Overtopping failure in levees

Thesis

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Rijkswaterstaat
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Photograph by Henri Cormont [2].

TU Delft

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by

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Abstract

The aim of this research is to provide a mathematical model that describes the physics in a levee when waves are overtopping a flood embankment. Ideally, this numerical simulation can replace empirical methods based on overtopping simulations and provide more insight into the physical process of an overtopping flow on a levee. This could prove to be useful for the design and maintenance of flood barriers.

Different interpretations of the stress tensor of pore water have each lead to distinct systems of partial differential equations. For each interpretation, the resulting system has been solved, using a finite element analysis in combination with a time-stepping method, in order to assess the validity of the imposed definitions. However, only one definition lead to a mathematical framework that yielded trustworthy results. In this final mathematical framework, both the hydrostatic water pressure and the gravitational force have been disregarded, resulting in a system only consisting of variables such as soil particles displacements, pore water velocities and a distribution function $\xi(t)$. The distribution function $\xi(t)$ represents the fraction of the exerted wave stress on the surface carried by the pore water. By definition, the fraction can vary over time, which stands in contrast to state of the art models.

The applicability of the results is limited, since the problem is simplified to a one-dimensional setting in which only normal stresses are exerted. The mathematical framework could theoretically be extended to multiple dimensions. However, it remains contestable whether it sufficiently simulates the physics in a levee. Further research is needed to show whether the extension holds when shear stresses are present and whether the same distribution function $\xi(t)$ can be applied to non-axial directions. In conclusion, this research is a proof of concept and serves as a stepping stone for more research. The used code can be found at https://github.com/HaveMersie/Overtoppingfailure.

Preface

Hereby I proudly present my thesis “Overtopping failure in levees”, which is a continuation of the PhD thesis of Myron van Damme and paper of Myron van Damme and Dennis den Ouden-van der Horst, but from a more mathematical perspective. It has been written to fulfill the graduation requirements of the MSc Applied Mathematics at the TU Delft. This research has been conducted from September 2020 to August 2021.

The project was undertaken at Rijkswaterstaat, where I did my graduate internship. The research has been difficult, especially since soil mechanics was completely new to me. However, I think that the physical model now has a more rigid mathematical foundation and on top of that the numerical results that were retrieved in the end, agreed with expectations from experiments in practice.

I would like to thank all my supervisors, but Myron van Damme and Dennis den Ouden-van der Horst in particular. Myron has always been very involved in the project and I deeply enjoyed our discussions and clashes between different perspectives. I learned a lot from them and I hope Myron did as well. A similar involvement I experienced from Dennis. Dennis was always willing to help me further when I was stuck or read through anything I sent him, continuously providing useful- and detailed feedback. Furthermore, our coffee breaks on-campus have been a delight after working at home for such a long time.

Another special thanks to my friends, especially those with whom I have studied together during my thesis period: Victor, Andreas, Rosa, Boet, Charlotte, Laurens, Tim, Jorn, Gijs and Mukkund. You have made working a lot easier for me, which was much needed during this pandemic. Apart from friends who I studied with, some other close people have helped me in other ways by simply providing some distraction from my research. In this regard I want to thank Matthijs and Karolina in particular. Lastly, I want to give my parents and brother a big thank you for always being there when I needed them and keeping me motivated.

I hope you will enjoy reading this thesis.

Michaël Mersie
Delft, August 2021

Nomenclature

$\bar{v}_i$	The expected value of the displacement of the pore water in the x_i-direction in a tube.
$\boldsymbol{n}$	The normal vector, pointing outwards.
δW_σ	Virtual work performed by body forces.
δW_g	Virtual work performed by body forces.
δW_g	Virtual work performed by internal and external forces.
ϵ_{ij}	The strain tensor for soil.
γ_w	The hydraulic conductivity.
μ	The dynamic viscosity of the pore water.
ω	Vorticity of the displacement field of the soil particles.
Ω_p	The part of the domain consisting of pore water.
Ω_p	The part of the domain consisting of soil particles.
ω_v	Vorticity of the displacement field of the pore water.
ρ_p	The density of the soil skeleton.
ρ_s	The density of the porous soil.
ρ_w	The density of the ground water.
σ_{ij}	The stress tensor for the soil skeleton.
σ_{ij}^w	The stress tensor for the pore water.
$\tilde{\sigma}_{ij}$	The stress tensor for an unsaturated soil skeleton.
$\tilde{\sigma}_{ij}^w$	The stress tensor for a fully saturated soil skeleton.
$\tilde{u}_i$	The local displacement of the soil particles in the x_i-direction in a tube.
$\tilde{v}_i$	The local displacement of the pore water in the x_i-direction in a tube.
G	The shear modulus.
g	The gravitational constant.
K	The compression modulus.
K_s	The calibration constant.
P	The water pressure.
p	The porosity of the soil.
q_i	The specific displacement in the x_i-direction.
S_p	The surface of domain Ω_p.
S_w	The surface of domain Ω_w.
u_i	The displacement of the soil particles in the x_i-direction.
v_i	The displacement of the pore water in the x_i-direction.

Contents

1
Introduction

Grass covers have shown to offer erosion protection to river levees and flood embankments. Overtopping waves can trigger erosion of the levee slope. In this thesis it is hypothesized that the hydrodynamic load that acts on a levee, induced by the overtopping flow, can result in deformation of the porous medium. This can lead to failure due to 'head-cut', 'roll-up' and 'collapse', that are thoroughly explained by Le et al. [8]. According to Steendam et al. [13], little research has been conducted with respect to the erosive effect of overtopping flow on dike slopes, mostly because of complications of scale models and the costs of overtopping tests. There have been some recent developments with respect to the other side of the spectrum. Van Bergeijk et al. [14] have made it possible to predict stresses originating from an arbitrary wave. In other words, this model focuses on the external load on the levee. Moreover, with the development of the wave overtopping simulator (Van der Meer et al. [17]), more practical field research has been conducted in the last years. This is a simulator which is positioned on an isolated part of a flood embankment. For a given time, the simulator lets waves flow over the slope. At the same time, measurements are made that are used to provide a solid basis between the endured stresses and the reason of failure.

At the same time, theoretical models have been developed that predict water pressures in porous media. One of these methods is the PORO-WSSI model by Ye et al. [5] that is based on the dynamic response in porous seabeds. The idea is that overtopping waves can be described as a summation of harmonic waves. Hence the model could be applied to overtopping waves as well. This method requires the a priori assumption that the pore water pressures match the hydrodynamic pressures under the waves, with the result that the effective stresses are assumed to be 0 on the surface. In other words, the assumption is made that the pore water instantaneously absorbs the full hydrodynamic surface load. Since the model results do not match the measured reality, it is often assumed that in practice the pore water cannot assumed to be incompressible [6]. This is not a new assumption, since compressibility of the pore water is also proposed by Verruijt [18]. By accounting for the compressibility in tests, the model outcomes are fitted to the test results. In practice this is done by including a calibrated Skempton coefficient. This method has been the state of the art approach for some time now. However, this approach is rather questionable, since tests are being tweaked to match the model outcomes and vice versa.

In a new proposed model by Van Damme and Den Ouden-van der Horst [15], these a priori assumptions have been disregarded, in pursuit of a process based approach to more accurately determine the effect of overtopping flow. Instead of assuming that the pore water absorbs the full hydrodynamic surface load, momentum balance equations are taken as boundary conditions, to enforce that the momentum balance equation will be valid on the whole domain. This makes the system more complex, but also more similar to a real situation. Furthermore, the assumed compressibility of the pore water is questionable. It is not a reasonable assumption that pore water in a seabed contains air, since the air has had all the time to dissolve in the sea water over time.

In this thesis, the new proposed model by Van Damme and Den Ouden-van der Horst [15] is derived and numerically solved with the use of a Finite Element solver. Furthermore, a different approach is attempted, where a Finite Element solver is used to directly solve the momentum equations of the soil particles and pore water, by using balance of volume. Lastly, the author has derived new assumptions on which a modified model is based on. This model will also be solved with a Finite Element solver and a golden-section search. In case further research shows that the final model is accurate enough, this model can provide significant insight in the way flood embankments fail. On top of that, the model can be implemented for other interesting applications related with flows over porous media, e.g. oxygen uptake in lung tissue. Being able to have a better understanding of the oxygen uptake in lung tissue can also provide more insight in processes that oxygen uptake is a result of, such as diffusion capacity, lung volume, breathing pattern, et cetera. (Lin et al. [9]). Another important application could be to analyze the effect of harmonic waves on the sand layers on top of buried offshore pipelines. As described by Martin et al. [10], the sand layer on top of these shallowly buried pipes can be affected by vibrations of the pipeline and by waves running over the layer. When the sand layer disappears, the pipe might experience an undesirable uplift. The overtopping waves model could quantify the pore water pressures in the sand layer, thus providing more insight in this phenomenon.

In Chapter 2, the model describing the physics in the levee will be derived, including the boundary- and initial conditions. Since this model did not yield any numerical results, a simplified version was analyzed in Chapter 3 and the original model was subsequently modified and solved. In Chapter 4, an attempt was made to directly solve the momentum balance equations. In Chapter 5, some assumptions have been disregarded and changed, with a new model as a result. In this chapter, the new model was numerically solved as well. In Chapter 6, conclusions will be drawn and some possible extensions and open questions will be discussed for future research.

2
Physical Model

2.1. Notation

Firstly, some notation will be introduced to improve the understandability of this thesis. As is often done in soil mechanics, indices are replaced by the variable that belongs to the specific index. E.g. u_2 will be written as u_y, the component in the y-direction and **not** the partial derivative of u with respect to y. The same thing is done for tensors, e.g. σ_{12} is written as σ_{xy}. Partial derivatives will simply be written in the classical way, e.g. $\frac{\partial u_y}{\partial y}$. The Einstein summation convention is often used, to make expressions more concise. In short this means that a repeated index represents a summation over this index, i.e.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{2.1}$$

2.2. Assumptions

In order to derive at the system which describes the physics in the levee, some assumptions are made. It is possible that these assumptions will be withdrawn in future research, as an extension to the original goal of this thesis. The assumptions are:

- The densities of the soil particles and pore water are taken to be constant.
- The advective acceleration of the soil particle matrix is taken to be zero.
- The advective acceleration of the pore water is taken to be zero.
- Soil particles are incompressible.
- Pore water particles are incompressible

2.3. Values of parameters

For convenience, the parameters are chosen to have the same value throughout this thesis, unless stated otherwise. It is not important for the proof of concept of this thesis what the actual values of the parameters are, as long as they have the right order of magnitude. The used values for the parameters are given by

$$p = 0.4, \tag{2.2}$$
$$\rho_p = 2650 \text{ kgm}^{-3}, \tag{2.3}$$
$$\rho_w = 997 \text{ kgm}^{-3}, \tag{2.4}$$
$$K = 50 \cdot 10^6 \text{ Nm}^{-2}, \tag{2.5}$$
$$G = 10.7 \cdot 10^6 \text{ Nm}^{-2}, \tag{2.6}$$
$$\gamma_w = 10^4 \text{ Nm}^{-3}, \tag{2.7}$$
$$Z = 3 \text{ m}, \tag{2.8}$$
$$\mu = 1.307 \cdot 10^{-3} \text{ Nsm}^{-2}. \tag{2.9}$$

2.4. Definitions

The definitions of the stress tensors are of major importance, since they dictate the form of the resulting system of partial differential equations. The stress tensor for pore water, σ^w_{ij}, is defined by Falconer [3]. The stress tensor for the soil particles is based on the one defined by Verruijt [20] when the coefficients are defined as

$$\alpha = 2G, \tag{2.10}$$
$$\beta = K - \frac{2}{3}G, \tag{2.11}$$

where $K[\text{Nm}^{-2}]$ is the compression modulus and $G[\text{Nm}^{-2}]$ is the shear modulus. However, in contrast with Verruijt's definition, an additional frictional water term is added. The stress tensors are therefore defined as:

$$\sigma_{ii} = -\left(\beta\frac{\partial u_j}{\partial x_j} + \alpha\frac{\partial u_i}{\partial x_i}\right), \qquad \sigma^w_{ii} = \mu\left(2\frac{\partial^2 v_i}{\partial x_i \partial t} - \frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t}\right) - P, \tag{2.12}$$
$$\sigma_{ij}|_{i\neq j} = -\frac{\alpha}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \qquad \sigma^w_{ij}|_{i\neq j} = \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \tag{2.13}$$

where $P[\text{Nm}^{-2}]$ denotes the hydrostatic water pressure, given by $P = \frac{1}{3}(\sigma^w_{xx} + \sigma^w_{yy} + \sigma^w_{zz})$, which is a negative stress. $u_i[m]$ represents the displacement of the soil particles in the x_i-direction, whereas $v_i[m]$ represents the displacement of the pore water in the x_i-direction. The strain tensor, denoted by ϵ^p, is defined as

$$\epsilon^p_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}. \tag{2.14}$$

Furthermore, note that the viscosity $\mu\left[\text{Nsm}^{-2}\right]$ is the dynamic viscosity, since the stresses $\sigma^w_{ij}|_{i\neq j}[\text{Nm}^{-2}]$ would otherwise not be of the right dimension. Now that the definitions are set, the system of equations that describe the physics in the levee, will be derived.

2.5. Volume balance equation

To arrive at the volume balance equations, some observations have been made. First of all, both the pore water and the soil particles are assumed incompressible. A volume balance equation can be derived when we assume that a change of volume can only be induced by adding water or taking water out of the porous medium. The density formula for the porous medium is given by

$$\rho_s = \rho_p(1-p) + \rho_w p, \tag{2.15}$$

where $\rho_s[\text{kgm}^{-3}]$ is the density of the porous medium, $\rho_p[\text{kgm}^{-3}]$ is the density of the soil matrix, $\rho_w[\text{kgm}^{-3}]$ is the density of the pore water and p is the porosity. The density ρ_s can only change when p changes, since the densities ρ_p and ρ_w are assumed constant. The change in porosity with respect to time is induced by the flux of the pore water; hence the volume balance equation for incompressible pore water, in Cartesian coordinates, is given by

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}\left(p\frac{\partial v_i}{\partial t}\right) = 0, \tag{2.16}$$

where v_i represent the displacements of the pore water in the three different directions. This is similar to the mass balance equations stated by Bui et al. [1], with the exception that in this expression the spatial gradient of the void fractions over the distribution is not considered to be negligible. In a similar manner, the volume balance equation for incompressible soil particles is given by

$$\frac{\partial(1-p)}{\partial t} + \frac{\partial}{\partial x_i}\left[(1-p)\frac{\partial u_i}{\partial t}\right] = 0, \tag{2.17}$$

where u_i represent the displacements of the soil particles in the three different directions. Now we impose that soil particles and pore water are mixed so well, that functions u_i and v_i are defined everywhere on the domain of interest. As a consequence, we can sum Equations (2.16) and (2.17) to arrive at the volume balance equation for the porous medium

$$\frac{\partial}{\partial x_i}\left\{p\left[\frac{\partial(v_i-u_i)}{\partial t}\right]\right\} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t}\right) = 0. \tag{2.18}$$

This volume balance equation will be used several times throughout this thesis. In order to derive useful partial differential equations, the expression for the virtual work will be derived. Before the derivation of the virtual work can be made, the essential boundary conditions need to be stated.

2.6. Essential boundary conditions

When a simple rectangular prism is taken as the domain of interest, given by $\Omega = [0,L]\times[0,B]\times[-Z,0]$, some essential boundary conditions can be formulated.

Following the boundary conditions for a porous seabed (Ye et. al [5]), the bottom of the levee, i.e. at $z=-Z$, is both rigid and impermeable. As a consequence, the soil should not be allowed to sink here, hence the vertical displacement should be set to zero, i.e. $u_z = 0$. Because of the impermeability, the pore water should not be able to infiltrate the bottom boundary, hence it also has to hold that $v_z = 0$ is on $z=-Z$. On the sides of the rectangular prism, we can impose essential boundary conditions as well. When L and B are sufficiently large, it is expected that the displacements at these boundaries in the normal direction will be negligible. Thus we would have that for $x=0$ and $x=L$ it holds that $u_x = 0$. Analogously it has to hold that for $y=0$ and $y=B$, $u_y=0$. These essential conditions will be put to use in the next section, where the virtual work will be derived.

2.7. Virtual work

There are several forces working on- and in the soil, that do virtual work. One of these forces is the gravitational force, of which the virtual work is given by

$$\delta\hat{W}_g = \int_{\Omega_p}\rho_p g u_z^* \, d\Omega + \int_{\Omega_w}\rho_w g v_z^* \, d\Omega, \tag{2.19}$$

where Ω_p is the part of the domain consisting of soil particles, Ω_w is the part of the domain consisting of pore water, $g\text{ms}^{-2}$ is the gravitational constant and $u_i^*[m]$ and $v_i^*[m]$ are the virtual displacements of the soil particles and pore water respectively in the three different directions. Another external force working on the soil matrix and the pore water is the force exerted by the overtopping wave. When we denote $\boldsymbol{F}_{\text{ext}}$ as the external stress exerted by the wave, the virtual work due to this stress is given by

$$\begin{aligned}
\delta W_F &= \oint_{S_p}\boldsymbol{F}_{\text{ext}}\cdot\boldsymbol{u}^*\,dS + \oint_{S_w}\boldsymbol{F}_{\text{ext}}\cdot\boldsymbol{v}^*\,dS, && (2.20)\\
&= \oint_{z=0\cap S_p}\boldsymbol{F}_{\text{ext}}\cdot\boldsymbol{u}^*\,dS + \oint_{z=0\cap S_w}\boldsymbol{F}_{\text{ext}}\cdot\boldsymbol{v}^*\,dS, && (2.21)\\
&= \oint_{z=0\cap S_p}\left(u_x^*F_{xz}+u_y^*F_{yz}+u_z^*F_{zz}\right)dS + \oint_{z=0\cap S_w}\left(v_x^*F_{xz}+v_y^*F_{yz}+v_z^*F_{zz}\right)dS. && (2.22)
\end{aligned}$$

The virtual work done by the internal forcing, denoted by δW_σ, is given by Van Damme and den Ouden-van der Horst [15] as

$$\delta W_\sigma = -\int_{\Omega_p} \epsilon^p_{ij}\tilde{\sigma}_{ij}\, d\Omega - \int_{\Omega_w} \epsilon^w_{ij}\tilde{\sigma}^w_{ij}\, d\Omega, \tag{2.23}$$

Substituting the strain tensor gives:

$$\begin{aligned}
\delta W_\sigma = &-\frac{1}{2}\int_{\Omega_p}\left\{2\frac{\partial u^*_x}{\partial x}\tilde{\sigma}_{xx} + \left(\frac{\partial u^*_x}{\partial y} + \frac{\partial u^*_y}{\partial x}\right)\tilde{\sigma}_{xy} + \left(\frac{\partial u^*_x}{\partial z} + \frac{\partial u^*_z}{\partial x}\right)\tilde{\sigma}_{xz}\right\} d\Omega && (2.24)\\
&-\frac{1}{2}\int_{\Omega_p}\left\{\left(\frac{\partial u^*_y}{\partial x} + \frac{\partial u^*_x}{\partial y}\right)\tilde{\sigma}_{yx} + 2\frac{\partial u^*_y}{\partial y}\tilde{\sigma}_{yy} + \left(\frac{\partial u^*_y}{\partial z} + \frac{\partial u^*_z}{\partial y}\right)\tilde{\sigma}_{yz}\right\} d\Omega && (2.25)\\
&-\frac{1}{2}\int_{\Omega_p}\left\{\left(\frac{\partial u^*_z}{\partial x} + \frac{\partial u^*_x}{\partial z}\right)\tilde{\sigma}_{zx} + \left(\frac{\partial u^*_z}{\partial y} + \frac{\partial u^*_y}{\partial z}\right)\tilde{\sigma}_{zy} + 2\frac{\partial u^*_z}{\partial z}\tilde{\sigma}_{zz}\right\} d\Omega && (2.26)\\
&-\frac{1}{2}\int_{\Omega_w}\left\{2\frac{\partial v^*_x}{\partial x}\tilde{\sigma}^w_{xx} + \left(\frac{\partial v^*_x}{\partial y} + \frac{\partial v^*_y}{\partial x}\right)\tilde{\sigma}^w_{xy} + \left(\frac{\partial v^*_x}{\partial z} + \frac{\partial v^*_z}{\partial x}\right)\tilde{\sigma}^w_{xz}\right\} d\Omega && (2.27)\\
&-\frac{1}{2}\int_{\Omega_w}\left\{\left(\frac{\partial v^*_y}{\partial x} + \frac{\partial v^*_x}{\partial y}\right)\tilde{\sigma}^w_{yx} + 2\frac{\partial v^*_y}{\partial y}\tilde{\sigma}^w_{yy} + \left(\frac{\partial v^*_y}{\partial z} + \frac{\partial v^*_z}{\partial y}\right)\tilde{\sigma}^w_{yz}\right\} d\Omega && (2.28)\\
&-\frac{1}{2}\int_{\Omega_w}\left\{\left(\frac{\partial v^*_z}{\partial x} + \frac{\partial v^*_x}{\partial z}\right)\tilde{\sigma}^w_{zx} + \left(\frac{\partial v^*_z}{\partial y} + \frac{\partial v^*_y}{\partial z}\right)\tilde{\sigma}^w_{zy} + 2\frac{\partial v^*_z}{\partial z}\tilde{\sigma}^w_{zz}\right\} d\Omega. && (2.29)
\end{aligned}$$

