Stability of numerical schemes used for pricing green bonds

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Sustainable Finance

Green financing is a growing phenomenon. More and more companies are moving from financing projects with regular bonds (brown bonds) to green bonds. In a moment, I will state what a green bond is. One important reason for this shift is copying behavior. Researchers have shown that governments, but also regular companies, are more likely to opt for green (or greener) choices when their trading partners or competitor countries also do so. Think for example about putting a price on carbon emission.

A green bond is an example of a green financial derivative. The green bond market is relatively new, it was only invented in 2007. Unfortunately, for a very long time, there was no clear definition of a green bond. Some researchers describe a green bond as “any type of bond instrument where the proceeds will be exclusively applied to finance new or existing eligible green projects.” Others think that green bonds are financial securities to finance projects to minimize the impact of greenhouse gas emissions, but yet another group of people say that green bonds are earmarked to finance only projects with environmental benefits. All these definitions may sound similar, but they are not exactly the same. It is important to have one clear definition of a green bond to avoid greenwashing. Greenwashing means that the ‘greenness’ of a bond is exaggerated.

Last year in November, the European Green Bond Standard released a document that stated a strict list of requirements that green bonds have to satisfy in order to be actually green.

It is nice to add that studies have shown that investors are willing to pay a little more to obtain a green bond rather than a brown bond with the same payments. This premium is called a greenium.

Pricing Green Bonds

Juriaan Rutten! [1] has derived a PDE to find a price for a green bond:

$$\frac{\partial V}{\partial t} + (\mu c - \lambda_1 \sigma_c c)\frac{\partial V}{\partial c} + \left(\alpha(\beta - r) - \lambda_2 \sigma_r \sqrt{r}\right)\frac{\partial V}{\partial r} + \frac{1}{2}\left(\sigma_c^2 c^2 \frac{\partial^2 V}{\partial c^2} + \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + 2c\rho\sigma_c\sigma_r\sqrt{r}\frac{\partial^2 V}{\partial r \partial c}\right) - rV = 0. \tag{1}$$

For this PDE, we have that

- c: carbon price,
- r: risk-free interest rate,
- σ_c, σ_r: volatility of the carbon price and the risk-free rate.

This PDE looks somewhat similar to the Black-Scholes equation, but here we also take the carbon price into account. Moreover, since we have a factor $\sqrt{r}$, we only allow non-negative values for the risk-free rate.

This PDE cannot be solved analytically. Therefore, we have to find a numerical solution. This can be done by making use of the finite difference method and choosing a time-stepping method, either explicit or implicit. Implicit methods are more stable than explicit methods, but we do have to solve a system of linear equations to make a step forward in time. These systems are often of the form $A\mathbf{u} = \mathbf{f}$, where A is a sparse, invertible matrix. This system can be solved by computing A^{-1} and multiplying A^{-1} from the left with $\mathbf{f}$, but unfortunately, from a computational viewpoint, this is a heavy task. Therefore, we will consider iterative solution methods. For this type of solution method, we want to obtain a sequence of solutions, denoted by $\{\mathbf{u}^k\}_{k\geq 0}$, that has the real solution $\mathbf{u}$ as a limit, i.e.,

$$\lim_{k\to\infty} \mathbf{u}^k = \mathbf{u}.$$

The numerical methods that we will consider are

- Conjugate Gradient,
- BiCG-Stab,
- GMRES.

Stability is an important factor that should be taken into account. For this master's project, we will examine the stability conditions for the explicit and implicit time-stepping methods by making use of Von Neumann's analysis. This type of analysis was developed by the Hungarian mathematician John von Neumann and makes use of Fourier Analysis.

Some results have already been found for the stability of numerical methods that are used for pricing regular options. It is not obvious that these results are still valid for PDE (1). For example, the changing volatility might change the stability conditions, such as the optimal time step size Δt. Also, the boundary conditions for green derivatives are different from normal derivatives because we take the carbon price into account.

References

[1] Juriaan Rutten. Green bond valuation: A numerical mathematics perspective. *https://repository.tudelft.nl/islandora/object/uuid:655fce08-75b7-4e90-9f22-3f792f9a8a99?collection=education*, 2024.