Stability of the numerical scheme used for pricing green bonds

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Definition: Zero Coupon Bond (ZCB)

A zero-coupon bond with a value at time t with maturity time T, denoted by $B(t, T)$, is a financial instrument that can be bought at time $t = 0$ for a price of $B(0, T)$, and pays one unit of currency (euro, dollar, etc.) at maturity time T, i.e., $B(T, T) = 1$.

Figure: Payments for a zero-coupon bond with maturity time T.

Option Pricing Theory

- Constant interest r:

$$e^{r(T-t)}B(t,T) = 1 \Longrightarrow B(t,T) = e^{-r(T-t)}.$$

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- Continuously differentiable interest: $r : [0,T] \to \mathbb{R}$:

$$B(t,T) = e^{-\int_t^T r(s)ds}.$$

Option Pricing Theory

Definition: European call option

A European call option gives an option holder the right, but not the obligation, to buy an asset at a pre-specified maturity time T for a pre-specified strike price K from the option writer.

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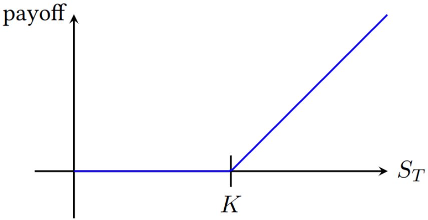

Figure: The pay-off diagram of a European call option: $(S_T - K)^+ := \max\{S_T - K, 0\}$.

Option Pricing Theory

Definition: European put option

A European put option gives an option holder the right, but not the obligation, to sell an asset at a pre-specified maturity time T for a pre-specified strike price K from the option writer.

Option Pricing Theory

Definition: European put option

A European put option gives an option holder the right, but not the obligation, to sell an asset at a pre-specified maturity time T for a pre-specified strike price K from the option writer.

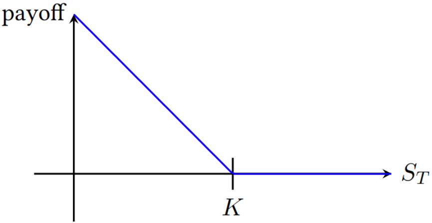

Figure: The pay-off diagram of a European put option: $(K - S_T)^+ := \max\{K - S(T), 0\}$.

Option Pricing Theory

- Value of a call option: V_{call}.
- Value of a put option: V_{put}.

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Theorem: Put-call parity

Let S be some asset, $V_i(S,t)$ the value of an option at time t for $i \in \{\text{call, put}\}$ with S being the underlying asset, K the strike price and $T-t$ the time to maturity. Moreover, let r be the risk-free interest rate and assume that r is constant. If $S(t)$ is the price of the asset at time t, we have the following equality:

$$V_{\text{call}}(t,S) + Ke^{-r(T-t)} = V_{\text{put}}(t,S) + S(t).$$

Option Pricing Theory

- Black-Scholes model

Figure: From left to right: Robert Merton, Myron Scholes, and Fischer Black.

Option Pricing Theory

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \\ \forall S \geq 0 : V(T, S) = F(S). \end{cases}$$

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Parameter analysis:

- V: the value of the call or put option,
- S: the value of the asset (or stock),
- r: risk-free interest rate,
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Boundary condition:

- For a call option: $F(S) = (S_T - K)^+$.
- For a put option: $F(S) = (K - S_T)^+$.

Option Pricing Theory

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$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Moreover, let Φ be the CDF of the standard normal distribution.

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- Put option:

$$V_{\text{put}}(t, S) = Ke^{-r(T-t)}\Phi\left(-d_2\right) - S(t)\Phi\left(-d_1\right).$$

Option Pricing Theory

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- Advantages:
  - Quick
  - Fair price for writer and holder of an option.

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About the Black-Scholes model:

- Advantages:
  - Quick
  - Fair price for writer and holder of an option.
- Disadvantage:
  - r and σ are assumed to be constant.
  - Trading is not a continuous process.
  - Dividend payments are considered absent.