Because of the symmetry of the stress tensor this simplifies to

$$\begin{aligned}
\delta W_\sigma = &-\int_{\Omega_p}\left\{\frac{\partial u^*_x}{\partial x}\tilde{\sigma}_{xx} + \left(\frac{\partial u^*_x}{\partial y} + \frac{\partial u^*_y}{\partial x}\right)\tilde{\sigma}_{xy} + \left(\frac{\partial u^*_x}{\partial z} + \frac{\partial u^*_z}{\partial x}\right)\tilde{\sigma}_{xz}\right\} d\Omega && (2.30)\\
&-\int_{\Omega_p}\left\{\frac{\partial u^*_y}{\partial y}\tilde{\sigma}_{yy} + \left(\frac{\partial u^*_y}{\partial z} + \frac{\partial u^*_z}{\partial y}\right)\tilde{\sigma}_{yz}\right\} d\Omega && (2.31)\\
&-\int_{\Omega_p}\left\{\frac{\partial u^*_z}{\partial z}\tilde{\sigma}_{zz}\right\} d\Omega && (2.32)\\
&-\int_{\Omega_w}\left\{\frac{\partial v^*_x}{\partial x}\tilde{\sigma}^w_{xx} + \left(\frac{\partial v^*_x}{\partial y} + \frac{\partial v^*_y}{\partial x}\right)\tilde{\sigma}^w_{xy} + \left(\frac{\partial v^*_x}{\partial z} + \frac{\partial v^*_z}{\partial x}\right)\tilde{\sigma}^w_{xz}\right\} d\Omega && (2.33)\\
&-\int_{\Omega_w}\left\{\frac{\partial v^*_y}{\partial y}\tilde{\sigma}^w_{yy} + \left(\frac{\partial v^*_y}{\partial z} + \frac{\partial v^*_z}{\partial y}\right)\tilde{\sigma}^w_{yz}\right\} d\Omega && (2.34)\\
&-\int_{\Omega_w}\left\{\frac{\partial v^*_z}{\partial z}\tilde{\sigma}^w_{zz}\right\} d\Omega. && (2.35)
\end{aligned}$$

Using Theorem 1 of Appendix A gives

$$\delta W_\sigma = \int_{\Omega_p} \begin{pmatrix} u_x^* \\ 0 \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}_{xx} + \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}_{xy} + \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} \cdot \nabla\tilde{\sigma}_{xz} + \begin{pmatrix} 0 \\ u_y^* \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}_{yy} + \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} \cdot \nabla\tilde{\sigma}_{yz} + \begin{pmatrix} 0 \\ 0 \\ u_z^* \end{pmatrix} \cdot \nabla\tilde{\sigma}_{zz} d\Omega \tag{2.36}$$

$$+ \int_{\Omega_w} \begin{pmatrix} v_x^* \\ 0 \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{xx} + \begin{pmatrix} v_y^* \\ v_x^* \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{xy} + \begin{pmatrix} v_z^* \\ 0 \\ v_x^* \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{xz} + \begin{pmatrix} 0 \\ v_y^* \\ 0 \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{yy} + \begin{pmatrix} 0 \\ v_z^* \\ v_y^* \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{yz} + \begin{pmatrix} 0 \\ 0 \\ v_z^* \end{pmatrix} \cdot \nabla\tilde{\sigma}^w_{zz} d\Omega \tag{2.37}$$

$$- \oint_{S_p} \left\{ \begin{pmatrix} u_x^* \\ 0 \\ 0 \end{pmatrix} \tilde{\sigma}_{xx} + \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \tilde{\sigma}_{xy} + \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} \tilde{\sigma}_{xz} + \begin{pmatrix} 0 \\ u_y^* \\ 0 \end{pmatrix} \tilde{\sigma}_{yy} + \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} \tilde{\sigma}_{yz} + \begin{pmatrix} 0 \\ 0 \\ u_z^* \end{pmatrix} \tilde{\sigma}_{zz} \right\} \cdot \boldsymbol{n} d\Gamma \tag{2.38}$$

$$- \oint_{S_w} \left\{ \begin{pmatrix} v_x^* \\ 0 \\ 0 \end{pmatrix} \tilde{\sigma}^w_{xx} + \begin{pmatrix} v_y^* \\ v_x^* \\ 0 \end{pmatrix} \tilde{\sigma}^w_{xy} + \begin{pmatrix} v_z^* \\ 0 \\ v_x^* \end{pmatrix} \tilde{\sigma}^w_{xz} + \begin{pmatrix} 0 \\ v_y^* \\ 0 \end{pmatrix} \tilde{\sigma}^w_{yy} + \begin{pmatrix} 0 \\ v_z^* \\ v_y^* \end{pmatrix} \tilde{\sigma}^w_{yz} + \begin{pmatrix} 0 \\ 0 \\ v_z^* \end{pmatrix} \tilde{\sigma}^w_{zz} \right\} \cdot \boldsymbol{n} d\Gamma \tag{2.39}$$

$$= \int_{\Omega_p} u_x^* \left(\frac{\partial\tilde{\sigma}_{xx}}{\partial x} + \frac{\partial\tilde{\sigma}_{xy}}{\partial y} + \frac{\partial\tilde{\sigma}_{xz}}{\partial z} \right) + u_y^* \left(\frac{\partial\tilde{\sigma}_{xy}}{\partial x} + \frac{\partial\tilde{\sigma}_{yy}}{\partial y} + \frac{\partial\tilde{\sigma}_{yz}}{\partial z} \right) + u_z^* \left(\frac{\partial\tilde{\sigma}_{xz}}{\partial x} + \frac{\partial\tilde{\sigma}_{yz}}{\partial y} + \frac{\partial\tilde{\sigma}_{zz}}{\partial z} \right) d\Omega \tag{2.40}$$

$$+ \int_{\Omega_w} v_x^* \left(\frac{\partial\tilde{\sigma}^w_{xx}}{\partial x} + \frac{\partial\tilde{\sigma}^w_{xy}}{\partial y} + \frac{\partial\tilde{\sigma}^w_{xz}}{\partial z} \right) + v_y^* \left(\frac{\partial\tilde{\sigma}^w_{xy}}{\partial x} + \frac{\partial\tilde{\sigma}^w_{yy}}{\partial y} + \frac{\partial\tilde{\sigma}^w_{yz}}{\partial z} \right) + v_z^* \left(\frac{\partial\tilde{\sigma}^w_{xz}}{\partial x} + \frac{\partial\tilde{\sigma}^w_{yz}}{\partial y} + \frac{\partial\tilde{\sigma}^w_{zz}}{\partial z} \right) d\Omega \tag{2.41}$$

$$- \oint_{S_p \cap z=-Z} \left\{ -u_x^* \tilde{\sigma}_{xz} - u_y^* \tilde{\sigma}_{yz} \right\} d\Gamma - \oint_{S_w \cap z=-Z} \left\{ -v_x^* \tilde{\sigma}^w_{xz} - v_y^* \tilde{\sigma}^w_{yz} \right\} d\Gamma \tag{2.42}$$

$$- \oint_{S_p \cap z=0} \left\{ u_x^* \tilde{\sigma}_{xz} + u_y^* \tilde{\sigma}_{yz} + u_z^* \tilde{\sigma}_{zz} \right\} d\Gamma - \oint_{S_w \cap z=0} \left\{ v_x^* \tilde{\sigma}^w_{xz} + v_y^* \tilde{\sigma}^w_{yz} + v_z^* \tilde{\sigma}^w_{zz} \right\} d\Gamma \tag{2.43}$$

$$- \oint_{S_p \cap x=0} \left\{ -u_y^* \tilde{\sigma}_{xy} - u_z^* \tilde{\sigma}_{xz} \right\} d\Gamma - \oint_{S_w \cap x=0} \left\{ -v_y^* \tilde{\sigma}^w_{xy} - v_z^* \tilde{\sigma}^w_{xz} \right\} d\Gamma \tag{2.44}$$

$$- \oint_{S_p \cap x=L} \left\{ u_y^* \tilde{\sigma}_{xy} + u_z^* \tilde{\sigma}_{xz} \right\} d\Gamma - \oint_{S_w \cap x=L} \left\{ v_y^* \tilde{\sigma}^w_{xy} + v_z^* \tilde{\sigma}^w_{xz} \right\} d\Gamma \tag{2.45}$$

$$- \oint_{S_p \cap y=0} \left\{ -u_x^* \tilde{\sigma}_{xy} - u_z^* \tilde{\sigma}_{yz} \right\} d\Gamma - \oint_{S_w \cap y=0} \left\{ -v_x^* \tilde{\sigma}^w_{xy} - v_z^* \tilde{\sigma}^w_{yz} \right\} d\Gamma \tag{2.46}$$

$$- \oint_{S_p \cap y=B} \left\{ u_x^* \tilde{\sigma}_{xy} + u_z^* \tilde{\sigma}_{yz} \right\} d\Gamma - \oint_{S_w \cap y=B} \left\{ v_x^* \tilde{\sigma}^w_{xy} + v_z^* \tilde{\sigma}^w_{yz} \right\} d\Gamma, \tag{2.47}$$

where it is used that on the boundaries with Dirichlet boundary conditions, the virtual displacement should be zero. The virtual work of the inertial forces is given by

$$\delta W_{\text{in}} = \int_{\Omega_p} u_i^* \left\{ \frac{\partial^2 \rho_p u_i}{\partial t^2} + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho_p \left(\frac{\partial u_j}{\partial t} \right)^2 \right) \right\} d\Omega + \int_{\Omega_w} v_i^* \left\{ \frac{\partial^2 \rho_w v_i}{\partial t^2} + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho_w \left(\frac{\partial v_j}{\partial t} \right)^2 \right) \right\} d\Omega. \tag{2.48}$$

The principle of virtual work states that the virtual work done by the internal- and external forces should equal the virtual work done by the inertial forces [7], which results in the following equality:

$$\oint_{z=0\cap S_p}\left(u_x^*F_{xz}+u_y^*F_{yz}+u_z^*F_{zz}\right)dS+\oint_{z=0\cap S_w}\left(v_x^*F_{xz}+v_y^*F_{yz}+v_z^*F_{zz}\right)dS \tag{2.49}$$

$$+\int_{\Omega_p}u_x^*\left(\frac{\partial\tilde{\sigma}_{xx}}{\partial x}+\frac{\partial\tilde{\sigma}_{xy}}{\partial y}+\frac{\partial\tilde{\sigma}_{xz}}{\partial z}\right)+u_y^*\left(\frac{\partial\tilde{\sigma}_{xy}}{\partial x}+\frac{\partial\tilde{\sigma}_{yy}}{\partial y}+\frac{\partial\tilde{\sigma}_{yz}}{\partial z}\right)+u_z^*\left(\frac{\partial\tilde{\sigma}_{xz}}{\partial x}+\frac{\partial\tilde{\sigma}_{yz}}{\partial y}+\frac{\partial\tilde{\sigma}_{zz}}{\partial z}\right)d\Omega \tag{2.50}$$

$$+\int_{\Omega_w}v_x^*\left(\frac{\partial\tilde{\sigma}^w_{xx}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{xy}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{xz}}{\partial z}\right)+v_y^*\left(\frac{\partial\tilde{\sigma}^w_{xy}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{yy}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{yz}}{\partial z}\right)+v_z^*\left(\frac{\partial\tilde{\sigma}^w_{xz}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{yz}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{zz}}{\partial z}\right)d\Omega \tag{2.51}$$

$$-\oint_{S_p\cap z=-Z}\left\{-u_x^*\tilde{\sigma}_{xz}-u_y^*\tilde{\sigma}_{yz}\right\}d\Gamma-\oint_{S_w\cap z=-Z}\left\{-v_x^*\tilde{\sigma}^w_{xz}-v_y^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.52}$$

$$-\oint_{S_p\cap z=0}\left\{u_x^*\tilde{\sigma}_{xz}+u_y^*\tilde{\sigma}_{yz}+u_z^*\tilde{\sigma}_{zz}\right\}d\Gamma-\oint_{S_w\cap z=0}\left\{v_x^*\tilde{\sigma}^w_{xz}+v_y^*\tilde{\sigma}^w_{yz}+v_z^*\tilde{\sigma}^w_{zz}\right\}d\Gamma \tag{2.53}$$

$$-\oint_{S_p\cap x=0}\left\{-u_y^*\tilde{\sigma}_{xy}-u_z^*\tilde{\sigma}_{xz}\right\}d\Gamma-\oint_{S_w\cap x=0}\left\{-v_y^*\tilde{\sigma}^w_{xy}-v_z^*\tilde{\sigma}^w_{xz}\right\}d\Gamma \tag{2.54}$$

$$-\oint_{S_p\cap x=L}\left\{u_y^*\tilde{\sigma}_{xy}+u_z^*\tilde{\sigma}_{xz}\right\}d\Gamma-\oint_{S_w\cap x=L}\left\{v_y^*\tilde{\sigma}^w_{xy}+v_z^*\tilde{\sigma}^w_{xz}\right\}d\Gamma \tag{2.55}$$

$$-\oint_{S_p\cap y=0}\left\{-u_x^*\tilde{\sigma}_{xy}-u_z^*\tilde{\sigma}_{yz}\right\}d\Gamma+\oint_{S_w\cap y=0}\left\{-v_x^*\tilde{\sigma}^w_{xy}-v_z^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.56}$$

$$-\oint_{S_p\cap y=B}\left\{u_x^*\tilde{\sigma}_{xy}+u_z^*\tilde{\sigma}_{yz}\right\}d\Gamma-\oint_{S_w\cap y=B}\left\{v_x^*\tilde{\sigma}^w_{xy}+v_z^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.57}$$

$$=\int_{\Omega_p}u_i^*\left\{\frac{\partial^2\rho_p u_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_j}{\partial t}\right)^2\right)\right\}d\Omega+\int_{\Omega_w}v_i^*\left\{\frac{\partial^2\rho_w v_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_j}{\partial t}\right)^2\right)\right\}d\Omega \tag{2.58}$$

Because the known stress tensors are only defined on the original domain Ω, it is necessary to go back to this original domain. In this thesis, two different approaches have been attempted. One approach is based on the original model by Van Damme and den Ouden-van der Horst [15] and an alternate approach is posed by the author. In the section, the original approach will be explained. The alternate approach will be elaborated in Chapter 5.

2.7.1. Original approach

Since it is presumed that the soil particles and pore water are perfectly mixed, it is arguable that an integral over domain Ω_p equals an integral over domain Ω multiplied by the fraction $(1-p)$ of the soil particles, i.e.

$$\int_{\Omega_p}\dots d\Omega=(1-p)\int_{\Omega}\dots d\Omega, \tag{2.59}$$

$$\int_{\Omega_w}\dots d\Omega=p\int_{\Omega}\dots d\Omega, \tag{2.60}$$

Applying this to Equation (2.49) results in

$$(1-p)\int_{\Omega} u_x^*\left(\frac{\partial\tilde{\sigma}_{xx}}{\partial x}+\frac{\partial\tilde{\sigma}_{xy}}{\partial y}+\frac{\partial\tilde{\sigma}_{xz}}{\partial z}\right)+u_y^*\left(\frac{\partial\tilde{\sigma}_{xy}}{\partial x}+\frac{\partial\tilde{\sigma}_{yy}}{\partial y}+\frac{\partial\tilde{\sigma}_{yz}}{\partial z}\right)+u_z^*\left(\frac{\partial\tilde{\sigma}_{xz}}{\partial x}+\frac{\partial\tilde{\sigma}_{yz}}{\partial y}+\frac{\partial\tilde{\sigma}_{zz}}{\partial z}\right)d\Omega \tag{2.61}$$

$$+p\int_{\Omega} v_x^*\left(\frac{\partial\tilde{\sigma}^w_{xx}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{xy}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{xz}}{\partial z}\right)+v_y^*\left(\frac{\partial\tilde{\sigma}^w_{xy}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{yy}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{yz}}{\partial z}\right)+v_z^*\left(\frac{\partial\tilde{\sigma}^w_{xz}}{\partial x}+\frac{\partial\tilde{\sigma}^w_{yz}}{\partial y}+\frac{\partial\tilde{\sigma}^w_{zz}}{\partial z}\right)d\Omega \tag{2.62}$$

$$-(1-p)\oint_{S\cap z=-Z}\left\{-u_x^*\tilde{\sigma}_{xz}-u_y^*\tilde{\sigma}_{yz}\right\}d\Gamma-p\oint_{S\cap z=-Z}\left\{-v_x^*\tilde{\sigma}^w_{xz}-v_y^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.63}$$

$$-(1-p)\oint_{S\cap z=0}\left\{u_x^*(\tilde{\sigma}_{xz}-F_{xz})+u_y^*(\tilde{\sigma}_{yz}-F_{yz})+u_z^*(\tilde{\sigma}_{zz}-F_{zz})\right\}d\Gamma \tag{2.64}$$

$$-p\oint_{S\cap z=0}\left\{v_x^*(\tilde{\sigma}^w_{xz}-F_{xz})+v_y^*(\tilde{\sigma}^w_{yz}-F_{yz})+v_z^*(\tilde{\sigma}^w_{zz}-F_{zz})\right\}d\Gamma \tag{2.65}$$

$$-(1-p)\oint_{S\cap x=0}\left\{-u_y^*\tilde{\sigma}_{xy}-u_z^*\tilde{\sigma}_{xz}\right\}d\Gamma-p\oint_{S\cap x=0}\left\{-v_y^*\tilde{\sigma}^w_{xy}-v_z^*\tilde{\sigma}^w_{xz}\right\}d\Gamma \tag{2.66}$$

$$-(1-p)\oint_{S\cap x=L}\left\{u_y^*\tilde{\sigma}_{xy}+u_z^*\tilde{\sigma}_{xz}\right\}d\Gamma-p\oint_{S\cap x=L}\left\{v_y^*\tilde{\sigma}^w_{xy}+v_z^*\tilde{\sigma}^w_{xz}\right\}d\Gamma \tag{2.67}$$

$$-(1-p)\oint_{S\cap y=0}\{-u_x^*\tilde{\sigma}_{xy}-u_z^*\tilde{\sigma}_{yz}\}d\Gamma-p\oint_{S\cap y=0}\left\{-v_x^*\tilde{\sigma}^w_{xy}-v_z^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.68}$$

$$-(1-p)\oint_{S\cap y=B}\{u_x^*\tilde{\sigma}_{xy}+u_z^*\tilde{\sigma}_{yz}\}d\Gamma-p\oint_{S\cap y=B}\left\{v_x^*\tilde{\sigma}^w_{xy}+v_z^*\tilde{\sigma}^w_{yz}\right\}d\Gamma \tag{2.69}$$

$$=(1-p)\int_{\Omega} u_i^*\left\{\frac{\partial^2\rho_p u_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_j}{\partial t}\right)^2\right)\right\}d\Omega+p\int_{\Omega} v_i^*\left\{\frac{\partial^2\rho_w v_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_j}{\partial t}\right)^2\right)\right\}d\Omega. \tag{2.70}$$

To find the momentum equations for the soil particles and the pore water, extensions for the unknown stress tensors are needed.

2.7.2. An extension for unknown stress tensors $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}^w_{ij}$

In order to retrieve useful relations, an extension is needed for these stress tensors to an arbitrary domain Θ, containing both soil particles and pore water. $\tilde{\sigma}_{ij}$ should be defined such that the total energy of σ_{ij} on an arbitrary domain Θ is equivalent to the total energy of $\tilde{\sigma}_{ij}$ on $\Theta_p \subset \Theta$, i.e.

$$\int_{\Theta_p}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta=\int_{\Theta}\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}d\Theta. \tag{2.71}$$

This makes sense because of the observation that σ_{ij} should theoretically only have a contribution on the fraction of Θ containing soil particles. Even though the Einstein summation convention is used in Expression (2.71), this expression also holds element wise, since the derivation should also hold in the one-dimensional case. In order to arrive at an extension for $\tilde{\sigma}_{ij}$, it makes sense to approximate the stress tensor by averaging. Taking an infinitely small element Θ we use an averaging for the integrand $\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}$:

$$\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}\approx\frac{1}{|\Theta|}\int_{\Theta}\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}d\Theta. \tag{2.72}$$

Using Requirement (2.71) gives

$$\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}\approx\frac{1}{|\Theta|}\int_{\Theta_p}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta. \tag{2.73}$$

Since Θ (and hence Θ_p) is an infinitely small domain, the assumption is made that the integrand $\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}^p_{ij}$ is constant on this small domain. This simplifies the integral to:

$$\frac{1}{|\Theta|}\int_{\Theta_p}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta\approx\frac{|\Theta_p|}{|\Theta|}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}, \tag{2.74}$$

$$=\frac{(1-p)|\Theta|}{|\Theta|}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}, \tag{2.75}$$

$$=(1-p)\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}. \tag{2.76}$$

Because Θ is an arbitrary small region, it can be concluded that

$$\sigma_{ij} \approx (1-p)\tilde{\sigma}_{ij}, \tag{2.77}$$

on the whole domain Ω, by combining Equations (2.73) and (2.74). The same thing can be done for σ^w_{ij}, resulting in the extension

$$\sigma^w_{ij} \approx p\tilde{\sigma}^w_{ij}, \tag{2.78}$$

on the whole domain Ω. These extensions can be utilized to express the partial derivative of the unknown stress tensors in the known stress tensors. Writing out this partial derivative gives

$$\frac{\partial\tilde{\sigma}_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\sigma_{ij}}{1-p}\right), \tag{2.79}$$

$$= \frac{(1-p)\frac{\partial\sigma_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_j}\sigma_{ij}}{(1-p)^2}, \tag{2.80}$$

$$\approx \frac{(1-p)\frac{\partial\sigma_{ij}}{\partial x_j}}{(1-p)^2}, \tag{2.81}$$

$$= \frac{1}{1-p}\frac{\partial\sigma_{ij}}{\partial x_j}, \tag{2.82}$$

where the approximation is justified by the fact that the partial derivatives in space of the porosity are nearly zero, while the porosity is somewhere around 0.4. A similar thing can be done for $\frac{\partial\tilde{\sigma}^w_{ij}}{\partial x_j}$. Using the extension for the stress tensor results in:

$$\int_\Omega u^*_x\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\right)+u^*_y\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\right)+u^*_z\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)d\Omega \tag{2.83}$$

$$+p\int_\Omega v^*_x\left(\frac{\partial\sigma^w_{xx}}{\partial x}+\frac{\partial\sigma^w_{xy}}{\partial y}+\frac{\partial\sigma^w_{xz}}{\partial z}\right)+v^*_y\left(\frac{\partial\sigma^w_{xy}}{\partial x}+\frac{\partial\sigma^w_{yy}}{\partial y}+\frac{\partial\sigma^w_{yz}}{\partial z}\right)+v^*_z\left(\frac{\partial\sigma^w_{xz}}{\partial x}+\frac{\partial\sigma^w_{yz}}{\partial y}+\frac{\partial\sigma^w_{zz}}{\partial z}\right)d\Omega \tag{2.84}$$

$$-\oint_{S\cap z=-Z}\left\{-u^*_x\sigma_{xz}-u^*_y\sigma_{yz}\right\}d\Gamma-\oint_{S\cap z=-Z}\left\{-v^*_x\sigma^w_{xz}-v^*_y\sigma^w_{yz}\right\}d\Gamma \tag{2.85}$$

$$-\oint_{S\cap z=0}\left\{u^*_x(\sigma_{xz}-(1-p)F_{xz})+u^*_y(\sigma_{yz}-(1-p)F_{yz})+u^*_z(\sigma_{zz}-(1-p)F_{zz})\right\}d\Gamma \tag{2.86}$$

$$-\oint_{S\cap z=0}\left\{v^*_x(\sigma^w_{xz}-pF_{xz})+v^*_y(\sigma^w_{yz}-pF_{yz})+v^*_z(\sigma^w_{zz}-pF_{zz})\right\}d\Gamma \tag{2.87}$$

$$-\oint_{S\cap x=0}\left\{-u^*_y\sigma_{xy}-u^*_z\sigma_{xz}\right\}d\Gamma-\oint_{S\cap x=0}\left\{-v^*_y\sigma^w_{xy}-v^*_z\sigma^w_{xz}\right\}d\Gamma \tag{2.88}$$

$$-\oint_{S\cap x=L}\left\{u^*_y\sigma_{xy}+u^*_z\sigma_{xz}\right\}d\Gamma-\oint_{S\cap x=L}\left\{v^*_y\sigma^w_{xy}+v^*_z\sigma^w_{xz}\right\}d\Gamma \tag{2.89}$$

$$-\oint_{S\cap y=0}\left\{-u^*_x\sigma_{xy}-u^*_z\sigma_{yz}\right\}d\Gamma-\oint_{S\cap y=0}\left\{-v^*_x\sigma^w_{xy}-v^*_z\sigma^w_{yz}\right\}d\Gamma \tag{2.90}$$

$$-\oint_{S\cap y=B}\left\{u^*_x\sigma_{xy}+u^*_z\sigma_{yz}\right\}d\Gamma-\oint_{S\cap y=B}\left\{v^*_x\sigma^w_{xy}+v^*_z\sigma^w_{yz}\right\}d\Gamma \tag{2.91}$$

$$=(1-p)\int_\Omega u^*_i\left\{\frac{\partial^2\rho_p u_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_j}{\partial t}\right)^2\right)\right\}d\Omega+p\int_\Omega v^*_i\left\{\frac{\partial^2\rho_w v_i}{\partial t^2}+\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_j}{\partial t}\right)^2\right)\right\}d\Omega \tag{2.92}$$

Since the displacements are virtual, it should hold that:

$$\frac{\partial\sigma_{ij}}{\partial x_j}-(1-p)\frac{\partial^2\rho_p u_i}{\partial t^2}-(1-p)\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_j}{\partial t}\right)^2\right)=0, \tag{2.93}$$

$$\frac{\partial\sigma^w_{ij}}{\partial x_j}-p\frac{\partial^2\rho_w v_i}{\partial t^2}-p\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_j}{\partial t}\right)^2\right)=0, \tag{2.94}$$

for $\boldsymbol{x} \in \Omega$, which are the six momentum equations for the soil particles and the pore water respectively ($i \in \{1,2,3\}$). Furthermore, since the displacements on the boundary are virtual as well, we conclude that it has to hold that for $z = -Z$:

$$\sigma_{xz} = 0, \tag{2.95}$$
$$\sigma_{yz} = 0, \tag{2.96}$$
$$\sigma^w_{xz} = 0, \tag{2.97}$$
$$\sigma^w_{yz} = 0. \tag{2.98}$$

For $z = 0$:

$$\sigma_{xz} = (1-p)F_{xz}, \tag{2.99}$$
$$\sigma_{yz} = (1-p)F_{yz}, \tag{2.100}$$
$$\sigma_{zz} = (1-p)F_{zz}, \tag{2.101}$$
$$\sigma^w_{xz} = pF_{xz}, \tag{2.102}$$
$$\sigma^w_{yz} = pF_{yz}, \tag{2.103}$$
$$\sigma^w_{zz} = pF_{zz}. \tag{2.104}$$

For $x = 0$:

$$\sigma_{xy} = 0, \tag{2.105}$$
$$\sigma_{xz} = 0, \tag{2.106}$$
$$\sigma^w_{xy} = 0, \tag{2.107}$$
$$\sigma^w_{xz} = 0, \tag{2.108}$$

For $x = L$:

$$\sigma_{xy} = 0, \tag{2.109}$$
$$\sigma_{xz} = 0, \tag{2.110}$$
$$\sigma^w_{xy} = 0, \tag{2.111}$$
$$\sigma^w_{xz} = 0. \tag{2.112}$$

Clearly the system is heavily affected by the boundary condition at $z = 0$, since this is the boundary where the wave exerts it stress.

2.7.3. Boundary condition at $z = 0$

The boundary condition at $z = 0$ is the reason why in this research it has been decided to differentiate from the state of the art models. In the model by Ye et. al [5] it is assumed that the water carries the full load of the wave, i.e. it is presumed that for $z = 0$:

$$\sigma_{xz} = 0, \tag{2.113}$$
$$\sigma_{yz} = 0, \tag{2.114}$$
$$\sigma_{zz} = 0, \tag{2.115}$$
$$\sigma^w_{xz} = F_{xz}, \tag{2.116}$$
$$\sigma^w_{yz} = F_{yz}, \tag{2.117}$$
$$\sigma^w_{zz} = F_{zz}. \tag{2.118}$$

It is very unlikely that the soil particles on the surface will not be influenced by the wave stress, so this seems like an inaccurate way of modelling this problem. In comparison with the PORO–WSSI model, the boundary Condition (2.99) is already more probable. However even this boundary condition poses some problems, because it fixes the fraction of the wave stress being carried by the pore water. In practice it is expected that the fraction of the load being carried is not constant, but varies over time. In order to capture this by a boundary condition, the boundary conditions are summed such that the fluxes of the stresses are not fixed anymore. This means that we end up with three boundary conditions at $z = 0$, namely:

$$\sigma_{xz} + \sigma^w_{xz} = F_{xz}, \tag{2.119}$$
$$\sigma_{yz} + \sigma^w_{yz} = F_{yz}, \tag{2.120}$$
$$\sigma_{zz} + \sigma^w_{zz} = F_{zz}. \tag{2.121}$$

Because by doing this three boundary conditions are lost, additional boundary conditions have to be formulated to make the system well defined. This boundary condition will be formulated in section 2.12.1. Note that this is not the mathematically most sound way of stating things. This is exactly why the distinction is made between the original approach and the approach by the author. This more formalized approach will be discussed in Chapter 5. For now we will continue with boundary Conditions (2.119) - (2.121). To express these boundary conditions in the variables that we want to solve for, the stress tensors have to be substituted. Hence the definition of the stress tensors directly influence the resulting momentum equations, expressed in the variables. However, within the research group there was some discussion about the definition of the stress tensors, which will be elaborated on in the next section.

2.7.4. Discussion fluid tensile stress

This disagreement within the research group was mainly about the meaning of variable P in the fluid tensile stress, given by

$$\sigma^w_{ii} = \mu\left(2\frac{\partial^2 v_i}{\partial x_i \partial t} - \frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t}\right) - P. \tag{2.122}$$

One belief is that the gradient of the velocity of the pore water is negligibly small and the approximation

$$\sigma^w_{ii} = -P, \tag{2.123}$$

should be used. Applying this yields the original model, as also stated in the prior literature report [11]. This is worked out in Sections 2.7.5 - 2.13. Furthermore, this assumption is also used in Chapter 4, in which an attempt is made to solve the momentum equations directly. According to Falconer [3], this variable P is the hydrostatic pressure. This would mean that at $z = -Z$, the value of P should be $-\rho_w pgZ$ when the virtual gravitational work is included or simply 0 when the virtual gravitational work is disregarded. However, following this train of thought, P would have a known profile that only changes when the surface at $z = 0$ is being pressed down.

However, within the research group there was some skepticism towards the definition of Falconer. An idea was that P was in fact a variable for the total water pressure. This means that when the gravitational terms are disregarded, it would be 0 at $z = -Z$ (there is no pore water displacement so there cannot be any dynamic water pressure either) or $-\rho_w pgZ$ when the gravitational terms are taken into account. The difference between the previous definition of P, is simply that P does not have a known profile in this case, but is a variable.

Another opinion is that the hydrostatic pressure could be neglected when the virtual work done by the gravitational force is not taken into account. This would boil down to using

$$\sigma^w_{ii} = \mu\left(2\frac{\partial^2 v_i}{\partial x_i \partial t} - \frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t}\right), \tag{2.124}$$

as a stress tensor. This will be worked out in Chapter 5. In the next section we will continue with the derivation of the simplified momentum equation by using Assumption (2.123).

2.7.5. Simplified Momentum Equations

In state of the art models often the momentum equations are solved on the domain of interest. Directly solving the Momentum Equations on the domain will be attempted in Chapter 4. If this approach would work, it could be easily extended to a three-dimensional setting, because of the similarities between the momentum equations. However, the approach by Van Damme and Den Ouden-van der Horst [15] makes use of the curl- and divergence-operators, applied to the momentum equations. Taking the curl in a three-dimensional setting results in three non-trivial equations. However, only one of these equations can be put in a useful partial differential equation. Moreover, analysis of the intersection of a flood embankment would already provide lots of insight, so for the moment there is little incentive to implement a three-dimensional model. In a two-dimensional model, only the x-direction and z-direction have to be used, i.e. for $i = 1$ and $i = 3$.

Before applying these operators on the momentum equations, some simplifications of the momentum Equations (2.93) and (2.94) can be made. When the last term of momentum Equation (2.93) is considered, assuming that ρ_p is constant in space, we have that:

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_i}{\partial t}\right)^2\right) = \frac{1}{2}\rho_p\frac{\partial}{\partial x_i}\left(\left(\frac{\partial u_i}{\partial t}\right)^2\right), \tag{2.125}$$

$$= \rho_p\frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial x_i \partial t}, \tag{2.126}$$

$$= \rho_p\frac{\partial x_i}{\partial t}\frac{\partial^2 u_i}{\partial x_i \partial t}, \tag{2.127}$$

$$\tag{2.128}$$

in which we see an advective acceleration term. Obviously the same thing can be done for $\frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_i}{\partial t}\right)^2\right)$. Often the advective accelerations are negligible compared to the contribution of the particle-particle and particle-water interaction. Substituting the tensor, using approximation (2.123), in the momentum equation of the soil particles gives

$$\frac{\partial^2 \rho_p(1-p)u_x}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial z}\right)\right] - \rho_p(1-p)g_x \tag{2.129}$$

$$-\frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{2.130}$$

$$\frac{\partial^2 \rho_p(1-p)u_z}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial z}\right)\right] - \rho_p(1-p)g_z \tag{2.131}$$

$$+\frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0. \tag{2.132}$$

The last term of these expressions is called the Darcy term. In the next section it will be explained where this term finds its origin.

Darcy term

The Darcy term $\frac{\gamma_w}{K_s}\frac{\partial p(v_i - u_i)}{\partial t}$ appears as a result of taking the partial derivative of the $\mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_j \partial t}\right)$-term with $i = 1$ and $j = 3$. In order to understand this, we note that the pore water flows through small 'tubes'. The radii of these 'tubes' are randomly distributed, according to Van Damme [16]. Even though for a given radius the pore water velocity profile is known, the pore water velocity and hence the pore water displacements have stochastic values, because of the randomness of the tube geometry. Hence, locally the stochastic displacements are denoted by $\tilde{u}_i$ and $\tilde{v}_i$. If we observe one 'tube' in the soil, the velocity of the pore water $\dot{\tilde{v}}_x$ is parabolic, as can be seen in Figure 2.1.

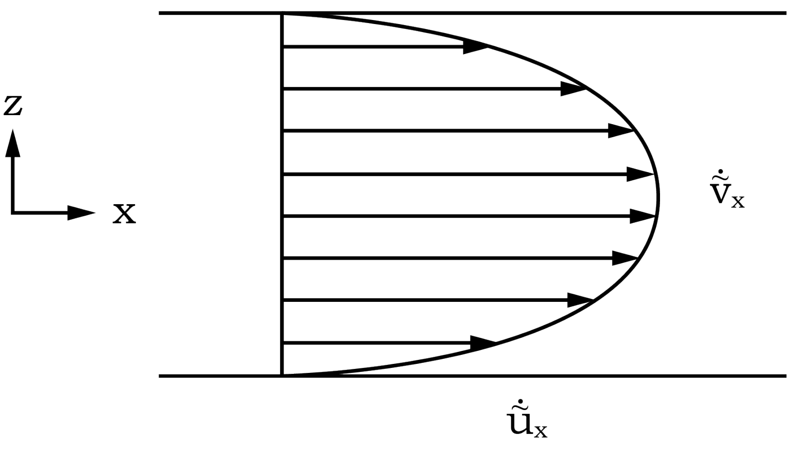


Figure 2.1: A horizontal 'tube' in the soil matrix

The stress tensors, defined in Section 2.4, are based on an infinitely small element and subsequently averaged such that they hold for the whole soil matrix. The same thing will be done for the partial derivative of the stress tensor. Taking the partial derivative to z results in

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{v}_x}{\partial z^2}+\frac{\partial^2 \tilde{v}_z}{\partial x \partial z}\right) \approx C\frac{d\bar{v}_x}{dt}, \tag{2.133}$$

since there is barely any perpendicular acceleration (it is assumed that the 'tube' nearly has a constant width, so there will hardly be any pore water moving inward) and because the second derivative of a parabolic profile is a constant. In this expression $C \in \mathbb{R}$ and $\bar{v}_x$ is the average pore water velocity in the tube, i.e. a resulting pore water velocity of a random draw from the radius distribution. The soil contains a fraction p 'tubes', that all have a parabolic profile. This profile depends on the relative velocity with respect to the tube wall. To extrapolate this local constant to an expression that is valid for the whole soil matrix, we return back to the deterministic displacements. The constant is hence proportional to $p(v_x - u_x)$. Observe that this constant is only constant in space (within one tube), not in time. Note that v_x and u_x can be seen as the expected value of the horizontal displacements in a tube. The proportionality is made explicit by introducing a calibration constant K_s, which explains the Darcy term $\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}$ at the end of the expression. This calibration constant K_s has the units $\left[\frac{m}{s}\right]$ and equals the pressure gradient after multiplication by the specific density of the pore water. Obviously the same analysis can be done for $i=3, j=1$. Hence the momentum balance equations for the soil particles become

$$\rho_p(1-p)\frac{\partial^2 u_x}{\partial t^2}-\rho_p(1-p)g_x-\frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}=0, \tag{2.134}$$

$$\rho_p(1-p)\frac{\partial^2 u_z}{\partial t^2}-\rho_p(1-p)g_z+\frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t}=0. \tag{2.135}$$

Similarly, the momentum balance equations for the pore water can be found. Note that the sign of the Darcy term should change sign, since action equals minus reaction. Hence the momentum balance equations for both the soil particles and pore water are reduced to

$$\rho_p(1-p)\frac{\partial^2 u_x}{\partial t^2}-\rho_p(1-p)g_x-\frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}=0, \tag{2.136}$$

$$\rho_p(1-p)\frac{\partial^2 u_z}{\partial t^2}-\rho_p(1-p)g_z+\frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.137}$$

$$\rho_w p\frac{\partial^2 v_x}{\partial t^2}+\rho_w p g_x+\frac{\partial P}{\partial x}+\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}=0, \tag{2.138}$$

$$\rho_w p\frac{\partial^2 v_z}{\partial t^2}+\rho_w p g_z+\frac{\partial P}{\partial z}+\frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t}=0. \tag{2.139}$$

Since the analysis has only been made for a two-dimensional setting, it is the right time to introduce the domain of interest, which can be seen in Figure 2.2. Note that Γ_1 is the boundary at $z=-Z$, Γ_2 is the boundary at $x=L$, Γ_3 is the boundary at $z=0$ and Γ_4 is the boundary at $x=0$. Both of these notations will be used interchangeably. Note that even though we are interested in displacements of soil particles and pore water, the computational domain is fixed. This is because the displacements are expected to be small. A moving computational domain could be interesting for further research, but goes beyond the scope of this thesis.

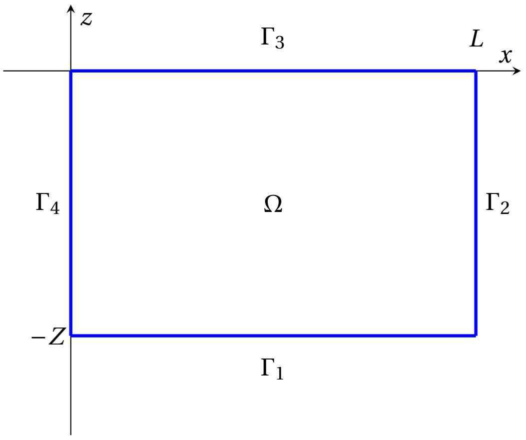


Figure 2.2: The domain of interest Ω.

To obtain the partial differential equations of the original model, the curl and divergence operators are applied on the four momentum Equations (2.136) - (2.139). The curl will be applied to find a partial differential equation for the vorticity.

2.8. Vorticity equation

Taking the curl of the Momentum Equations (2.136)-(2.137) results in

$$\frac{\partial}{\partial z}\left(\rho_p(1-p)\frac{\partial^2 u_x}{\partial t^2}-\rho_p(1-p)g_x-\frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}\right) \tag{2.140}$$

$$-\frac{\partial}{\partial x}\left(\rho_p(1-p)\frac{\partial^2 u_z}{\partial t^2}-\rho_p(1-p)g_z+\frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)-\frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t}\right)=0. \tag{2.141}$$

Working out the partial derivatives gives

$$\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial z}\right)-\frac{\partial}{\partial z}\left(\rho_p(1-p)g_x\right)-\frac{\alpha}{2}\frac{\partial^2}{\partial z^2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-(\beta+\alpha)\frac{\partial^2}{\partial z\partial x}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-\frac{\gamma_w}{K_s}\frac{\partial^2 p(v_x-u_x)}{\partial z\partial t} \tag{2.142}$$

$$-\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial x}\left(\rho_p(1-p)g_z\right)-\frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)+(\beta+\alpha)\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)+\frac{\gamma_w}{K_s}\frac{\partial^2 p(v_z-u_z)}{\partial x\partial t}=0. \tag{2.143}$$

Re-arranging the terms gives the equation

$$\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-\frac{\alpha}{2}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)\right]-\frac{\gamma_w p}{K_s}\left(\frac{\partial}{\partial t}\left[\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right]\right) \tag{2.144}$$

$$=\frac{\gamma_w p}{K_s}\left(\frac{\partial}{\partial t}\left[\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right]\right). \tag{2.145}$$

When the vorticity of the water displacement vector is defined as

$$\omega_v=\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}, \tag{2.146}$$

and the vorticity for the soil matrix is defined analogously, the constitutive equation for the vorticity can be written as

$$\rho_p(1-p)\frac{\partial^2\omega}{\partial t^2}+\frac{\gamma_w p}{K_s}\frac{\partial\omega}{\partial t}-\frac{\alpha}{2}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial z^2}\right)=\frac{\gamma_w p}{K_s}\frac{\partial\omega_v}{\partial t}. \tag{2.147}$$

Analogously a constitutive relation for ω_v can be found by taking the curl of the Momentum Balance Equations (2.138) and (2.139) of the pore water, given by:

$$\frac{\partial \rho_w p \omega_v}{\partial t^2} + \frac{\gamma_w p}{K_s}\frac{\partial \omega_v}{\partial t} = \frac{\gamma_w p}{K_s}\frac{\partial \omega}{\partial t}. \tag{2.148}$$

However, the momentum balance equation for soil states that $\omega = \omega_v$, so the only thing that remains is

$$\rho_p(1-p)\frac{\partial^2 \omega}{\partial t^2} - \frac{\alpha}{2}\Delta\omega = 0. \tag{2.149}$$

Now we have found a partial differential equation that describes the vorticity. In the next section, a partial differential equation for the volumetric strain will be derived.

2.9. Volumetric strain equation

To find a partial differential equation describing the volumetric strain, the divergence operator is used instead of the curl operator. Firstly, we define specific displacements q_x and q_z as:

$$q_x = p(v_x - u_x), \tag{2.150}$$
$$q_z = p(v_z - u_z). \tag{2.151}$$

Note that the volumetric strain, by definition, is equal to the divergence of the displacement vector [12], so in the two-dimensional case this becomes

$$\epsilon_{\text{vol}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}. \tag{2.152}$$

Taking the divergence of momentum Equations (2.136) and (2.137) gives the following equality:

$$\frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\frac{\partial u_x}{\partial x}\right) - \frac{\alpha}{2}\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} - \frac{\gamma_w}{K_s}\frac{\partial^2 q_x}{\partial x \partial t} \tag{2.153}$$

$$+\frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\frac{\partial u_z}{\partial z}\right) + \frac{\alpha}{2}\frac{\partial^2}{\partial z \partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} - \frac{\gamma_w}{K_s}\frac{\partial^2 q_z}{\partial z \partial t} = 0, \tag{2.154}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\epsilon_{\text{vol}}\right) - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) - \frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z}\right) = 0, \tag{2.155}$$

$$\Rightarrow \rho_p(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) - \frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z}\right) = 0, \tag{2.156}$$

where it is assumed that u_x and u_z are sufficiently smooth, which enables changing the order of differentiation. Substituting the volume balance Equation (2.18) results in

$$\rho_p(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) = 0. \tag{2.157}$$

This partial differential equation describes the volumetric strain over time. Finally, a relation between the volumetric strain and the water pressure will be derived.

2.10. Pressure equation

A relation for the pressure P needs to be derived as well. Taking the divergence of the momentum balance equations of the pore water gives

$$\frac{\partial}{\partial x}\left(\rho_w p\frac{\partial^2 v_x}{\partial t^2}\right)+\frac{\partial}{\partial x}\rho_w p g_x+\frac{\partial^2 P}{\partial x^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x-u_x)}{\partial t}+\frac{\partial}{\partial z}\left(\rho_w p\frac{\partial^2 v_z}{\partial t^2}\right)+\frac{\partial}{\partial z}\rho_w p g_z+\frac{\partial^2 P}{\partial z^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.158}$$

$$\Rightarrow \rho_w\frac{\partial}{\partial x}\left(\frac{\partial^2 p v_x}{\partial t^2}\right)+\frac{\partial^2 P}{\partial x^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x-u_x)}{\partial t}+\rho_w\frac{\partial}{\partial z}\left(\frac{\partial^2 p v_z}{\partial t^2}\right)+\frac{\partial^2 P}{\partial z^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.159}$$

$$\Rightarrow \rho_w\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\frac{\partial p v_x}{\partial t}+\frac{\partial}{\partial z}\frac{\partial p v_z}{\partial t}\right)+\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x-u_x)}{\partial t}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.160}$$

$$\Rightarrow -\rho_w\frac{\partial^2 p}{\partial t^2}+\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x-u_x)}{\partial t}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.161}$$

$$\Rightarrow -\rho_w\frac{\partial}{\partial t}\left[(1-p)\frac{\partial \epsilon_{\text{vol}}}{\partial t}\right]+\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x-u_x)}{\partial t}+\frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z-u_z)}{\partial t}=0, \tag{2.162}$$

$$\Rightarrow -\rho_w(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2}+\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}-\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t}=0. \tag{2.163}$$

The volume balance equation implies that

$$\frac{\partial p}{\partial t}=(1-p)\frac{\partial \epsilon_{\text{vol}}}{\partial t}, \tag{2.164}$$

with

$$\nabla\cdot\frac{\partial p v_i}{\partial t}=-\frac{\partial p}{\partial t}. \tag{2.165}$$

Equation (2.165) is used for Equality (2.161) and Equation (2.164) is used for Equality (2.162). For Equality (2.163) it is used that, when the functions are sufficiently smooth, Equation (2.18) can be rewritten as:

$$\frac{\partial}{\partial t}\left\{\left[\frac{\partial p(v_i-u_i)}{\partial x_i}\right]\right\}=-\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right)=\frac{\partial \epsilon_{\text{vol}}}{\partial t}. \tag{2.166}$$

So we end up with expression

$$-\frac{\partial^2 P}{\partial x^2}-\frac{\partial^2 P}{\partial z^2}=-\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t}-\rho_w(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2}. \tag{2.167}$$

Hence, when the solution for ϵ_{vol} is known, the pressure P can directly be determined.

2.11. Relations for the displacement

Vorticity, volumetric strain and displacements can be related by working out

$$-\frac{\partial w}{\partial z}-\frac{\partial \epsilon_{\text{vol}}}{\partial x}=-\frac{\partial^2 u_x}{\partial z^2}+\frac{\partial^2 u_z}{\partial z\partial x}-\frac{\partial^2 u_x}{\partial x^2}-\frac{\partial^2 u_z}{\partial x\partial z}, \tag{2.168}$$

$$=-\frac{\partial^2 u_x}{\partial x^2}-\frac{\partial^2 u_x}{\partial z^2}, \tag{2.169}$$

where we have assumed sufficiently smoothness.

This can be done analogously for $\frac{\partial w}{\partial x}-\frac{\epsilon_{\text{vol}}}{\partial z}$, assuming that u_x is sufficiently smooth, which results in the following set of equations:

$$\begin{cases} -\frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} & = -\frac{\partial w}{\partial z} - \frac{\partial \epsilon_{\text{vol}}}{\partial x}, \\ -\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} & = \frac{\partial w}{\partial x} - \frac{\partial \epsilon_{\text{vol}}}{\partial z}. \end{cases} \tag{2.170}$$

These relations are used as fourth and fifth equations of our system, since these expressions are more useful than the definitions of ω and ϵ_{vol}. This is because it is more straightforward for these partial differential equations to formulate a weak formulation. Now there are five partial differential equations that describe five parameters. The only thing that is left are stating the boundary conditions and initial conditions.

2.12. Boundary conditions

Since the model is based on a two-dimensional setting, the essential boundary conditions and the natural boundary conditions, found in Section 2.7, only need to be given in two dimensions as well. When the stress tensors are substituted, we have that for $z = -Z$:

$$-\frac{\alpha}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \mu\left(\frac{\partial^2 v_x}{\partial z \partial t} + \frac{\partial^2 v_z}{\partial x \partial t}\right) = 0, \tag{2.171}$$

$$\mu\left(\frac{\partial^2 v_x}{\partial z \partial t} + \frac{\partial^2 v_z}{\partial x \partial t}\right) = 0, \tag{2.172}$$

$$u_z = 0, \tag{2.173}$$

$$v_z = 0, \tag{2.174}$$

which is equivalent to saying that for $z = -Z$ we have that

$$\frac{\partial u_x}{\partial z} = 0, \tag{2.175}$$

$$\frac{\partial^2 v_x}{\partial z \partial t} = 0, \tag{2.176}$$

$$u_z = 0, \tag{2.177}$$

$$v_z = 0. \tag{2.178}$$

Hence by definition, it also holds that $\omega = 0$ for $z = -Z$.

For $x = 0$:

$$-\frac{\alpha}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \mu\left(\frac{\partial^2 v_x}{\partial z \partial t} + \frac{\partial^2 v_z}{\partial x \partial t}\right) = 0, \tag{2.179}$$

$$\mu\left(\frac{\partial^2 v_x}{\partial z \partial t} + \frac{\partial^2 v_z}{\partial x \partial t}\right) = 0, \tag{2.180}$$

$$u_x = 0, \tag{2.181}$$

$$v_x = 0, \tag{2.182}$$

which is equivalent to saying that for $x = 0$ we have that

$$\frac{\partial u_z}{\partial x} = 0, \tag{2.183}$$

$$\frac{\partial^2 v_z}{\partial x \partial t} = 0, \tag{2.184}$$

$$u_x = 0, \tag{2.185}$$

$$v_x = 0, \tag{2.186}$$

Analogously, for $x = L$ it has to hold that

$$\frac{\partial u_z}{\partial x} = 0, \tag{2.187}$$

$$\frac{\partial^2 v_z}{\partial x \partial t} = 0, \tag{2.188}$$

$$u_x = 0, \tag{2.189}$$

$$v_x = 0, \tag{2.190}$$

For $z = 0$:

$$-\alpha\frac{\partial u_z}{\partial z} - \beta\epsilon_{\text{vol}} - P = F_{zz} \tag{2.191}$$

$$-\frac{\alpha}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = F_{xz}. \tag{2.192}$$

Since the partial differential equations that were found in the previous sections do not contain water displacements v_i anymore, additional boundary conditions are needed. Furthermore, an additional boundary condition is still needed for $z = 0$, as discussed before.

2.12.1. Additional boundary conditions

For $x = 0$, $x = L$ and $z = -Z$ it is expected that, when L and Z are sufficiently large, the effect by the wave stress flattens out. This is why on these boundaries a homogeneous Neumann boundary condition is assumed for both P and ϵ_{vol}.

At $z = 0$, i.e. at the top of the levee, the boundary conditions are only related with F_{xz} and F_{zz}. Both of these functions are assumed to be known at $z = 0$ and represent the hydrodynamic loads, induced by the overtopping waves of interest. As we have seen from the virtual work, the total stress component is determined by the water pressure and the effective stress, i.e. $F_{zz} = \sigma^w_{zz} + \sigma_{zz}$. Substituting the expression for σ_{zz} gives the boundary condition

$$F_{zz}|_{z=0} = -P|_{z=0} - \beta\epsilon_{\text{vol}}|_{z=0} - \alpha\frac{\partial u_z}{\partial z}|_{z=0}. \tag{2.193}$$

There is no Darcy friction term present on the surface of the domain [19]. Since the shear stress is a weighted average of the shear stresses experienced between all soil particles in two directions, we have that

$$F_{xz}|_{z=0} = F_{zx}|_{z=0} = \frac{\alpha}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)|_{z=0}, \tag{2.194}$$

due to symmetry of the stress tensor. It can easily be checked that the boundary condition can equivalently be written as

$$F_{xz}|_{z=0} = \frac{\alpha}{2}w|_{z=0} - \alpha\frac{\partial u_x}{\partial z}|_{z=0}. \tag{2.195}$$

The momentum balance equation for the pore water in the vertical direction is given by:

$$\frac{\partial^2 \rho_p(1-p)u_z}{\partial t^2} + \frac{\partial^2 \rho_w p v_z}{\partial t^2} + \frac{\partial F_{xz}}{\partial x} = -\frac{\partial F_{zz}}{\partial z}. \tag{2.196}$$

In the analytical approach by Van Damme and Den Ouden-van der Horst [15] the accelerations cannot be ignored since this would violate the existence of an analytical solution. However, numerical analysis provides more flexibility, so the accelerations will initially be neglected. This results in the simpler relation

$$\frac{\partial F_{xz}}{\partial x} = -\frac{\partial F_{zz}}{\partial z}. \tag{2.197}$$

Substituting the partial derivatives of the other two boundary conditions at $z = 0$ and using the definition of ϵ_{vol} results in an equivalent expression for the third boundary condition at $z = 0$, given by

$$-(\alpha+\beta)\frac{\partial\epsilon_{\text{vol}}}{\partial z} + \frac{\alpha}{2}\frac{\partial\omega}{\partial x} - \frac{\partial P}{\partial z} = 0. \tag{2.198}$$

The only things left are the initial conditions of the variables.

2.13. Initial conditions

Since the partial differential equations of the model are of the second order in time, two initial conditions for both the vorticity and volumetric strain are necessary. We make the assumption that at $t = 0$ no hydrodynamic load is present on the soil. In other words, there will not be a shear stress on the surface, so the initial vorticity will be equal to zero as well, i.e. $w|_{t=0} = 0$. Furthermore, in order to create a second initial condition to the vorticity, it is assumed that the first overtopping wave will only arrive after some time. Hence the vorticity will initially not change over time, so it can be imposed that $\frac{\partial w}{\partial t}|_{t=0} = 0$. Furthermore, it is assumed that at $t = 0$ effective stresses are absent and as a consequence the volumetric strain will be zero, i.e. $\epsilon_{\text{vol}} = 0$. Since the dynamic water pressure is also assumed to be zero at $t = 0$, it has to hold that $\frac{\partial \epsilon_{\text{vol}}}{\partial t}|_{t=0} = 0$. In summary, the initial conditions are

$$w|_{t=0} = \frac{\partial w}{\partial t}|_{t=0} = 0, \tag{2.199}$$

$$\epsilon_{\text{vol}}|_{t=0} = \frac{\partial \epsilon_{\text{vol}}}{\partial t}|_{t=0} = 0. \tag{2.200}$$

2.14. Complete system

In this chapter, a full derivation of the complete system that describes the physics in the levee is given. In conclusion, the system can be written as

$$\text{for } \boldsymbol{x} \in \Omega \begin{cases} \rho_p (1-p) \frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s} \frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\alpha+\beta) \frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} - (\alpha+\beta) \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} & = 0, \\ \rho_p (1-p) \frac{\partial^2 w}{\partial t^2} - \frac{\alpha}{2} \frac{\partial^2 w}{\partial x^2} - \frac{\alpha}{2} \frac{\partial^2 w}{\partial z^2} & = 0, \\ \rho_w (1-p) \frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s} \frac{\partial \epsilon_{\text{vol}}}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} & = 0, \\ \frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} & = 0, \\ -\frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} & = 0, \end{cases} \tag{2.201}$$

With the boundary conditions

$$\text{for } x = 0 \text{ and } x = L: \begin{cases} u_x & = 0, \\ \frac{\partial u_z}{\partial x} & = 0, \\ w & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial x} & = 0, \\ \frac{\partial P}{\partial x} & = 0. \end{cases} \tag{2.202}$$

$$\text{For } z = 0: \begin{cases} \frac{\alpha}{2} \omega - \alpha \frac{\partial u_x}{\partial z} & = F_{xz}, \\ -\beta \epsilon_{\text{vol}} - \alpha \frac{\partial u_z}{\partial z} - P & = F_{zz}, \\ -(\alpha+\beta) \frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\alpha}{2} \frac{\partial \omega}{\partial x} - \frac{\partial P}{\partial z} & = 0. \end{cases} \tag{2.203}$$

$$\text{For } z = -Z: \begin{cases} u_z & = 0, \\ \frac{\partial u_x}{\partial z} & = 0, \\ \frac{\partial P}{\partial z} & = 0, \\ \omega & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial z} & = 0. \end{cases} \tag{2.204}$$

Finally, we have the initial conditions:

$$\begin{cases} \dot{\epsilon}_{\text{vol}}|_{t=0} = \epsilon_{\text{vol}}|_{t=0} & = 0, \\ \dot{\omega}|_{t=0} = \omega|_{t=0} & = 0. \end{cases} \tag{2.205}$$

Note that both $-(\alpha+\beta)$, $-\frac{\alpha}{2}$ and -1 are negative constants, which strengthens the idea that the partial differential equations are solvable. In the following section, a numerical approach will be extensively worked out in order to solve the system.

3

One-dimensional stationary solution

Since the numerical framework of the original model, extensively worked out in the literature report, did not yield any valuable results, it is assumed that the original model is either inconsistent or consistent but with infinite solutions. In order to analyse the problem of the model, an attempt is made to find a stationary solution of a simplified model. As a test case, the shear stress will be set to zero. Furthermore, when the normal stress exerted by the wave is chosen to be a function solely of time, no changes will occur in the x-direction. The derivatives with respect to x will be zero and there will not be any displacements in the x-direction either. As a result, the two-dimensional system can be reduced to a one-dimensional model, given by

$$\text{for } \boldsymbol{x} \in \Omega \begin{cases} (1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{\rho_p K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - \frac{\alpha+\beta}{\rho_p}\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} & = 0, \\ \rho_w (1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - \frac{\partial^2 P}{\partial z^2} & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial z} - \frac{\partial^2 u_z}{\partial z^2} & = 0, \end{cases} \tag{3.1}$$

with boundary conditions

$$\text{For } z = 0\text{: } \begin{cases} -\beta\epsilon_{\text{vol}} - \alpha\frac{\partial u_z}{\partial z} - P & = F_{zz}(t), \\ -(\alpha+\beta)\frac{\partial \epsilon_{\text{vol}}}{\partial z} - \frac{\partial P}{\partial z} & = 0. \end{cases} \tag{3.2}$$

$$\text{For } z = -Z\text{: } \begin{cases} u_z & = 0, \\ \frac{\partial P}{\partial z} & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial z} & = 0. \end{cases} \tag{3.3}$$

Finally, we have the initial conditions:

$$\Big\{ \dot{\epsilon}_{\text{vol}}|_{t=0} = \epsilon_{\text{vol}}|_{t=0} \quad = 0. \tag{3.4}$$

In order to do some checks, $F_{zz}(t)$ is chosen such that both the value and the gradient on $t = 0$ are equal to zero. Furthermore, for $t \to \infty$ the normal stress will be constant, say $F < 0$. The solution of the one-dimensional system should tend to a stationary solution, belonging to $F_{zz} = F$. To find this stationary solution, the stationary one-dimensional system has to be solved.

3.1. Stationary one-dimensional model

Noting that in this one-dimensional test case without a shear stress, it holds that

$$\epsilon_{\text{vol}} = \frac{\partial u_z}{\partial z}, \tag{3.5}$$

the stationary one-dimensional system is given by

$$\text{for } \boldsymbol{x} \in \Omega \begin{cases} \frac{\partial^3 u_z^S}{\partial z^3} & = 0, \\ \frac{d^2 P^S}{dz^2} & = 0, \end{cases} \tag{3.6}$$

with boundary conditions

$$\text{For } z=0: \begin{cases} -(\alpha+\beta)\frac{du_z^S}{dz} - P^S &= F, \\ -(\alpha+\beta)\frac{\partial^2 u_z^S}{\partial z^2} - \frac{dP^S}{dz} &= 0. \end{cases} \tag{3.7}$$

$$\text{For } z=-Z: \begin{cases} u_z^S &= 0, \\ \frac{\partial P^S}{\partial z} &= 0, \\ \frac{\partial^2 u_z^S}{\partial z^2} &= 0. \end{cases} \tag{3.8}$$

It can immediately be seen from the partial differential equations that

$$u_z^S = K_1 z^2 + K_2 z + K_3, \tag{3.9}$$

$$P^S = K_4 z + K_5, \tag{3.10}$$

with $K_i \in \mathbb{R}$ for $i = 1,\ldots,5$. Using the Set of boundary conditions (3.7), it has to hold that $K_1 = K_4 = 0$ and $K_3 = K_2 Z$, resulting in

$$u_z^S = K_2 z + K_2 Z, \tag{3.11}$$

$$P^S = K_5. \tag{3.12}$$

Even though the first equation of the Set of boundary conditions (3.8) relates K_2 and K_5 by

$$-(\alpha+\beta)K_2 + K_5 = F, \tag{3.13}$$

the second equation of the Set of boundary conditions (3.8) holds for all values of K_2 and K_5. This simply means that an equation is missing to solve the stationary one-dimensional problem, i.e. there are infinitely many stationary solutions. One may expect that when a system has infinitely many stationary solutions, it is most likely that the system itself has infinitely many solutions as well. This would explain why the numerical approach did not yield any good results. Other combinations of boundary conditions can be tried to see whether this would make the system well-posed.

3.2. Changing boundary conditions at $z = -Z$

One could argue that if Z is large enough, a homogeneous Dirichlet boundary condition for P or a homogeneous Dirichlet boundary condition for ϵ_{vol} are justified. Imposing these different sets of boundary conditions at least makes sure that the stationary one-dimensional system is well-posed.

3.2.1. Boundary conditions alternative 1

When the Neumann boundary condition of ϵ_{vol} is changed to a Dirichlet boundary condition, the boundary conditions are given by

$$\text{For } z=-Z: \begin{cases} u_z^S &= 0, \\ \frac{\partial P^S}{\partial z} &= 0, \\ \frac{\partial u_z^S}{\partial z} &= 0, \end{cases} \tag{3.14}$$

which results in the stationary solution

$$u_z^S = 0, \tag{3.15}$$

$$P^S = F. \tag{3.16}$$

This scenario is very unlikely, since there would be no deformation of the soil whatsoever in the limit case. Furthermore this goes against the observations that in the limit situation the pore water hardly carries any load.

When the system was solved numerically, the time integration matrix turned out to be singular, in other words this system is ill-posed.

3.2.2. Boundary conditions alternative 2

Another possibility is changing the boundary condition for P to a Dirichlet boundary condition as well. This gives the set of boundary conditions

$$\text{For } z = -Z\text{:} \begin{cases} u_z^S & = 0, \\ P^S & = 0, \\ \frac{\partial u_z^S}{\partial z} & = 0, \end{cases} \tag{3.17}$$

which results in the stationary solution

$$u_z^S = -\frac{F}{2(\alpha+\beta)(Z-1)}z^2 - \frac{FZ}{(\alpha+\beta)(Z-1)}z - \frac{FZ^2}{2(\alpha+\beta)(Z-1)}, \tag{3.18}$$

$$P^S = \frac{F}{1-Z}z + \frac{F}{1-Z}. \tag{3.19}$$

Again, this stationary solution is not very likely, since in a stationary situation we expected the stationary water pressure P^S to be zero. When the system was solved numerically, the time integration matrix turned out to be singular, in other words this system is ill-posed as well.

3.2.3. Boundary conditions alternative 3

$$\text{For } z = -Z\text{:} \begin{cases} u_z^S & = 0, \\ P^S & = 0, \\ \frac{\partial^2 u_z^S}{\partial z^2} & = 0, \end{cases} \tag{3.20}$$

which results in the statinary solution

$$u_z^S = -\frac{F}{\alpha+\beta}z - \frac{F}{\alpha+\beta}Z, \tag{3.21}$$

$$P^S = 0. \tag{3.22}$$

This is a stationary solution that would agree with observations in reality, where in a limit case the soil particles carry the load. Applying these boundary conditions to a numerical scheme yields some results. The numerical approximation will be elaborated on in the next section.

3.3. Numerical approximation

In order to make a finite element approximation, a weak formulation is needed. To obtain the weak formulation for the system, the three equations are being multiplied by test functions $\eta^{\epsilon_\text{vol}}$, η^P and η^{u_z} respectively. Subsequently the equation is integrated over the domain $-Z \le z \le 0$. The weak formulation is then given by

$$\begin{cases} \rho_p(1-p)\int_{-Z}^0 \eta^{\epsilon_\text{vol}} \frac{\partial^2 \epsilon_\text{vol}}{\partial t^2}\,dz + \frac{\gamma_w}{K_s}\int_{-Z}^0 \eta^{\epsilon_\text{vol}} \frac{\partial \epsilon_\text{vol}}{\partial t}\,dz + \eta^{\epsilon_\text{vol}}(0)\frac{\partial P}{\partial z}(0) + (\alpha+\beta)\int_{-Z}^0 \frac{\partial \eta^{\epsilon_\text{vol}}}{\partial z}\frac{\partial \epsilon_\text{vol}}{\partial z}\,dz & = 0, \\ \rho_w(1-p)\int_{-Z}^0 \eta^P \frac{\partial^2 \epsilon_\text{vol}}{\partial t^2}\,dz + \frac{\gamma_w}{K_s}\int_{-Z}^0 \eta^P \frac{\partial \epsilon_\text{vol}}{\partial t}\,dz - \eta^P(0)\frac{\partial P}{\partial z}(0) + \int_{-Z}^0 \frac{\partial \eta^P}{\partial z}\frac{\partial P}{\partial z}\,dz & = 0, \\ \int_{-Z}^0 \eta^{u_z}\frac{\partial \epsilon_\text{vol}}{\partial z}\,dz + \frac{\eta^{u_z}(0)P(0)}{\alpha+\beta} + \int_{-Z}^0 \frac{\partial \eta^{u_z}}{\partial z}\frac{\partial u_z}{\partial z}\,dz & = -\frac{\eta^{u_z}(0)F(t)}{\alpha+\beta}. \end{cases} \tag{3.23}$$

Substituting the Galerkin approximations for the variables, given by

$$\epsilon_\text{vol} = \sum_{j=1}^n a_j(t)\eta_j(z), \tag{3.24}$$

$$P = \sum_{j=1}^n b_j(t)\eta_j(z), \tag{3.25}$$

$$u_z = \sum_{j=1}^n c_j(t)\eta_j(z), \tag{3.26}$$

results in the following Galerkin equations:

$$
\begin{cases}
\sum_{j=1}^{n}\left\{\ddot{a}_j(t)\rho_p(1-p)\int_{-Z}^{0}\eta_i\eta_j dz+\dot{a}_j(t)\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j dz+b_j(t)\eta_i(0)\frac{\partial\eta_j}{\partial z}(0)+a_j(t)(\alpha+\beta)\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}dz\right\} & =0,\\
\sum_{j=1}^{n}\left\{\ddot{a}_j(t)\rho_w(1-p)\int_{-Z}^{0}\eta_i\eta_j dz+\dot{a}_j(t)\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j dz-b_j(t)\eta_i(0)\frac{\partial\eta_j}{\partial z}(0)+b_j(t)\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}dz\right\} & =0,\\
\sum_{j=1}^{n}\left\{a_j(t)\int_{-Z}^{0}\eta_i\frac{\partial\eta_j}{\partial z}dz+b_j(t)\frac{\eta_i(0)\eta_j(0)}{\alpha+\beta}+c_j(t)\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}dz\right\} & =-\frac{\eta_i(0)F(t)}{\alpha+\beta}.
\end{cases}
\tag{3.27}
$$

Writing this in matrix form results in

$$M_{aa}\ddot{\boldsymbol{a}}(t)+W_{aa}\dot{\boldsymbol{a}}(t)+S_{aa}\boldsymbol{a}(t)+S_{ab}\boldsymbol{b}(t)=\boldsymbol{0}, \tag{3.28}$$

$$M_{ba}\ddot{\boldsymbol{a}}(t)+W_{ba}\dot{\boldsymbol{a}}(t)+S_{bb}\boldsymbol{b}(t)=\boldsymbol{0}, \tag{3.29}$$

$$S_{ca}\boldsymbol{a}(t)+S_{cb}\boldsymbol{b}(t)+S_{cc}\boldsymbol{c}(t)=\boldsymbol{f}(t), \tag{3.30}$$

where the element matrices are given by

$$M_{aa}^{e_k}=\rho_p(1-p)\int_{e_k}\eta_i\eta_j d\Omega, \tag{3.31}$$

$$W_{aa}^{e_k}=\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j d\Omega, \tag{3.32}$$

$$S_{aa}^{e_k}=(\alpha+\beta)\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}d\Omega, \tag{3.33}$$

$$M_{ba}^{e_k}=\rho_w(1-p)\int_{e_k}\eta_i\eta_j d\Omega, \tag{3.34}$$

$$W_{ba}^{e_k}=\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j d\Omega, \tag{3.35}$$

$$S_{bb}^{e_k}=\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}d\Omega, \tag{3.36}$$

$$S_{ca}^{e_k}=\int_{e_k}\eta_i\frac{\partial\eta_j}{\partial z}d\Omega, \tag{3.37}$$

$$S_{cc}^{e_k}=\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}d\Omega, \tag{3.38}$$

where e_k denotes an internal element, i.e. $e_k=[z_{k-1},z_k]$ and $(i,j)\in\{k-1,k\}^2$.. Computing these element matrices, using the Theorem by Holand et al. [4] from Appendix A gives

$$M_{aa}^{e_k}=\rho_p(1-p)\frac{z_k-z_{k-1}}{6}(1+\delta_{ij}), \tag{3.39}$$

$$W_{aa}^{e_k}=\frac{\gamma_w}{K_s}\frac{z_k-z_{k-1}}{6}(1+\delta_{ij}), \tag{3.40}$$

$$S_{aa}^{e_k}=\frac{\alpha+\beta}{z_k-z_{k-1}}(-1+2\delta_{ij}), \tag{3.41}$$

$$M_{ba}^{e_k}=\rho_w(1-p)\frac{z_k-z_{k-1}}{6}(1+\delta_{ij}), \tag{3.42}$$

$$W_{ba}^{e_k}=\frac{\gamma_w}{K_s}\frac{z_k-z_{k-1}}{6}(1+\delta_{ij}), \tag{3.43}$$

$$S_{bb}^{e_k}=\frac{1}{z_k-z_{k-1}}(-1+2\delta_{ij}), \tag{3.44}$$

$$S_{ca}^{e_k}=\frac{1}{2}(-1+2\delta_{j,k}), \tag{3.45}$$

$$S_{cc}^{e_k}=\frac{1}{z_k-z_{k-1}}(-1+2\delta_{ij}), \tag{3.46}$$

where $(i,j)\in\{k-1,k\}^2$.

The element vector is equal to zero. The boundary element matrix is given by

$$S_{ab}^{be_k} = \eta_i(0)\frac{\partial \eta_j}{\partial z}, \tag{3.47}$$

$$S_{bb}^{be_k} = -\eta_i(0)\frac{\partial \eta_j}{\partial z}, \tag{3.48}$$

$$S_{cb}^{be_k} = \frac{\eta_i(0)\eta_j(0)}{\alpha+\beta}, \tag{3.49}$$

$$\tag{3.50}$$

where $i = n, j \in \{n-1, n\}$. Computing these values gives

$$S_{ab}^{be_k} = \frac{1}{z_k - z_{k-1}}(2\delta_{j,n} - 1), \tag{3.51}$$

$$S_{bb}^{be_k} = -\frac{1}{z_k - z_{k-1}}(2\delta_{j,n} - 1), \tag{3.52}$$

$$S_{cb}^{be_k} = \frac{1}{\alpha+\beta}\delta_{j,n}, \tag{3.53}$$

$$\tag{3.54}$$

where $i = n, j \in \{n-1, n\}$. The boundary element vector is simply given by

$$f^{be_k} = -\frac{\eta_i(0)F_{zz}(t)}{\alpha+\beta}, \tag{3.55}$$

or computed

$$f_n^{be_k} = -\frac{F_{zz}(t)}{\alpha+\beta}. \tag{3.56}$$

By introducing $\boldsymbol{\chi}(t) = \begin{pmatrix}\boldsymbol{a}(t)\\ \dot{\boldsymbol{a}}(t)\end{pmatrix}$ and $\boldsymbol{\theta}(t) = \begin{pmatrix}\boldsymbol{b}(t)\\ \boldsymbol{c}(t)\end{pmatrix}$, the system can equivalently be written as:

$$M_{\chi\chi}\dot{\boldsymbol{\chi}}(t) = S_{\chi\chi}\boldsymbol{\chi}(t) + S_{\chi\theta}\boldsymbol{\theta}(t), \tag{3.57}$$

$$S_{\theta\chi}\boldsymbol{\theta}(t) = M_{\theta\chi}\dot{\boldsymbol{\chi}}(t) + \boldsymbol{f}_\theta(t) + \tilde{\boldsymbol{f}}, \tag{3.58}$$

where

$$M_{\chi\chi} = \begin{pmatrix} I & \varnothing \\ W_{aa} & M_{aa}\end{pmatrix}, \tag{3.59}$$

$$S_{\chi\chi} = \begin{pmatrix} \varnothing & I \\ -S_{aa} & \varnothing\end{pmatrix}, \tag{3.60}$$

$$S_{\chi\theta} = \begin{pmatrix} \varnothing & \varnothing \\ -S_{ab} & \varnothing\end{pmatrix}, \tag{3.61}$$

$$S_{\theta\chi} = \begin{pmatrix} S_{bb} & \varnothing \\ S_{cb} & S_{cc}\end{pmatrix}, \tag{3.62}$$

$$M_{\theta\chi} = \begin{pmatrix} -W_{ba} & -M_{ba} \\ -S_{ca} & \varnothing\end{pmatrix}, \tag{3.63}$$

$$\boldsymbol{f}_\theta(t) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f}(t)\end{pmatrix}. \tag{3.64}$$

Note that $\tilde{\boldsymbol{f}}(t)$ can be introduced to include a inhomogeneous Dirichlet boundary condition at $z = -Z$ for variable P. Now that the numerical system is stated as a time-dependent system and a quasi-time-dependent system, a time-stepping method can be implemented.

Dirichlet boundary conditions

The two Dirichlet boundary conditions at $z = -Z$ are imposed in the numerical scheme by setting the corresponding rows of matrices $S_{\theta\chi}, M_{\theta\chi}$ and $\tilde{f}(t)$ to zero rows and subsequently putting pivots in these rows of matrix $S_{\chi\theta}$. The only thing that is left is a numerical integration of the system.

3.3.1. Numerical integration

Since at this stage the mere concern is whether a numerical solution exists, accuracy is of minor importance. Hence a first-order time-integration method should suffice for now. In order to not have to worry about the size of a time step, an unconditionally stable method is preferred. The Backward Euler method is hence a suitable candidate. Applying Backward Euler on the time-dependent system gives

$$M_{\chi\chi}\boldsymbol{\chi}^{n+1} = M_{\chi\chi}\boldsymbol{\chi}^{n} + \Delta t S_{\chi\chi}\boldsymbol{\chi}^{n+1} + \Delta t S_{\chi\theta}\boldsymbol{\theta}^{n+1} \tag{3.65}$$

Evaluating the quasi-time-dependent system on $t = t_{n+1}$ simply gives the relation

$$S_{\theta\chi}\boldsymbol{\theta}^{n+1} = M_{\theta\chi}\dot{\boldsymbol{\chi}}^{n+1} + \tilde{\boldsymbol{f}}^{n+1}. \tag{3.66}$$

Moreover, when the time-dependent-system is also evaluated at $t = t_{n+1}$ and it is assumed that matrix $M_{\chi\chi}$ is invertible, it holds that

$$\dot{\boldsymbol{\chi}}^{n+1} = M_{\chi\chi}^{-1}S_{\chi\chi}\boldsymbol{\chi}^{n+1} + M_{\chi\chi}^{-1}S_{\chi\theta}\boldsymbol{\theta}^{n+1}. \tag{3.67}$$

Combining equations (3.66) and (3.67) results in

$$S_{\theta\chi}\boldsymbol{\theta}^{n+1} = M_{\theta\chi}M_{\chi\chi}^{-1}S_{\chi\chi}\boldsymbol{\chi}^{n+1} + M_{\theta\chi}M_{\chi\chi}^{-1}S_{\chi\theta}\boldsymbol{\theta}^{n+1} + \tilde{\boldsymbol{f}}^{n+1} \tag{3.68}$$

The Equalities (3.65) and (3.68) can be written as one numerical scheme:

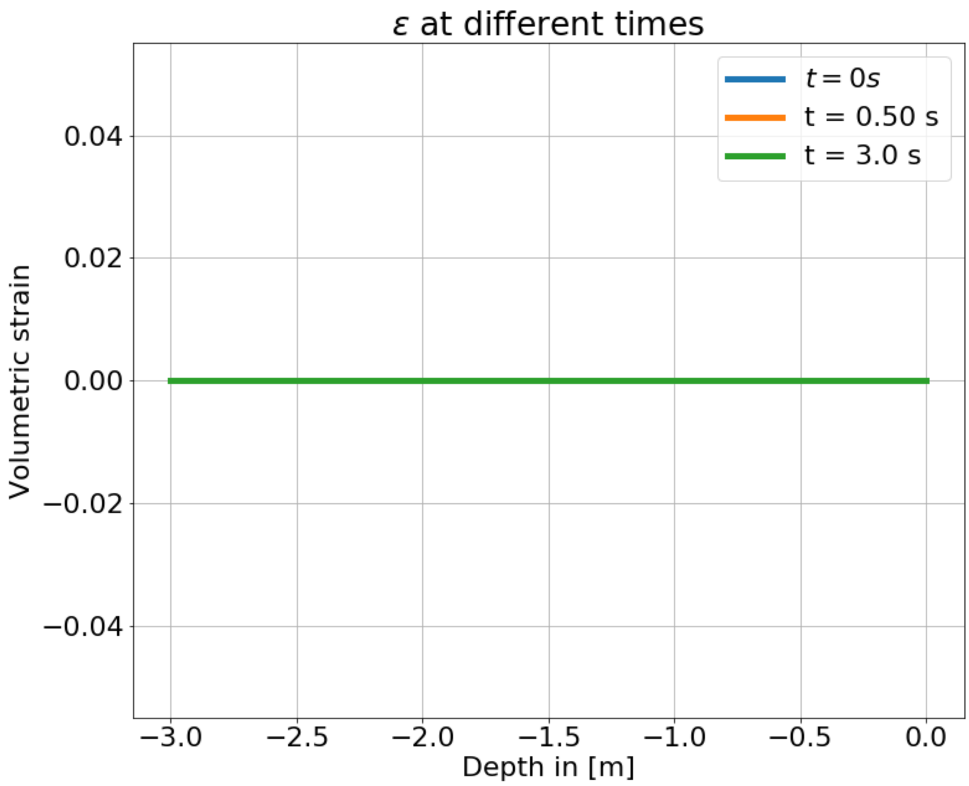
$$\begin{pmatrix} M_{\chi\chi} - \Delta t S_{\chi\chi} & -\Delta t S_{\chi\theta} \\ M_{\theta\chi}M_{\chi\chi}^{-1}S_{\chi\chi} & M_{\theta\chi}M_{\chi\chi}^{-1}S_{\chi\theta} - S_{\theta\chi} \end{pmatrix} \begin{pmatrix} \chi^{n+1} \\ \theta^{n+1} \end{pmatrix} = \begin{pmatrix} M_{\chi\chi} & \varnothing \\ \varnothing & \varnothing \end{pmatrix} \begin{pmatrix} \chi^{n} \\ \theta^{n} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -\tilde{\boldsymbol{f}}^{n+1} \end{pmatrix}. \tag{3.69}$$

It is of interest whether the system approaches the stationary solution, which was found by analysing the stationary system. Hence $F_{zz}(t)$, the perpendicular stress exerted on the surface, is chosen to be a trigonometric function that stays constant once it reaches its maximum absolute value, i.e. Hence $F_{zz}(t)$ needs to be equal to a constant, say F_{zz}, for $t \to \infty$, but also needs to satisfy the initial conditions. The wave stress is thus chosen to be:

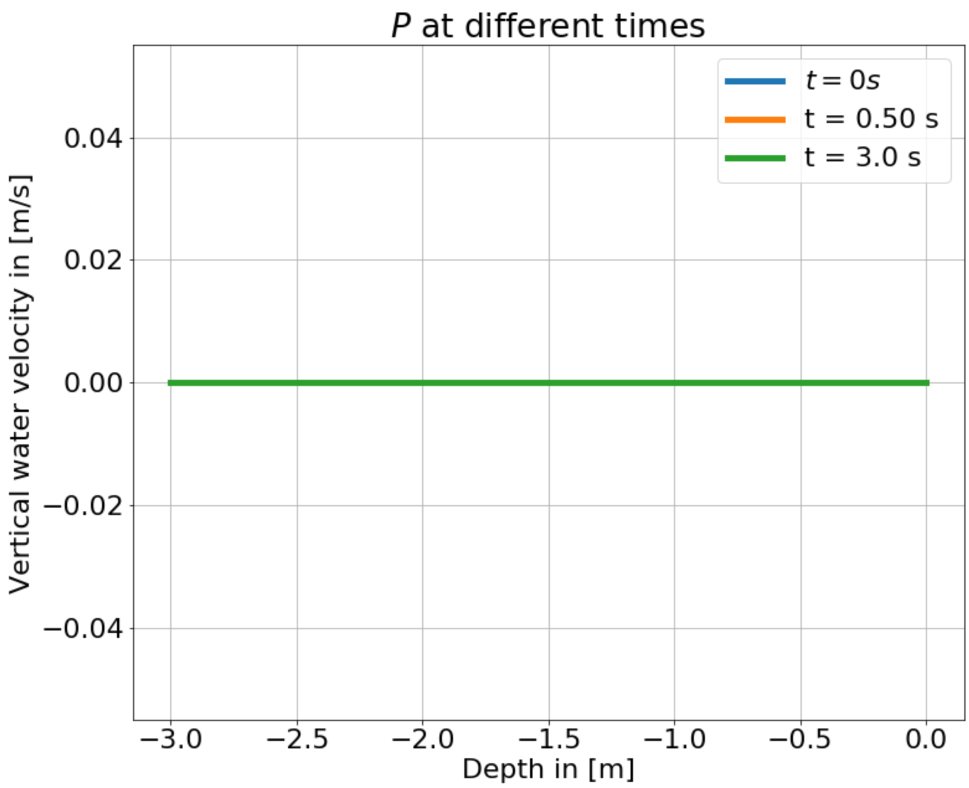
$$F(t) = \begin{cases} -F_{zz}(1 - \cos(t)) & \text{for } t < \pi \\ -2F_{zz} & \text{for } t \geq \pi \end{cases} \tag{3.70}$$

where the value of F_{zz} is chosen as $F_{zz} = 10^4$. Note that $F(t) \leq 0$ for $t \geq 0$, since the wave is exerting a downward stress.

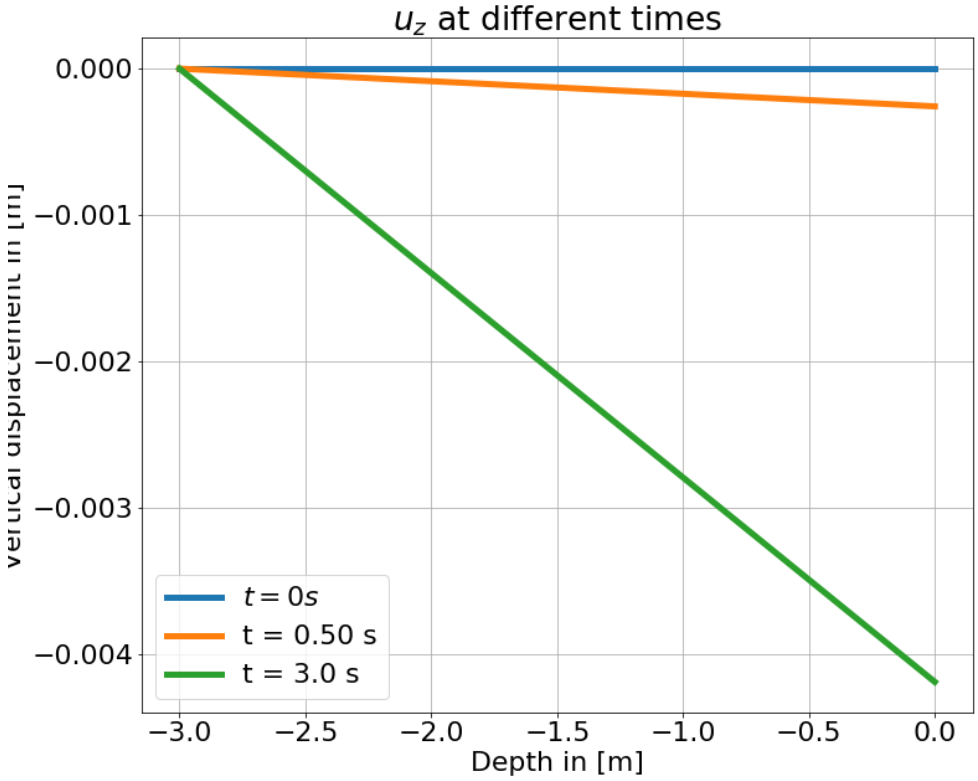
The three variables ϵ_{vol}, P and u_z are plotted at three times: the starting time, at a time where the wave has just commenced, $t = 0.5$, and a moment where the stress has already reached its peak, $t = 3.0$. The number of integration points is chosen to be $n = 1000$ and the time step size is chosen as $\Delta t = 0.01$.



(a) ϵ_{vol} at different times



(b) P at different times



(c) u_z at different times

Figure 3.1: The variables at times $t = 0$, $t = 0.5$ and $t = 3.0$.

The numerical results, depicted in Figure 3.1, show that once again the pore water hardly carries any load, even when the wave stress just starts exerting its stress on the soil. As a matter of fact it remains zero throughout the whole simulation, which is why only one line can be seen. This does not agree with findings from wave overtopping simulations. It can however be seen that the numerical solution indeed reaches the stationary solution. Thus it is likely that there is not something wrong with the solving of the system, but probably with the system itself, since it fails to capture reality in a proper manner. It is important to note that imposing a inhomogeneous Dirichlet boundary condition for P at $z = -Z$ yields a similar unrealistic result. In the next section, an attempt is made to solve the momentum equations for both the soil particles and the pore water directly, without applying the divergence- and curl operator first.

4

Balance of volume

A logical approach for solving the momentum Equations (2.137) and (2.139) is assuming that the volume balance (2.18) holds and deriving a direct relation between u_z and V_z. Since the one-dimensional volume balance states that

$$\frac{\partial}{\partial z}\left\{p\left[V_z - \frac{\partial u_z}{\partial t}\right]\right\} + \frac{\partial}{\partial z}\frac{\partial u_z}{\partial t} = 0, \tag{4.1}$$

integration with respect to z gives

$$pV_z - p\frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial t} = A(t), \tag{4.2}$$

$$\Rightarrow V_z = \frac{A(t)}{p} - \frac{1-p}{p}\frac{\partial u_z}{\partial t}. \tag{4.3}$$

where $A(t)$ is constant in z, but can depend on time. However, since we have homogeneous Dirichlet boundary conditions for u_z and v_z on $z = -Z$, it should hold that

$$\frac{\partial u_z}{\partial t}(t, -Z) = 0, \tag{4.4}$$

$$V_z(t, -Z) = 0. \tag{4.5}$$

When evaluating Equation (4.3) on $z = -Z$ and for an arbitrary time, this means that

$$V_z(t, -Z) = \frac{A(t)}{p} - \frac{1-p}{p}\frac{\partial u_z}{\partial t}(t, -Z), \tag{4.6}$$

$$\Rightarrow 0 = \frac{A(t)}{p} - \frac{1-p}{p}\cdot 0, \tag{4.7}$$

which can only hold when $A(t) = 0$ for all $t \geq 0$. The direct relation between u_z and V_z is hence

$$V_z(t, z) = -\frac{1-p}{p}\frac{\partial u_z}{\partial t}(t, z). \tag{4.8}$$

By substituting this expression, a new system can be formulated, only using variables u_z and P. Note that this can only be done in one dimension, since in multiple dimensions the volume balance does not provide enough information for a direct, useful substitution.

4.1. Resulting system from substitution

Substituting Relation (4.8) in the momentum Equations (2.137) and (2.139) gives the new system

$$(1-p)\rho_p \frac{\partial^2 u_z}{\partial t^2} - (\alpha+\beta)\frac{\partial^2 u_z}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial u_z}{\partial t} = 0, \tag{4.9}$$

$$-(1-p)\rho_w \frac{\partial^2 u_z}{\partial t^2} - \frac{dP}{dz} - \frac{\gamma_w}{K_s}\frac{\partial u_z}{\partial t} = 0, \tag{4.10}$$

with Dirichlet boundary conditions

$$u_z(t,-Z) = 0, \tag{4.11}$$

$$P(t,-Z) = -\rho_w p g Z. \tag{4.12}$$

On $z=0$ we have that

$$-P - (\alpha+\beta)\frac{\partial u_z}{\partial z} = F_{zz}(t), \tag{4.13}$$

where $F_{zz}(t)$ is the perpendicular stress exerted on the surface. For the test case, the same Function (3.70) is used. In the next sections, two different approaches will be analysed to attempt to solve the resulting system analytically.

4.2. Analytical solution 1

An analytical solution can be found by carrying out a set of steps, as will be done in the following subsections. However, some notes regarding boundary- and initial conditions will be made.

4.2.1. Assuming $P(t,z)$ to be known

In order to analytically solve this, we first assume $P(t,z)$ to be a known function, which enables the solving of $u(t,z)$ in terms of $P(t,z)$ and $F_{zz}(t,z)$. Subsequently an expression for $P(t,z)$ will be found, using the second partial differential equation. Hence on boundary $z=0$, we have boundary condition

$$(\alpha+\beta)\frac{\partial u_z}{\partial z}(t,0) = P(t,0) + F_{zz}(t). \tag{4.14}$$

Lastly, we should consider the initial conditions, which are given by

$$u(0,z) = 0, \tag{4.15}$$

$$\frac{\partial u_z}{\partial t}(0,z) = 0. \tag{4.16}$$

In order to solve this system, the quasi-stationary problem needs to be solved first.

4.2.2. Quasi-stationary problem

The quasi-stationary problem is given by:

$$-(\alpha+\beta)\frac{\partial^2 u_z^S}{\partial z^2} = 0, \tag{4.17}$$

with $u_z^S = u_z^S(t,z)$ for $t \geq 0, -Z \leq z \leq 0$ subject to

$$u_z^S(t,-Z) = 0, \tag{4.18}$$

$$(\alpha+\beta)\frac{\partial u_z^S}{\partial z}(t,0) = P(t,0) + F_{zz}(t). \tag{4.19}$$

The solution of this problem is linear in z and given by

$$u_z^S(t,z) = \frac{P(t,0) + F_{zz}(t)}{\alpha+\beta}(z+Z). \tag{4.20}$$

The next step is to formulate the transient problem.

4.2.3. Transient problem

We introduce:

$$u_z(t,z) = u_z^S(t,z) + u_z^T(t,z), \tag{4.21}$$

which results in

$$(1-p)\rho_p \frac{\partial^2 u_z^T}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial u_z^T}{\partial t} - (\alpha+\beta)\frac{\partial^2 u_z^T}{\partial z^2} = -(1-p)\rho_p \frac{\partial^2 u_z^S}{\partial t^2} - \frac{\gamma_w}{K_s}\frac{\partial u_z^S}{\partial t}, \tag{4.22}$$

and conditions

$$u_z^T(0,z) = -u_z^S(0,z), \tag{4.23}$$

$$= -\frac{P(0,z)+F_{zz}(0)}{\alpha+\beta}(z+Z), \tag{4.24}$$

$$= 0, \tag{4.25}$$

$$\frac{\partial u_z^T(0,z)}{\partial t} = -\frac{u_z^S}{\partial t}(0,z), \tag{4.26}$$

$$= -\frac{\frac{P(0,z)}{\partial t} + F'_{zz}(0)}{\alpha+\beta}(z+Z), \tag{4.27}$$

$$= 0, \tag{4.28}$$

$$u_z^T(t,-Z) = 0, \tag{4.29}$$

$$u_z^T(t,0) = 0. \tag{4.30}$$

Since it is expected that the shape of the transient solution and the shape of the homogeneous transient solution are the same, the homogeneous transient problem will be analysed.

4.2.4. Homogeneous transient problem

The homogeneous transient problem is given by

$$(1-p)\rho_p \frac{\partial^2 u_z^{HT}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial u_z^{HT}}{\partial t} - (\alpha+\beta)\frac{\partial^2 u_z^{HT}}{\partial z^2} = 0, \tag{4.31}$$

with conditions

$$u_z^{HT}(0,z) = 0, \tag{4.32}$$

$$\frac{\partial u_z^{HT}(0,z)}{\partial t} = 0, \tag{4.33}$$

$$u_z^{HT}(t,-Z) = 0, \tag{4.34}$$

$$u_z^{HT}(t,0) = 0 \tag{4.35}$$

An attempt will be made to analytically solve this system by using separation of variables.

4.2.5. Separation of variables

The assumption is made that u_z^{HT} can be expressed as a multiplication of functions that depend on only one variable, i.e.

$$u_z^{HT} = Z_1(z)T_1(t). \tag{4.36}$$

These expressions are substituted in the boundary conditions.

Boundary conditions

Substitution in the boundary conditions gives

$$Z_1'(0)T_1(t) = 0, \tag{4.37}$$

$$Z_1(-Z)T_1(t) = 0, \tag{4.38}$$

for $t \geq 0$. Since we are not interested in trivial solutions, it has to hold that

$$Z_1'(0) = 0, \tag{4.39}$$

$$Z_1(-Z) = 0. \tag{4.40}$$

The separation of variables will now be applied to the momentum equation of the soil particles.

4.2.6. Momentum equation soil

Substitution of Expression (4.103) in the momentum Equation (4.9) for the soil particles gives

$$(1-p)\rho_p Z_1(z)T_1(t)'' - (\alpha+\beta)Z_1''(z)T_1(t) + \frac{\gamma_w}{K_s}Z_1(z)T_1(t)' = 0. \tag{4.41}$$

Dividing this equation by $Z_1(z)T_1(t)$ gives

$$(1-p)\rho_p \frac{T_1(t)''}{T_1(t)} - (\alpha+\beta)\frac{Z_1''(z)}{Z_1(z)} + \frac{\gamma_w}{K_s}\frac{T_1(t)'}{T_1(t)} = 0. \tag{4.42}$$

For this to hold, there should be a constant, say $\lambda \in \mathbb{R}$, such that

$$(1-p)\rho_p \frac{T_1(t)''}{T_1(t)} + \frac{\gamma_w}{K_s}\frac{T_1(t)'}{T_1(t)} = -\lambda, \tag{4.43}$$

$$-(\alpha+\beta)\frac{Z_1''(z)}{Z_1(z)} = \lambda, \tag{4.44}$$

which is equivalent to saying

$$(1-p)\rho_p T_1(t)'' + \frac{\gamma_w}{K_s}T_1(t)' + \lambda T_1(t) = 0, \tag{4.45}$$

$$-(\alpha+\beta)Z_1''(z) - \lambda Z_1(z) = 0. \tag{4.46}$$

The differential equation for $Z_1(z)$ will be analysed, where a distinction is made between three cases: $\lambda = 0$, $\lambda = \mu^2$ and $\lambda = -\mu^2$ with $\mu > 0$.

Case 1: $\lambda = 0$

The ordinary differential equation reduces to:

$$Z_1''(z) = 0, \tag{4.47}$$

which has the general solution

$$Z_1(z) = C_1 z + C_2, \tag{4.48}$$

where $C_1, C_2 \in \mathbb{R}$. However, the only values that satisfy the boundary Conditions (4.104) and (4.105) for $Z_1(z)$ are $C_1 = C_2 = 0$. This would mean that $u_z^{HT}T(t,z) \equiv 0$, the trivial solution.

Case 2: $\lambda = -\mu^2$

The general solution for the differential equation is

$$Z_1(z) = C_3 e^{\frac{\mu z}{\sqrt{\alpha+\beta}}} + C_4 e^{-\frac{\mu z}{\sqrt{\alpha+\beta}}} \tag{4.49}$$

with $C_3, C_4 \in \mathbb{R}$. The boundary conditions give the relations

$$C_3 - C_4 = 0, \tag{4.50}$$

$$C_3 e^{-\frac{\mu Z}{\sqrt{\alpha+\beta}}} + C_4 e^{\frac{\mu Z}{\sqrt{\alpha+\beta}}} = 0, \tag{4.51}$$

which can only hold for $C_3 = C_4 = 0$ and hence this case does not yield a non-trivial solution.

Case 3: $\lambda = \mu^2$

The general solution for the differential equation is

$$Z_1(z) = C_5 \cos\left(\frac{\mu z}{\sqrt{\alpha+\beta}}\right) + C_6 \sin\left(\frac{\mu z}{\sqrt{\alpha+\beta}}\right) \tag{4.52}$$

with $C_5, C_6 \in \mathbb{R}$. The boundary conditions give that

$$C_6 = 0, \tag{4.53}$$

$$C_5 \cos\left(\frac{-\mu Z}{\sqrt{\alpha+\beta}}\right) = 0. \tag{4.54}$$

Since for a non-trivial solution it is required that $C_5 \neq 0$, it has to hold that

$$\cos\left(\frac{-\mu Z}{\sqrt{\alpha+\beta}}\right) = 0, \tag{4.55}$$

which is the case for

$$\frac{-\mu Z}{\sqrt{\alpha+\beta}} = \frac{\pi}{2} + k\pi, \tag{4.56}$$

$$\Rightarrow \mu_k = \frac{\pi\sqrt{\alpha+\beta}}{2Z} + k\frac{\pi\sqrt{\alpha+\beta}}{Z}, \tag{4.57}$$

where $k \in \mathbb{N}_0$. Hence the solution of the homogeneous transient problem is of the form

$$u_z^{HT}(t,z) = \sum_{k=0}^{\infty} T_k^H(t) \cos\left(\left(\frac{\pi}{2Z} + k\frac{\pi}{Z}\right)z\right), \tag{4.58}$$

where functions $T_k^H(t)$ are yet to be determined. However, these functions are not important, since only the form of function $u_z^{HT}(t,z)$ will be used, not the actual solution.

4.2.7. Solution of transient problem

The Ansatz is used that the transient solution should have the same form as u_z^{HT}, giving

$$u_z^T(t,z) = \sum_{k=0}^{\infty} T_k(t) \cos\left(\left(\frac{\pi}{2Z} + k\frac{\pi}{Z}\right)z\right) \tag{4.59}$$

Substitution in the Equations (4.89) gives

$$\sum_{k=0}^{\infty} \left(\rho_p(1-p)T_k''(t) + \frac{\gamma_w}{K_s} T_k'(t) + (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2 T_k(t)\right) \cos\left(\frac{\pi+2k\pi}{2Z}z\right) = h(t,z), \tag{4.60}$$

where $h(t,z)$ is the notation for the right hand side of the transient problem, i.e.

$$h(t,z) = -(1-p)\rho_p \frac{\partial^2 u_z^S}{\partial t^2} - \frac{\gamma_w}{K_s}\frac{\partial u_z^S}{\partial t}. \tag{4.61}$$

Note that the cosine functions are orthogonal. Furthermore, it holds that

$$\int_{-Z}^{0} \cos^2\left(\frac{\pi+2k\pi}{2Z}z\right)^2 dz = \frac{Z}{2}, \tag{4.62}$$

for all $k \in \mathbb{N}_0$. Hence, multiplying Equation (4.127) by $\cos\left(\frac{\pi+2l\pi}{2Z}z\right)$ and integrating over z from $z=-Z$ to $z=0$, gives that

$$\frac{Z}{2}\left(\rho_p(1-p)T_k''(t) + \frac{\gamma_w}{K_s}T_k'(t) + (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2 T_k(t)\right) = \int_{-Z}^{0} h(t,z)\cos\left(\frac{\pi+2k\pi}{2Z}z\right)dz, \tag{4.63}$$

for $k \in \mathbb{N}_0$. For convenience we now define

$$D_1 = \rho_p(1-p), \tag{4.64}$$

$$D_2 = \frac{\gamma_w}{K_s}, \tag{4.65}$$

$$D_{3,k} = (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2, \tag{4.66}$$

$$h_k(t) = \frac{2}{Z}\int_{-Z}^{0} h(t,z)\cos\left(\frac{\pi+2k\pi}{2Z}z\right)dz. \tag{4.67}$$

Using this short notation, the resulting ordinary differential equation can be written as

$$D_1 T_k''(t) + D_2 T_k'(t) + D_{3,k} T_k(t) = h_k(t), \tag{4.68}$$

$$T_k(0) = 0, \tag{4.69}$$

$$T_k'(0) = 0. \tag{4.70}$$

The determinant of the characteristic equation is given by

$$D_k = D_2^2 - 4D_1 D_{3,k}. \tag{4.71}$$

Three distinct cases should be considered: $D<0$, $D=0$ and $D>0$.

Case 1: $D_k < 0$

For $D_k < 0$ the solution for $T_k(t)$ is given by

$$T_k(t) = -\frac{2}{\sqrt{-D_k}}\int_0^t h_k(\tilde{t})\sin\left(\frac{\sqrt{-D_k}(\tilde{t}-t)}{2D_1}\right)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}d\tilde{t}. \tag{4.72}$$

Case 2: $D_k = 0$

For $D_k = 0$ the solution for $T_k(t)$ is given by

$$T_k(t) = -\frac{1}{A}\int_0^t h_k(\tilde{t})(\tilde{t}-t)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}d\tilde{t}. \tag{4.73}$$

Case 3: $D_k > 0$

For $D_k > 0$ the solution for $T_k(t)$ is given by

$$T_k(t) = -\frac{2}{\sqrt{D_k}}\int_0^t h_k(\tilde{t})\sinh\left(\frac{\sqrt{D_k}(\tilde{t}-t)}{2D_1}\right)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}d\tilde{t}. \tag{4.74}$$

Taking this altogether, the solution for the system can be formulated.

4.2.8. Solution

In conclusion, the solution of the system can be written as

$$u_z(t,z) = \frac{P(t,z)+F_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right), \tag{4.75}$$

where $T_k(t)$ are given by Expressions (4.72), (4.73) or (4.74), depending on the value of D_k.

4.2.9. Resulting partial differential equation

When Expression (4.75) for $u_z(t,z)$ is substituted in the partial differential Equation (4.10), it results in a partial differential equation for $P(t,z)$, given by:

$$-(1-p)\rho_w\left[\frac{\frac{\partial^2 P(t,z)}{\partial t^2}+F''_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T''_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right)\right] \tag{4.76}$$

$$-\frac{\partial P}{\partial z} - \frac{\gamma_w}{K_s}\left[\frac{\frac{\partial P(t,z)}{\partial t}+F'_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T'_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right)\right] = 0, \tag{4.77}$$

or equivalently

$$-\frac{(1-p)\rho_w}{\alpha+\beta}\frac{\partial^2 P(t,z)}{\partial t^2}(z+Z) - \frac{\gamma_w}{K_s(\alpha+\beta)}\frac{\partial P(t,z)}{\partial t}(z+Z) - \frac{\partial P}{\partial z} \tag{4.78}$$

$$= \frac{(1-p)\rho_w}{\alpha+\beta}F''_{zz}(t)(z+Z) + (1-p)\rho_w\sum_{k=0}^{\infty} T''_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right) + \frac{\gamma_w}{K_s(\alpha+\beta)}F'_{zz}(t)(z+Z) + \frac{\gamma_w}{K_s}\sum_{k=0}^{\infty} T'_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right). \tag{4.79}$$

However, since the functions $T_k(t)$ explicitly depend on function $P(t,z)$ as well, it is not clear how this partial differential equation should be solved. Hence, another analytical approach has also been attempted, which will be described in the next section.

4.3. Analytical solution 2

In this section, similar steps will be carried out as done before. However, some adaptations to the boundary conditions are made with the hope of finding an analytically solvable system.

4.3.1. Rewriting the boundary condition

Note that by introducing a distribution function $0 \le \xi(t) \le 1$, solely depending on time, we can write this boundary condition into two separate boundary conditions:

$$-P = \xi(t)F_{zz}(t), \tag{4.80}$$

$$(\alpha+\beta)\frac{\partial u_z}{\partial z} = (1-\xi(t))F_{zz}(t). \tag{4.81}$$

This $\xi(t)$ function comes a bit out of the blue, but a formal mathematical reasoning will be given in Chapter 5. The function $\xi(t)$ represents the fraction of the force being carried by the pore water, whereas the soil particles carry fraction $1-\xi(t)$. Again, the initial conditions are given by

$$u(0,z) = 0, \tag{4.82}$$

$$\frac{\partial u_z}{\partial t}(0,z) = 0. \tag{4.83}$$

In order to solve this system, the quasi-stationary problem needs to be solved first.

4.3.2. Quasi-stationary problem

The quasi-stationary problem is given by:

$$-(\alpha+\beta)\frac{\partial^2 u_z^S}{\partial z^2} = 0, \tag{4.84}$$

with $u_z^S = u_z^S(t,z)$ for $t > 0, -Z < z < 0$ subject to

$$u_z^S(t,-Z) = 0, \tag{4.85}$$

$$(\alpha+\beta)\frac{\partial u_z^S}{\partial z}(t,0) = \xi(t)F_{zz}(t). \tag{4.86}$$

The solution of the quasi-stationary problem is hence given by

$$u_z^S(t,z) = \frac{\xi(t)F_{zz}(t)}{\alpha+\beta}(z+Z). \tag{4.87}$$

The next step is stating the transient problem.

4.3.3. Transient problem

By introducing

$$u_z(t,z) = u_z^S(t,z) + u_z^T(t,z), \tag{4.88}$$

we have that

$$(1-p)\rho_p\frac{\partial^2 u_z^T}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial u_z^T}{\partial t} - (\alpha+\beta)\frac{\partial^2 u_z^T}{\partial z^2} = -(1-p)\rho_p\frac{\partial^2 u_z^S}{\partial t^2} - \frac{\gamma_w}{K_s}\frac{\partial u_z^S}{\partial t}, \tag{4.89}$$

with conditions

$$\begin{aligned}
u_z^T(0,z) &= -u_z^S(0,z), && (4.90)\\
&= -\frac{\xi(0)F_{zz}(0)}{\alpha+\beta}(z+Z), && (4.91)\\
&= 0, && (4.92)\\
\frac{\partial u_z^T(0,z)}{\partial t} &= -\frac{u_z^S}{\partial t}(0,z), && (4.93)\\
&= -\frac{\xi'(0)F_{zz}(0)}{\alpha+\beta}(z+Z) - \frac{\xi(0)F'_{zz}(0)}{\alpha+\beta}(z+Z), && (4.94)\\
&= 0, && (4.95)\\
u_z^T(t,-Z) &= 0, && (4.96)\\
u_z^T(t,0) &= 0. && (4.97)
\end{aligned}$$

Since it is expected that the shape of the transient solution and the shape of the homogeneous transient solution are the same, the homogeneous transient problem will be analysed.

4.3.4. Homogeneous transient problem

The homogeneous transient problem is given by

$$(1-p)\rho_p\frac{\partial^2 u_z^{HT}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial u_z^{HT}}{\partial t} - (\alpha+\beta)\frac{\partial^2 u_z^{HT}}{\partial z^2} = 0, \tag{4.98}$$

with conditions

$$u_z^{HT}(0,z) = 0, \tag{4.99}$$
$$\frac{\partial u_z^{HT}(0,z)}{\partial t} = 0, \tag{4.100}$$
$$u_z^{HT}(t,-Z) = 0, \tag{4.101}$$
$$u_z^{HT}(t,0) = 0 \tag{4.102}$$

An attempt will be made to analytically solve this system by using separation of variables.

4.3.5. Separation of variables

The assumption is made that u_z^{HT} can be expressed using functions depending on only one variable, i.e.

$$u_z^{HT} = Z_1(z)T_1(t). \tag{4.103}$$

These expressions are substituted in the boundary conditions.

Boundary conditions

Substitution in the boundary conditions gives

$$Z_1'(0)T_1(t) = 0, \tag{4.104}$$
$$Z_1(-Z)T_1(t) = 0, \tag{4.105}$$

for $t \geq 0$. Since we are not interested in trivial solutions, it has to hold that

$$Z_1'(0) = 0, \tag{4.106}$$
$$Z_1(-Z) = 0. \tag{4.107}$$

The separation of variables will now be applied to the momentum equation of the soil particles.

4.3.6. Momentum equation soil

Substitution in the momentum Equation (4.9) for the soil particles gives

$$(1-p)\rho_p Z_1(z)T_1(t)'' - (\alpha+\beta)Z_1''(z)T_1(t) + \frac{\gamma_w}{K_s}Z_1(z)T_1(t)' = 0. \tag{4.108}$$

Dividing this equation by $Z_1(z)T_1(t)$ gives

$$(1-p)\rho_p \frac{T_1(t)''}{T_1(t)} - (\alpha+\beta)\frac{Z_1''(z)}{Z_1(z)} + \frac{\gamma_w}{K_s}\frac{T_1(t)'}{T_1(t)} = 0. \tag{4.109}$$

For this to hold, there should be a constant, say $\lambda \in \mathbb{R}$, such that

$$(1-p)\rho_p \frac{T_1(t)''}{T_1(t)} + \frac{\gamma_w}{K_s}\frac{T_1(t)'}{T_1(t)} = -\lambda, \tag{4.110}$$
$$-(\alpha+\beta)\frac{Z_1''(z)}{Z_1(z)} = \lambda, \tag{4.111}$$

which is equivalent to saying

$$(1-p)\rho_p T_1(t)'' + \frac{\gamma_w}{K_s}T_1(t)' + \lambda T_1(t) = 0, \tag{4.112}$$
$$-(\alpha+\beta)Z_1''(z) - \lambda Z_1(z) = 0. \tag{4.113}$$

The differential equation for $Z_1(z)$ will be analysed, where a distinction is made between three cases: $\lambda = 0$, $\lambda = \mu^2$ and $\lambda = -\mu^2$ with $\mu > 0$.

Case 1: $\lambda = 0$

The ordinary differential equation reduces to:

$$Z_1''(z) = 0, \tag{4.114}$$

which has the general solution

$$Z_1(z) = C_1 z + C_2, \tag{4.115}$$

where $C_1, C_2 \in \mathbb{R}$. However, the only values that satisfy the boundary Conditions (4.104) and (4.105) for $Z_1(z)$ are $C_1 = C_2 = 0$. This would mean that $u_z^{HT} T(t,z) \equiv 0$, the trivial solution.

Case 2: $\lambda = -\mu^2$

The general solution for the differential equation is

$$Z_1(z) = C_3 e^{\frac{\mu z}{\sqrt{\alpha+\beta}}} + C_4 e^{-\frac{\mu z}{\sqrt{\alpha+\beta}}} \tag{4.116}$$

with $C_3, C_4 \in \mathbb{R}$. The boundary conditions give the relations

$$C_3 - C_4 = 0, \tag{4.117}$$

$$C_3 e^{-\frac{\mu Z}{\sqrt{\alpha+\beta}}} + C_4 e^{\frac{\mu Z}{\sqrt{\alpha+\beta}}} = 0, \tag{4.118}$$

which can only hold for $C_3 = C_4 = 0$ and hence does not yield a non-trivial solution.

Case 3: $\lambda = \mu^2$

The general solution for the differential equation is

$$Z_1(z) = C_5 \cos\left(\frac{\mu z}{\sqrt{\alpha+\beta}}\right) + C_6 \sin\left(\frac{\mu z}{\sqrt{\alpha+\beta}}\right) \tag{4.119}$$

with $C_5, C_6 \in \mathbb{R}$. The boundary conditions give that

$$C_6 = 0, \tag{4.120}$$

$$C_5 \cos\left(\frac{-\mu Z}{\sqrt{\alpha+\beta}}\right) = 0. \tag{4.121}$$

Since for a non-trivial solution it is required that $C_5 \neq 0$, it has to hold that

$$\cos\left(\frac{-\mu Z}{\sqrt{\alpha+\beta}}\right) = 0, \tag{4.122}$$

which is the case for

$$\frac{-\mu Z}{\sqrt{\alpha+\beta}} = \frac{\pi}{2} + k\pi, \tag{4.123}$$

$$\Rightarrow \mu_k = \frac{\pi\sqrt{\alpha+\beta}}{2Z} + k\frac{\pi\sqrt{\alpha+\beta}}{Z}, \tag{4.124}$$

for $k \in \mathbb{N}_0$. Hence the solution of the homogeneous transient problem is of the form

$$u_z^{HT}(t,z) = \sum_{k=0}^{\infty} T_k^H(t) \cos\left(\left(\frac{\pi}{2Z} + k\frac{\pi}{Z}\right) z\right). \tag{4.125}$$

The form of the solution of the homogeneous transient problem will now be used to solve the transient problem.

4.3.7. Solution of transient problem

The Ansatz is used that the transient solution should have the same form as u_z^{HT}, giving

$$u_z^T(t,z) = \sum_{k=0}^{\infty} T_k(t)\cos\left(\left(\frac{\pi}{2Z} + k\frac{\pi}{Z}\right)z\right). \tag{4.126}$$

Substitution in the Equations (4.89) gives

$$\sum_{k=0}^{\infty}\left(\rho_p(1-p)T_k''(t) + \frac{\gamma_w}{K_s}T_k'(t) + (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2 T_k(t)\right)\cos\left(\frac{\pi+2k\pi}{2Z}z\right) = h(t,z), \tag{4.127}$$

where $h(t,z)$ is the notation for the right hand side of the transient problem, i.e.

$$h(t,z) = -(1-p)\rho_p\frac{\partial^2 u_z^S}{\partial t^2} - \frac{\gamma_w}{K_s}\frac{\partial u_z^S}{\partial t}. \tag{4.128}$$

Note that the cosine functions are orthogonal. Furthermore, it holds that

$$\int_{-Z}^{0}\cos^2\left(\frac{\pi+2k\pi}{2Z}z\right)^2 dz = \frac{Z}{2}, \tag{4.129}$$

for all $k \in \mathbb{N}_0$. Hence, multiplying Equation (4.127) by $\cos\left(\frac{\pi+2l\pi}{2Z}z\right)$ and integrating over z from $z=-Z$ to $z=0$, gives that

$$\frac{Z}{2}\left(\rho_p(1-p)T_k''(t) + \frac{\gamma_w}{K_s}T_k'(t) + (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2 T_k(t)\right) = \int_{-Z}^{0} h(t,z)\cos\left(\frac{\pi+2k\pi}{2Z}z\right)dz, \tag{4.130}$$

for $k \in \mathbb{N}_0$. For convenience we now define

$$D_1 = \rho_p(1-p), \tag{4.131}$$

$$D_2 = \frac{\gamma_w}{K_s}, \tag{4.132}$$

$$D_{3,k} = (\alpha+\beta)\left(\frac{\pi+2k\pi}{2Z}\right)^2, \tag{4.133}$$

$$h_k(t) = \frac{2}{Z}\int_{-Z}^{0} h(t,z)\cos\left(\frac{\pi+2k\pi}{2Z}z\right)dz. \tag{4.134}$$

Using this short notation, the resulting ordinary differential equation can be written as

$$D_1T_k''(t) + D_2T_k'(t) + D_{3,k}T_k(t) = h_k(t), \tag{4.135}$$

$$T_k(0) = 0, \tag{4.136}$$

$$T_k'(0) = 0. \tag{4.137}$$

The determinant of the characteristic equation is given by

$$D_k = D_2^2 - 4D_1D_{3,k}. \tag{4.138}$$

Three distinct cases should be considered: $D_k < 0$, $D_k = 0$ and $D_k > 0$.

Case 1: $D_k < 0$

For $D_k < 0$ the solution is given by

$$T_k(t) = -\frac{2}{\sqrt{-D_k}}\int_0^t h_k(\tilde{t})\sin\left(\frac{\sqrt{-D_k}(\tilde{t}-t)}{2D_1}\right)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}d\tilde{t}. \tag{4.139}$$

Case 2: $D_k = 0$

For $D_k = 0$ the solution is given by

$$T_k(t) = -\frac{1}{A}\int_0^t h_k(\tilde{t})(\tilde{t}-t)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}\,d\tilde{t}. \tag{4.140}$$

Case 3: $D_k > 0$

For $D_k > 0$ the solution is given by

$$T_k(t) = -\frac{2}{\sqrt{D_k}}\int_0^t h_k(\tilde{t})\sinh\left(\frac{\sqrt{D_k}(\tilde{t}-t)}{2D_1}\right)e^{\frac{D_2(\tilde{t}-t)}{2D_1}}\,d\tilde{t}. \tag{4.141}$$

Taking this altogether, the solution for the system can be formulated.

4.3.8. Solution

In conclusion, the solution for $u_z(t,z)$ can be written as

$$u_z(t,z) = \frac{\xi(t)F_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right), \tag{4.142}$$

where T_k is given by Expressions (4.139), (4.140) or (4.141), depending on the value of D_k. The solution for $u_z(t,z)$ will now be substituted in the momentum equation for the pore water.

4.4. Solution of $P(t,z)$

Substituting Solution (4.142) in partial differential Equation (4.10) gives the partial differential equation:

$$-(1-p)\rho_w\left(\frac{\xi''(t)F_{zz}(t)}{\alpha+\beta}(z+Z) + \frac{\xi'(t)F'_{zz}(t)}{\alpha+\beta}(z+Z) + \frac{\xi'(t)F'_{zz}(t)}{\alpha+\beta}(z+Z) + \frac{\xi(t)F''_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T''_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right)\right) \tag{4.143}$$

$$-\frac{\partial P}{\partial z} - \frac{\gamma_w}{K_s}\left(\frac{\xi'(t)F_{zz}(t)}{\alpha+\beta}(z+Z) + \frac{\xi(t)F'_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T'_k(t)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right)\right) = 0, \tag{4.144}$$

which can equivalently be written as

$$\frac{\partial P}{\partial z} = \left(-(1-p)\rho_w\frac{\xi''(t)F_{zz}(t)+2\xi'(t)F'_{zz}(t)+\xi(t)F''_{zz}(t)}{\alpha+\beta} - \frac{\gamma_w}{K_s}\frac{\xi'(t)F_{zz}(t)+\xi(t)F'_{zz}(t)}{\alpha+\beta}\right)(z+Z) \tag{4.145}$$

$$-\sum_{k=0}^{\infty}\left((1-p)\rho_w T''_k(t) + \frac{\gamma_w}{K_s}T'_k(t)\right)\cos\left(\left(\frac{\pi}{2Z}+k\frac{\pi}{Z}\right)z\right). \tag{4.146}$$

Integrating both sides to z gives that

$$P(t,z) = \left(-(1-p)\rho_w\frac{\xi''(t)F_{zz}(t)+2\xi'(t)F'_{zz}(t)+\xi(t)F''_{zz}(t)}{\alpha+\beta} - \frac{\gamma_w}{K_s}\frac{\xi'(t)F_{zz}(t)+\xi(t)F'_{zz}(t)}{\alpha+\beta}\right)\left[\frac{z^2}{2}+zZ\right] \tag{4.147}$$

$$-\sum_{k=0}^{\infty}\left((1-p)\rho_w T''_k(t) + \frac{\gamma_w}{K_s}T'_k(t)\right)\frac{2Z}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi z}{2Z}\right) + C_P(t). \tag{4.148}$$

Note that $P(t,z)$ has to satisfy

$$P(t,-Z) = -\rho_w pg, \tag{4.149}$$

$$P(t,0) = -\xi(t)F_{zz}(t) \tag{4.150}$$

Boundary Condition (4.150) gives that

$$C_P(t) = -\xi(t)F_{zz}(t). \tag{4.151}$$

As a result, it has to hold, according to boundary Condition (4.149), that

$$-\rho_w p g = \left((1-p)\rho_w \frac{\xi''(t)F_{zz}(t) + 2\xi'(t)F'_{zz}(t) + \xi(t)F''_{zz}(t)}{\alpha+\beta} - \frac{\gamma_w}{K_s}\frac{\xi'(t)F_{zz}(t) + \xi(t)F'_{zz}(t)}{\alpha+\beta}\right)\frac{Z^2}{2} \tag{4.152}$$

$$- \sum_{k=0}^{\infty}\left((1-p)\rho_w T''_k(t) + \frac{\gamma_w}{K_s}T'_k(t)\right)\frac{2Z}{(2k+1)\pi}\left(-1\right)^{k+1} - \xi(t)F_{zz}(t). \tag{4.153}$$

This resulting ordinary differential equation describes function $\xi(t)$.

4.5. Total solution

The solutions for u_z and P are given by

$$u_z(t,z) = \frac{\xi(t)F_{zz}(t)}{\alpha+\beta}(z+Z) + \sum_{k=0}^{\infty} T_k(t)\cos\left(\left(\frac{\pi}{2Z} + k\frac{\pi}{Z}\right)z\right), \tag{4.154}$$

$$P(t,z) = \left(-(1-p)\rho_w \frac{\xi''(t)F_{zz}(t) + 2\xi'(t)F'_{zz}(t) + \xi(t)F''_{zz}(t)}{\alpha+\beta} - \frac{\gamma_w}{K_s}\frac{\xi'(t)F_{zz}(t) + \xi(t)F'_{zz}(t)}{\alpha+\beta}\right)\left[\frac{z^2}{2} + zZ\right] \tag{4.155}$$

$$- \sum_{k=0}^{\infty}\left((1-p)\rho_w T''_k(t) + \frac{\gamma_w}{K_s}T'_k(t)\right)\frac{2Z}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi z}{2Z}\right) - \xi(t)F_{zz}(t), \tag{4.156}$$

where $\xi(t)$ is the solution of the ordinary differential Equation (4.152). However, ordinary differential Equation (4.152) is rather complicated and it is not trivial to solve this, since the functions $T_k(t)$ are dependent on function $\xi(t)$ as well. Therefore, this analytical approach does not yield an explicit solution, just like the analytical approach without the use of function $\xi(t)$. In the next section, an attempt was made to solve the system numerically.

4.6. Numerical solution

In order to retrieve a numerical solution, a numerical system has to be found by using the standard approach of stating a weak formulation and substituting the Galerkin approximations.

4.6.1. Numerical system

Multiplying the partial differential equations by test functions η^{u_z} and η^P respectively and integrating from $z=-Z$ to $z=0$ gives the weak formulation

$$(1-p)\rho_p \int_{-Z}^{0} \eta^{u_z}\frac{\partial^2 u_z}{\partial t^2}d\Omega - \frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}\frac{\partial u_z}{\partial t}d\Omega + (\alpha+\beta)\int_{-Z}^{0}\frac{\partial \eta^{u_z}}{\partial z}\frac{\partial u_z}{\partial z}d\Omega - (\alpha+\beta)\eta^{u_z}(0)\frac{\partial u}{\partial z}(0) = 0, \tag{4.157}$$

$$-(1-p)\rho_w \int_{-Z}^{0}\eta^P\frac{\partial^2 u_z}{\partial t^2}d\Omega + \frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^P\frac{\partial u_z}{\partial t}d\Omega - \int_{-Z}^{0}\eta^P\frac{\partial P}{\partial z}d\Omega = 0. \tag{4.158}$$

Using boundary condition (4.13) for u_z gives

$$(1-p)\rho_p \int_{-Z}^{0} \eta^{u_z}\frac{\partial^2 u_z}{\partial t^2}d\Omega - \frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}\frac{\partial u_z}{\partial t}d\Omega + (\alpha+\beta)\int_{-Z}^{0}\frac{\partial \eta^{u_z}}{\partial z}\frac{\partial u_z}{\partial z}d\Omega + \eta^{u_z}(0)P(0) = -\eta^{u_z}(0)F_{zz}(t), \tag{4.159}$$

$$-(1-p)\rho_w \int_{-Z}^{0}\eta^P\frac{\partial^2 u_z}{\partial t^2}d\Omega + \frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^P\frac{\partial u_z}{\partial t}d\Omega - \int_{-Z}^{0}\eta^P\frac{\partial P}{\partial z}d\Omega = 0. \tag{4.160}$$

Using the Galerkin approximations

$$u_z = \sum_{j=1}^{n} d_j(t)\eta_j, \tag{4.161}$$

$$P = \sum_{j=1}^{n} e_j(t)\eta_j, \tag{4.162}$$

gives the following Galerkin equations:

$$\sum_{j=1}^{n} \left\{ \ddot{d}_j(t)(1-p)\rho_p \int_{-Z}^{0} \eta_i\eta_j d\Omega - \dot{d}_j(t)\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j d\Omega + d_j(t)(\alpha+\beta)\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}d\Omega + e_j(t)\eta_i(0)\eta_j(0)\right\} = -\eta_i(0)F_{zz}(t), \tag{4.163}$$

$$\sum_{j=1}^{n} \left\{ \ddot{d}_j(t) - (1-p)\rho_w \int_{-Z}^{0} \eta_i\eta_j d\Omega + \dot{d}_j(t)\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j d\Omega - e_j(t)\int_{-Z}^{0}\eta_i\frac{\partial\eta_j}{\partial z}d\Omega\right\} = 0, \tag{4.164}$$

for $i = 1,\ldots,n$. In matrix form this can be put as

$$M_{dd}\ddot{\boldsymbol{d}}(t) + W_{dd}\dot{\boldsymbol{d}}(t) + S_{dd}\boldsymbol{d}(t) + S_{de}\boldsymbol{e}(t) = \boldsymbol{f}_d(t), \tag{4.165}$$

$$M_{ed}\ddot{\boldsymbol{d}}(t) + W_{ed}\dot{\boldsymbol{d}}(t) + S_{ee}\boldsymbol{e}(t) = \boldsymbol{0}. \tag{4.166}$$

The element matrices are given by

$$M_{dd}^{e_k} = (1-p)\rho_p \int_{e_k} \eta_i\eta_j d\Omega, \tag{4.167}$$

$$W_{dd}^{e_k} = -\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j d\Omega, \tag{4.168}$$

$$S_{dd}^{e_k} = (\alpha+\beta)\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}d\Omega, \tag{4.169}$$

$$M_{ed}^{e_k} = -(1-p)\rho_w \int_{e_k}\eta_i\eta_j d\Omega, \tag{4.170}$$

$$W_{ed}^{e_k} = \frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j d\Omega, \tag{4.171}$$

$$S_{ee}^{e_k} = -\int_{e_k}\eta_i\frac{\partial\eta_j}{\partial z}d\Omega, \tag{4.172}$$

where the element $e_k = [z_{k-1}, z_k]$. There is no contribution for the element vector. When the element matrices are computed, using Theorem 2 of Appendix A, we have that

$$M_{dd}^{e_k} = (1-p)\rho_p\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{4.173}$$

$$W_{dd}^{e_k} = -\frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{4.174}$$

$$S_{dd}^{e_k} = \frac{\alpha+\beta}{z_k - z_{k-1}}(-1+2\delta_{ij}), \tag{4.175}$$

$$M_{ed}^{e_k} = -(1-p)\rho_w\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{4.176}$$

$$W_{ed}^{e_k} = \frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{4.177}$$

$$S_{ee}^{e_k} = \frac{1}{2}(2\delta_{j,k-1} - 1), \tag{4.178}$$

The boundary element matrix is given by

$$S_{de}^{be_k} = \eta_i(0)\eta_j(0), \tag{4.179}$$

or computed

$$S_{de}^{be_k} = 1, \tag{4.180}$$

and the boundary element vector is given by

$$f_d = -\eta_i(0)F_{zz}(t), \tag{4.181}$$

or computed

$$f_d = -F_{zz}(t). \tag{4.182}$$

Similarly as before, by introducing vector $\boldsymbol{\psi}(t) = \begin{pmatrix}\boldsymbol{d}\\ \dot{\boldsymbol{d}}\end{pmatrix}$, the system can equivalently be written as

$$M_{\psi\psi}\dot{\boldsymbol{\psi}}(t) = S_{\psi\psi}\boldsymbol{\psi}(t) + S_{\psi e}\boldsymbol{e}(t) + \boldsymbol{f}_{\psi}(t), \tag{4.183}$$

$$S_{ee}\boldsymbol{e}(t) = M_{e\psi}\dot{\boldsymbol{\psi}}(t), \tag{4.184}$$

where

$$M_{\psi\psi} = \begin{pmatrix} I & \varnothing \\ W_{dd} & M_{dd}\end{pmatrix}, \tag{4.185}$$

$$S_{\psi\psi} = \begin{pmatrix} \varnothing & I \\ -S_{dd} & \varnothing\end{pmatrix}, \tag{4.186}$$

$$S_{\psi e} = \begin{pmatrix} \varnothing \\ -S_{de}\end{pmatrix}, \tag{4.187}$$

$$\boldsymbol{f}_{\psi}(t) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f}_d(t)\end{pmatrix}, \tag{4.188}$$

$$S_{ee} = S_{ee}, \tag{4.189}$$

$$M_{e\psi} = \begin{pmatrix} -W_{ed} & -M_{ed}\end{pmatrix}. \tag{4.190}$$

4.6.2. Numerical integration

For the numerical integration, the same argumentation holds as stated before. Applying the Backward Euler method gives

$$M_{\psi\psi}\boldsymbol{\psi}^{n+1} = M_{\psi\psi}\boldsymbol{\psi}^{n} + \Delta t S_{\psi\psi}\boldsymbol{\psi}^{n+1} + \Delta t S_{\psi e}\boldsymbol{e}^{n+1} + \Delta t \boldsymbol{f}_{\psi}^{n+1}. \tag{4.191}$$

When Equation (4.184) is evaluated in $t = t_{n+1}$, we have that

$$S_{ee}\boldsymbol{e}^{n+1} = M_{e\psi}\dot{\boldsymbol{\psi}}^{n+1} + \boldsymbol{f}_e. \tag{4.192}$$

where $\boldsymbol{f}_e$ contains the inhomogeneous Dirichlet boundary condition. Assuming that $M_{\psi\psi}$ is an invertible matrix, it holds that

$$\dot{\boldsymbol{\psi}}^{n+1} = M_{\psi\psi}^{-1}S_{\psi\psi}\boldsymbol{\psi}^{n+1} + M_{\psi\psi}^{-1}S_{\psi e}\boldsymbol{e}^{n+1} + M_{\psi\psi}^{-1}\boldsymbol{f}_{\psi}^{n+1}. \tag{4.193}$$

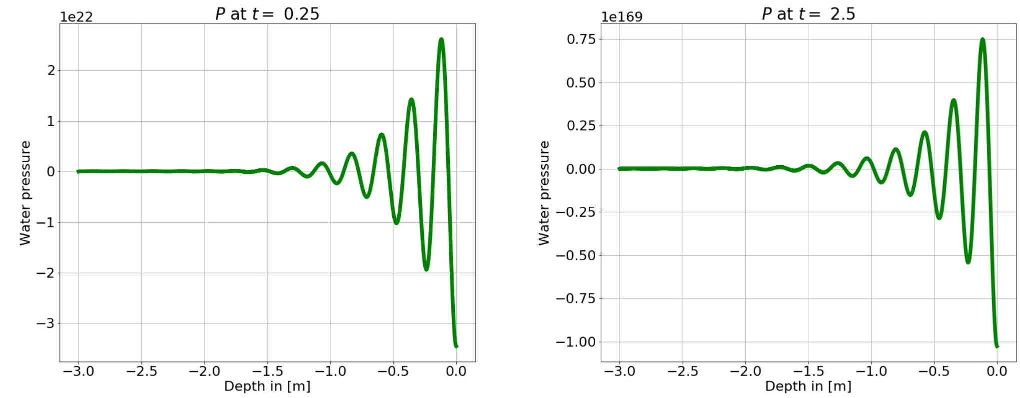
Combing Equation (4.192) and (4.193) gives

$$S_{ee}\boldsymbol{e}^{n+1} = M_{e\psi}M_{\psi\psi}^{-1}S_{\psi\psi}\boldsymbol{\psi}^{n+1} + M_{e\psi}M_{\psi\psi}^{-1}S_{\psi e}\boldsymbol{e}^{n+1} + M_{e\psi}M_{\psi\psi}^{-1}\boldsymbol{f}_{\psi}^{n+1} + \boldsymbol{f}_e. \tag{4.194}$$

The Equations (4.191) and (4.194) can be written in one numerical scheme as

$$\begin{pmatrix} M_{\psi\psi} - \Delta t S_{\psi\psi} & -\Delta t S_{\psi e} \\ M_{e\psi}M_{\psi\psi}^{-1}S_{\psi\psi} & M_{e\psi}M_{\psi\psi}^{-1}S_{\psi e} - S_{ee}\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}^{n+1}\\ \boldsymbol{e}^{n+1}\end{pmatrix} = \begin{pmatrix} M_{\psi\psi} & \varnothing \\ \varnothing & \varnothing\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}^{n}\\ \boldsymbol{e}^{n}\end{pmatrix} + \begin{pmatrix}\Delta t \boldsymbol{f}_{\psi}^{n+1} \\ -M_{e\psi}M_{\psi\psi}^{-1}\boldsymbol{f}_{\psi}^{n+1} - \boldsymbol{f}_e\end{pmatrix}. \tag{4.195}$$

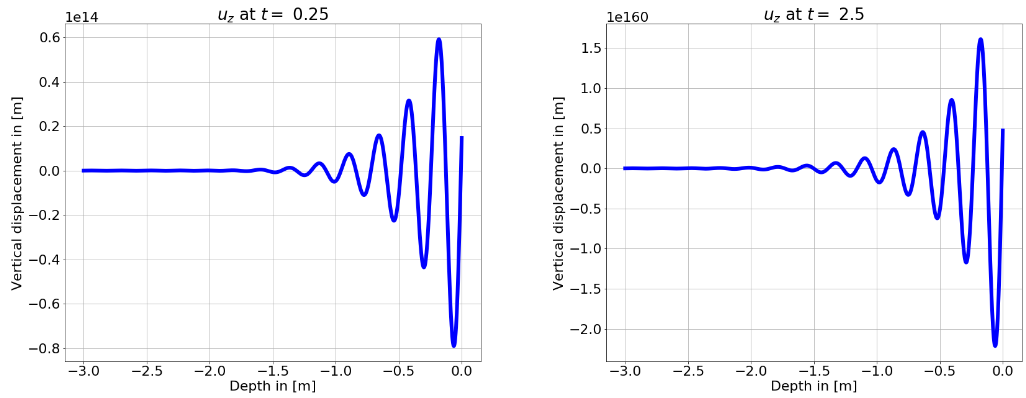
Solving the numerical system with $n = 1000$ and $\Delta t = 0.01$ gives Figures 4.1-4.2b. It can clearly be seen that the values of the parameters explode over time.



(a) P at $t = 0.25$

(b) P at $t = 2.5$

Figure 4.1: P at two different times



(a) u_z at $t = 0.25$

(b) u_z at $t = 2.5$

Figure 4.2: u_z at two different times

This system is clearly not in agreement with what happens in reality. All the methods up and till this point have not yielded any results that seem acceptable. So far we have always assumed that there is a balance of volume. What this means is that soil can only have a change of volume due to pore water flowing in or out. However, even though both the soil particles and the pore water are assumed incompressible, one could argue that the structure of the soil might induce a minimal change of volume. When soil granules move with respect to each other, pores could either expand or shrink. The soil will still be expected to resist change of the current configuration as much as possible, hence these volume changes will be very little. In the next chapter, the validity of the volume balance equation will be disregarded and alternatively the volume balance equation will be minimized by introducing a new function $\xi(t)$.

5

New approach with $\xi(t)$

By slightly adjusting some assumptions, a new model can be derived from similar expressions for the virtual work. Instead of Equation (2.59), the assumption can be made that the distribution of the stresses in a domain does not go proportionally with the porosity and is hence fixed, but actually varies over time. This can be done by introducing a new variable $0 \le \xi(t) \le 1$, which denotes the fraction of the stresses being carried by the water. This means we assume that,

$$\int_{\Omega_p} \dots d\Omega = (1-\xi(t)) \int_{\Omega} \dots d\Omega, \tag{5.1}$$

$$\int_{\Omega_w} \dots d\Omega = \xi(t) \int_{\Omega} \dots d\Omega. \tag{5.2}$$

On a similar note, new extensions for the stress tensors can be derived.

5.1. An extension for unknown stress tensors $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}_{ij}^{w}$

As seen before, the extension is based on conservation of energy, i.e.

$$\int_{\Theta_p} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Theta = \int_{\Theta} \frac{1}{2}\epsilon_{ij}^{p*} \sigma_{ij} d\Theta. \tag{5.3}$$

Taking an infinitely small element Θ we use an averaging for the integrand $\frac{1}{2}\epsilon_{ij}^{p*}\sigma_{ij}$:

$$\frac{1}{2}\epsilon_{ij}^{p*} \sigma_{ij} \approx \frac{1}{|\Theta|} \int_{\Theta} \frac{1}{2}\epsilon_{ij}^{p*} \sigma_{ij} d\Theta. \tag{5.4}$$

Using Requirement (5.3) gives

$$\frac{1}{2}\epsilon_{ij}^{p*} \sigma_{ij} \approx \frac{1}{|\Theta|} \int_{\Theta_p} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Theta. \tag{5.5}$$

The new Assumptions (5.1) and (5.2) can now be applied to the right hand side of Equation (5.5), which results in

$$\frac{1}{|\Theta|} \int_{\Theta_p} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Theta = \frac{(1-\xi(t))}{|\Theta|} \int_{\Theta} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Theta. \tag{5.6}$$

Since Θ is an infinitely small domain, the assumption is made that the integrand $\frac{1}{2}\epsilon_{ij}^{p*}\tilde{\sigma}_{ij}^{p}$ is constant on this small domain. This simplifies Equation (5.6) to

$$\frac{(1-\xi(t))}{|\Theta|} \int_{\Omega} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij} = \frac{(1-\xi(t))|\Theta|}{|\Theta|} \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij}, \tag{5.7}$$

$$= (1-\xi(t)) \frac{1}{2}\epsilon_{ij}^{p*} \tilde{\sigma}_{ij}. \tag{5.8}$$

Since Θ is an arbitrary small region, it can be concluded that

$$\sigma_{ij} \approx (1-\xi(t))\tilde{\sigma}_{ij}, \tag{5.9}$$

on the whole domain Ω, by combining Equations (5.6) and (5.8). The same thing can be done for σ^w_{ij}, resulting in the extension

$$\sigma^w_{ij} \approx \xi(t)\tilde{\sigma}^w_{ij}, \tag{5.10}$$

on the whole domain Ω.

Following the same derivation of the virtual work as in Section 2.7, but using the Equations (5.1), (5.2), (5.9) and (5.10), gives the following six momentum equations for soil particles and pore water respectively ($i \in \{1,2,3\}$):

$$\frac{\partial \sigma_{ij}}{\partial x_j} - (1-\xi)\frac{\partial^2 \rho_p u_i}{\partial t^2} - (1-\xi)\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_j}{\partial t}\right)^2\right) = 0, \tag{5.11}$$

$$\frac{\partial \sigma^w_{ij}}{\partial x_j} - \xi\frac{\partial^2 \rho_w v_i}{\partial t^2} - \xi\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_j}{\partial t}\right)^2\right) = 0, \tag{5.12}$$

for $\boldsymbol{x} \in \Omega$. Furthermore, it has to hold that for $z = -Z$:

$$\sigma_{xz} = 0, \tag{5.13}$$
$$\sigma_{yz} = 0, \tag{5.14}$$
$$\sigma^w_{xz} = 0, \tag{5.15}$$
$$\sigma^w_{yz} = 0. \tag{5.16}$$

For $z = 0$:

$$\sigma_{xz} = (1-\xi(t))F_{xz}, \tag{5.17}$$
$$\sigma_{yz} = (1-\xi(t))F_{yz}, \tag{5.18}$$
$$\sigma_{zz} = (1-\xi(t))F_{zz}, \tag{5.19}$$
$$\sigma^w_{xz} = \xi(t)F_{xz}, \tag{5.20}$$
$$\sigma^w_{yz} = \xi(t)F_{yz}, \tag{5.21}$$
$$\sigma^w_{zz} = \xi(t)F_{zz}. \tag{5.22}$$

For $x = 0$:

$$\sigma_{xy} = 0, \tag{5.23}$$
$$\sigma_{xz} = 0, \tag{5.24}$$
$$\sigma^w_{xy} = 0, \tag{5.25}$$
$$\sigma^w_{xz} = 0, \tag{5.26}$$

For $x = L$:

$$\sigma_{xy} = 0, \tag{5.27}$$
$$\sigma_{xz} = 0, \tag{5.28}$$
$$\sigma^w_{xy} = 0, \tag{5.29}$$
$$\sigma^w_{xz} = 0. \tag{5.30}$$

It is important to note that using new Assumptions (5.1) and (5.2) for the estimation of the integral and old extensions (2.77) and (2.78) of the stress tensor results in the original system, found in Section 2.7. Using the old Assumptions (2.59) and (2.60) for the estimation of the integral and the new Extensions (5.9) and (5.10) for the stress tensor results in the same system that was found in this section. In the next section, a test case will be analysed.

5.2. One-dimensional test case

For the one-dimensional test case, some assumptions are made. The wave stress is considered to only have a normal component, which is again given by Function (3.70). Because of symmetry, displacements and spatial derivatives in the $x-$ (and $y-$) direction can be neglected. Disregarding the virtual work done by the gravitational force and hence disregarding the hydrostatic pressure P, i.e. using stress tensors given by

$$\sigma_{ii} = -\left(\beta\frac{\partial u_j}{\partial x_j} + \alpha\frac{\partial u_i}{\partial x_i}\right), \qquad \sigma^w_{ii} = \mu\left(2\frac{\partial^2 v_i}{\partial x_i \partial t} - \frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t}\right), \tag{5.31}$$

$$\sigma_{ij}|_{i\neq j} = -\frac{\alpha}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \qquad \sigma^w_{ij}|_{i\neq j} = \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \tag{5.32}$$

Now we define V_z as

$$V_z = \frac{\partial v_z}{\partial t}. \tag{5.33}$$

Substituting the stress tensors and the expression for V_z in the momentum Equations (5.11) and (5.12) results in the system

$$(1-\xi(t))\rho_p \frac{\partial^2 u_z}{\partial t^2} - (\alpha+\beta)\frac{\partial^2 u_z}{\partial z^2} - p\frac{\gamma_w}{K_s}V_z + p\frac{\gamma_w}{K_s}\frac{\partial u_z}{\partial t} = 0, \tag{5.34}$$

$$\xi(t)\rho_w \frac{\partial V_z}{\partial t} - \frac{4}{3}\mu\frac{\partial^2 V_z}{\partial z^2} + p\frac{\gamma_w}{K_s}V_z - p\frac{\gamma_w}{K_s}\frac{\partial u_z}{\partial t} = 0, \tag{5.35}$$

with boundary conditions

$$-(\alpha+\beta)\frac{\partial u_z}{\partial z} = (1-\xi(t))F_{zz}(t), \tag{5.36}$$

$$-\frac{4}{3}\mu\frac{\partial V_z}{\partial z} = \xi(t)F_{zz}(t), \tag{5.37}$$

on $z=0$. On $z=-Z$ we have that

$$u_z = 0, \tag{5.38}$$

$$V_z = 0. \tag{5.39}$$

When $F_{zz}(t) = F$ is constant for $t \to \infty$, we expect the system to tend to a stationary solution. In the following section, the stationary solution of the system will be found.

5.3. Stationary solution

The stationary system is given by

$$-(\alpha+\beta)\frac{\partial^2 u_z}{\partial z^2} + p\frac{\gamma_w}{K_s}V_z - = 0, \tag{5.40}$$

$$-\frac{4}{3}\mu\frac{\partial^2 V_z}{\partial z^2} - p\frac{\gamma_w}{K_s}V_z = 0, \tag{5.41}$$

with boundary conditions

$$-(\alpha+\beta)\frac{\partial u_z}{\partial z} = (1-\xi(t))F_{zz}, \tag{5.42}$$

$$-\frac{4}{3}\mu\frac{\partial V_z}{\partial z} = \xi(t)F_{zz}, \tag{5.43}$$

on $z = 0$ and

$$u_z = 0, \tag{5.44}$$
$$V_z = 0, \tag{5.45}$$

on $z = -Z$. Since the determinant D of the characteristic equation of Equation (5.41) is equal to $D = -\frac{16(p\mu\gamma_w)}{3K_s} < 0$, the general stationary solution for V_z, denoted by V_z^S, is given by:

$$V_z^S = C_1 \cos\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) + C_2 \sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) \tag{5.46}$$

Boundary condition (5.43) gives that

$$C_2 = -\frac{1}{2}\sqrt{\frac{3K_s}{\mu p\gamma_w}}\xi^S F. \tag{5.47}$$

Boundary condition (5.45) gives that

$$C_1 = -C_2 \tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right), \tag{5.48}$$
$$= -\frac{1}{2}\sqrt{\frac{3K_s}{\mu p\gamma_w}}\xi^S F \tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right). \tag{5.49}$$

Hence, we have that

$$V_z^S = \frac{1}{2}\sqrt{\frac{3K_s}{\mu p\gamma_w}}\xi^S F \tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right)\cos\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) - \frac{1}{2}\sqrt{\frac{3K_s}{\mu p\gamma_w}}\xi^S F \sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) \tag{5.50}$$

Substituting this expression in differential equation (5.40) and integrating twice to z gives

$$u_z^S = -2\frac{K_s}{p\gamma_w}\sqrt{\frac{\mu K_s}{3p\gamma_w}}\xi^S F \tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right)\cos\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) + 2\frac{K_s}{p\gamma_w}\sqrt{\frac{\mu K_s}{3p\gamma_w}}\xi^S F \sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right) + C_3 z + C_4 \tag{5.51}$$

Boundary condition (5.42) gives that

$$C_3 = -\frac{1-\xi^S}{\alpha+\beta}F + \frac{K_s}{p\gamma_w}\xi^S F, \tag{5.52}$$

Dirichlet Boundary condition (5.44) gives that

$$C_4 = 2\frac{K_s}{p\gamma_w}\sqrt{\frac{\mu K_s}{3p\gamma_w}}\xi^S F \tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right)\cos\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right) \tag{5.53}$$
$$-2\frac{K_s}{p\gamma_w}\sqrt{\frac{\mu K_s}{3p\gamma_w}}\xi^S F \sin\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right) - \frac{1-\xi^S}{\alpha+\beta}FZ + \frac{K_s}{p\gamma_w}\xi^S FZ. \tag{5.54}$$

The functions u_z and V_z depend on the stationary parameter ξ^S, which can be found by minimizing the integral of the squared volume balance, i.e.

$$\min_{0\le\xi^S\le1}\int_{-Z}^{0}\left(p\frac{dV_z}{dz}\right)^2 dz, \tag{5.55}$$

where it is already used that $\frac{\partial u_z^S}{\partial t} = 0$. