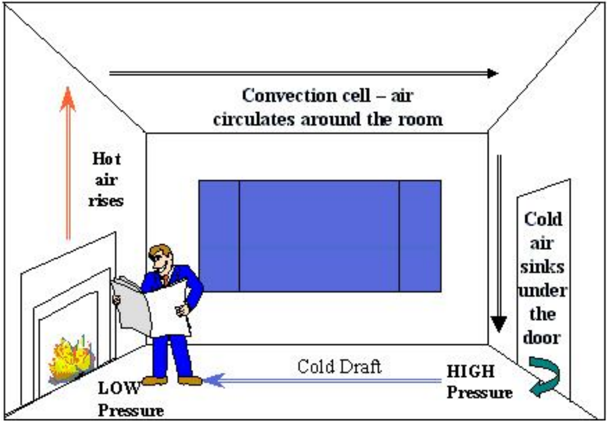


Figure: Heat causes movement of air

Heat causes pressure change (2)

Examples

- Balloon pops due to heat
- Warm air rising

Francis Bacon: *"Heat itself, its essence and quiddity is motion and nothing else."*

Combustion in an Engine

Pressure changes within a small space

- Modelled by wave equation
- Three Boundary Conditions
  - Opening in wall (constant pressure)
  - Inlet and solid walls (constant speed)
  - Outlet (acoustic impedance)
- Flame response

Helmholtz equation

Transformation from time to frequency domain.
The wave equation:

$$\nabla \cdot \left(\frac{1}{\rho_0} \nabla p_1 \right) - \frac{1}{\gamma p_0} \frac{\partial^2 p_1}{\partial t^2} = - \frac{\gamma - 1}{\gamma p_0} \frac{\partial q_1}{\partial t}$$

Becomes the Helmholtz equation:

$$\nabla \cdot \left(\frac{1}{\rho_0} \nabla \hat{p} \right) + \frac{\omega^2}{\gamma p_0} \hat{p} = i\omega \frac{\gamma - 1}{\gamma p_0} \hat{q}(x)$$

Internship

Figure: Signs at the entrance

Work

Figure: A Sunblade 200

CERFACS

CERFACS ...

- is located in Toulouse, France
- employs ca. 100 researchers
- works in 5 fields:
  - Parallel Algorithms
  - Electromagnetism
  - Aviation & Environment
  - Computational Fluid Dynamics
  - Climate Modelling and Global Change

Overview

Three different mathematical topics:

- Discretization of the equation
- Eigenvalue problems
- Solution methods
  - Arnoldi
  - Jacobi-Davidson

Discretization (1)

Equation:

$$\nabla \cdot \left(\frac{1}{\rho_0} \nabla \hat{p} \right) + \frac{\omega^2}{\gamma p_0} \hat{p} = i\omega \frac{\gamma - 1}{\gamma p_0} \hat{q}(x)$$

LHS: homogeneous Helmholtz equation

- Use finite elements and Galerkin method
- Use integration by parts

RHS: $\hat{q}(x) =$ Non-linear in ω

Discretization (2)

Galerkin method:

- divide the domain in elements
- define test functions ϕ_j
- define S_v: the set of vertices outside the boundary where the pressure is 0.
- approximate $\hat{p}$ by $\hat{p}(x) \approx \sum_{j:v_j \in S_v} \hat{p}_j \phi_j(x)$
- multiply the LHS with the test function
- integrate over the domain Ω

Discretization (3)

We started with:

$$\nabla \cdot \left(\frac{1}{\rho_0} \nabla \hat{p} \right) + \frac{\omega^2}{\gamma p_0} \hat{p} = 0.$$

We obtain $\forall k : v_k \in S_v$:

$$\int_\Omega \phi_k \nabla \left(\frac{1}{\rho_0} \nabla \cdot \sum_{j:v_j \in S_v} \hat{p}_j \phi_j(x) \right) dx +$$

$$\omega^2 \int_\Omega \frac{\phi_k}{\gamma p_0} \sum_{j:v_j \in S_v} \hat{p}_j \phi_j(x) dx = 0.$$

Interchange summation and integration:

$$\sum_{j:v_j \in S_v} \int_\Omega \frac{1}{\rho_0} \phi_k \nabla \cdot (\nabla \phi_j) dx \hat{p}_j + \omega^2 \sum_{j:v_j \in S_v} \int_\Omega \frac{1}{\gamma p_0} \phi_k \phi_j dx \hat{p}_j = 0.$$

Discretization (4)

- Integrate the first integral by parts
- determine contributions from boundary conditions

Final equation:

$$\forall k : v_k \in S_v : \sum_{j:v_j \in S_v} \left(-\int_\Omega \frac{1}{\rho_0} \nabla \phi_k \nabla \phi_j dx \hat{p}_j \right) +$$

$$\omega \sum_{j:v_j \in S_v} \left(i \int_{\partial\Omega_Z} \frac{1}{\rho_0 c_0 Z} \phi_k \phi_j d\xi \hat{p}_j \right) +$$

$$\omega^2 \sum_{j:v_j \in S_v} \left(\int_\Omega \frac{1}{\gamma p_0} \phi_k \phi_j dx \hat{p}_j \right) = 0$$

or in matrix notation:

$$AP + \omega B(\omega) P + \omega^2 CP = 0$$

Different degrees of reality

- No impedance: solve $AP + \omega^2 CP = 0$ (generalized)
- Assume $1/Z = 1/Z_0 + Z_1\omega + Z_2/\omega$:
  solve $\mathcal{A}P + \omega\mathcal{B}P + \omega^2\mathcal{C}P = 0$ (quadratic)
- RHS $\neq 0$: solve $(A - D(\omega))P + \omega B(\omega)P + \omega^2 CP = 0$ (fully non-linear)

Different types of Eigenvalue Problems

- Linear: $Ax = \lambda x$.
- Quadratic: $Ax + \lambda Bx + \lambda^2 Cx = 0$.
- Linearized Quadratic (if $C = I$):

$$\begin{pmatrix} -B & -A \\ I & 0 \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix} = \lambda \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda x \\ x \end{pmatrix}$$

- Non-linear: $T(\lambda)x = 0$.

Search space methods

- Problem: dimension N of the problem is big (>10000)
- Idea:
  - Look in a small subspace $\mathcal{W}$ of $\mathbb{R}^N$.
  - Use the approximation to construct a better subspace.
- How?
  - Construct base $v_1 \ldots v_k$ of $\mathcal{W}$.
  - Solve the projected eigenvalue problem: $W'T(\omega)Wu = 0$.
  - Check whether $T(\omega)u <$tol.
  - Not good enough? Improve search space.

Arnoldi's Method

- Choose starting vector v and max. subspace size k.
- Construct Krylov space: $\mathrm{span}(v, Av, \ldots A^k v)$.
- Project the eigenvalue problem on the Krylov space, and solve it.
- Restart with a different starting vector, if necessary.

Jacobi-Davidson

- Choose starting vector v and max. subspace size k.
- Set $W = v$, and solve the projected eigenvalue problem.
- Select a Ritz pair (ω, u), and calculate the residual $T(\omega)u$.
- Solve $t \perp u$ (approximately) from $(I - uu^*)(A - \theta I)(I - uu^*)t = -r$.
- Orthogonalize t against W and set $W = [W \;\; t]$.
- Restart after k iterations with the latest Ritz vector.

Differences (1)

- History
  - Arnoldi exists since 1951, widely implemented and optimized
  - Jacobi-Davidson exists since 1996, needs optimization
- Method:
  - Arnoldi constructs search space immediately (k iterations at once)
  - JD solves a small eigenvalue problem and an equation every iteration

Differences (2)

- Convergence speed
  - Arnoldi has linear convergence and a low workload per iteration
  - JD has quadratic convergence and a high workload per iteration
- Adaptability
  - Arnoldi is designed for linear problems
  - JD is designed for any problem

Implementations of the algorithms

Arnoldi has been implemented in ARPACK.
Jacobi-Davidson has been implemented by Gerard Sleijpen for linear and generalized problems, and by Martin van Gijzen and Jan-Willem van Leeuwen for quadratic problems.
Issues:

- Must be matrix-free (in Fortran), only use matvec-subroutine.
- Stopping criterion must be equal.
- Maximal search space size must be (nearly) optimal.
- We need meshes for the tests.

Meshes

- Matlab
  - Rectangle of 0.5m $\times$ 0.1m, 2 grid densities
- Fortran
  - Rectangle of 1m $\times$ 0.2m, 4 grid densities
  - Rectangular box of 1m $\times$ 0.2m $\times$ 0.1m, 4 grid densities
  - Combustion chamber ARRIUS, 22000 nodes

Stopping Criterion (1)

The Arnoldi Residual:

$$\|r_{AR}\|_2 = \left\| \begin{pmatrix} -B & -A \\ I & 0 \end{pmatrix} \begin{pmatrix} \omega p_{AR} \\ p_{AR} \end{pmatrix} - \omega \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \omega p_{AR} \\ p_{AR} \end{pmatrix} \right\|_2$$

The Jacobi-Davidson Residual:

$$\|r_{JD}\|_2 = \left\|Ap_{JD} + \omega Bp_{JD} + \omega^2 Cp_{JD}\right\|_2$$

$$= \left\| \begin{pmatrix} -B & -A \\ I & 0 \end{pmatrix} \begin{pmatrix} \omega p_{JD} \\ p_{JD} \end{pmatrix} - \omega \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \omega p_{JD} \\ p_{JD} \end{pmatrix} \right\|_2$$

$$\left\| \begin{pmatrix} \omega p_{JD} \\ p_{JD} \end{pmatrix} \right\|_2 = \sqrt{1 + |\omega|^2}, \quad \left\| \begin{pmatrix} \omega p_{AR} \\ p_{AR} \end{pmatrix} \right\|_2 = 1.$$

Stopping Criterion (2)

The Arnoldi (ARPACK) criterion:
$\|r_{AR}\|_2 < tol \cdot |\omega|$

The Jacobi-Davidson criterion:
$\|r_{JD}\|_2 < tol \cdot |\omega| \cdot \sqrt{1 + |\omega|^2}$

Maximal Search space size

Hard to predict the optimal value.
Tests:

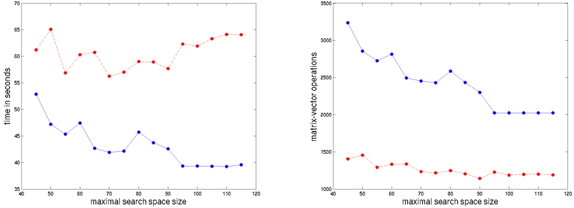


Figure: Left: CPU-time, Right: Matvecs

Results, overview

- Linear problems in MATLAB
- Quadratic and nonlinear problems in MATLAB
- Tests done with Fortran
- The ARRIUS chamber

Linear Problems, MATLAB

	Small	Large
AR	0.46 s	2.57 s
JD	3.18 s	65.35 s

Table: Results for linear problems using ARPACK and JDQZ

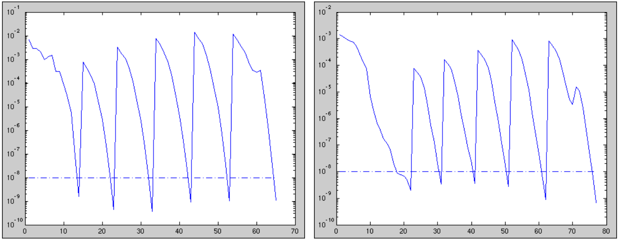


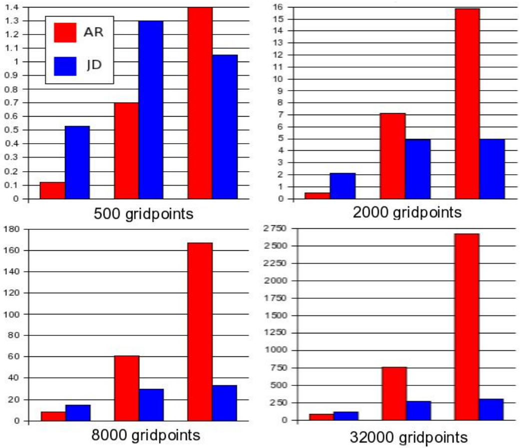
Figure: Convergence history for JDQZ

Quadratic and Nonlinear Problems, MATLAB

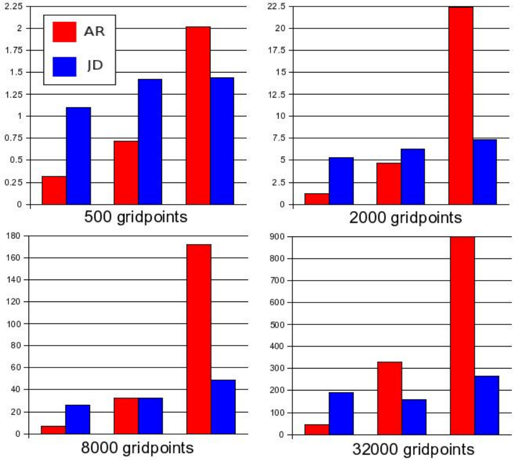
	Small	Large
AR (quad)	0.775 s	11.74 s
JD (quad)	1.20 s	26.70 s
AR (NL)	177.5 s	2451 s
JD (NL)	3.14 s	N/A

Table: Results for Quadratic and Nonlinear Problems

Fortran Results, 2D academic case



Fortran Results, 3D academic case



Fortran Results, 2D academic and ARRIUS

	$u = 0$		$y = 0.4+0.3$ i		$y = 3+2$ i	
	AR	JD	AR	JD	AR	JD
λ_1	55.23	53.64	523.35	57.43	1358.35	55.49
λ_5	62.78	70.19	528.77	135.34	1802.81	150.45
λ_{10}	76.49	116.46	746.36	272.4	2059.02	299.03

Table: Results for the 2D academic testcase with 8000 nodes

	$u = 0$		$y = 0.4+0.3$ i		$y = 3+2$ i	
	AR	JD	AR	JD	AR	JD
λ_1	131	131	1158	284	2106	310
λ_5	131	223	1211	1266	2302	1556
λ_{10}	197	513	1662	3127	7724	3644
Matvecs	2669	7291	8018	39882	37747	46857

Table: Results for the ARRIUS chamber with 22000 nodes

Conclusions

- Arnoldi is the best method for linear problems
- Jacobi-Davidson is better for quadratic problems
- Jacobi-Davidson has potential for nonlinear problems

Future Research

- Improve restart strategy
- Use preconditioning for the correction equation
- Improve JD method for nonlinear problems
- Implement parallel version

Questions

Questions?