Sustainable Finance

- Brown derivatives $\longrightarrow$ Green derivatives

Sustainable Finance

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- Copying behavior

Green bonds

Sustainable Finance

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- Multiple definitions

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Sustainable Finance

Key question: How can we determine the value of a green bond?

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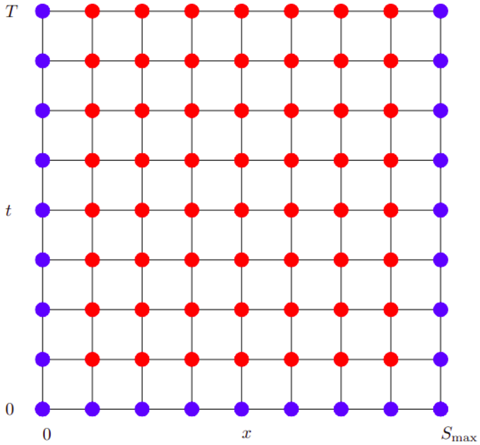

Figure 1: Finite difference grid $\{mh, nk\}_{m,n}$ for $0 \leq m \leq M$ and $0 \leq n \leq N$.

Time stepping methods:

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- Explicit
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Differences between explicit and implicit time methods:

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Differences between explicit and implicit time methods:

- Less computations are required for explicit methods.
- Implicit methods tend to be more stable than explicit methods.

Stability of the Numerical Methods

Consider the test equation for $\lambda < 0$:

$$\begin{cases} y' = \lambda y + g(t), t > t_0, \\ y(t_0) = y_0. \end{cases} \tag{1}$$

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Subtracting (1) from (2), we obtain

$$\begin{cases} \epsilon' = \tilde{y}' - y' = \lambda(\tilde{y} - y) = \lambda\epsilon, t > t_0, \\ \epsilon(t_0) = \epsilon_0, \end{cases} \tag{3}$$

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$$|Q(\lambda \Delta t)| \leq 1 \iff \text{Stability.}$$
$$|Q(\lambda \Delta t)| < 1 \iff \text{Absolute stability.}$$

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We also will consider von Neumann stability.

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$$\{\mathbf{u}^k\}_{k\geq 0} \text{ s.t. } \lim_{k\to\infty} \mathbf{u}^k = \mathbf{u}.$$

Numerical Analysis

Three iterative solution methods:

- Conjugate Gradient
  - A must be positive definite: $\forall \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\} : \mathbf{v}^\top A\mathbf{v} > 0$,
  - Short recurrences.

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- BiCG-STAB
  - Can be used for general matrices A,
  - Short recurrences,
  - No optimality property.
- GMRES
  - Can be used for general matrices A,
  - Long recurrences,
  - Optimality property.

Numerical Analysis

The PDE for pricing a green bond is derived by Juriaan Rutten:

$$\frac{\partial V}{\partial t} + (\mu c - \lambda_c \sigma_c c)\frac{\partial V}{\partial c} + \left(\alpha(\beta - r) - \lambda_r \sigma_r \sqrt{r}\right)\frac{\partial V}{\partial r} + \frac{1}{2}\left(\sigma_c^2 c^2 \frac{\partial^2 V}{\partial c^2} + \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + 2c\rho\sigma_c\sigma_r\sqrt{r}\frac{\partial^2 V}{\partial r \partial c}\right) - rV = 0.$$

- c: carbon price,
- r: risk-free interest rate,
- σ_c, σ_r: volatility of the carbon price and the risk-free rate.

Research Questions

Are the stability conditions for FE, BE, and CN still valid for this PDE?

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Are the stability conditions for FE, BE, and CN still valid for this PDE?

- How do the changing interest rate r and the volatility σ influence the stability of the numerical methods?
- What role do the boundary conditions play in when determining the stability?

Thank you for listening. Are there any questions?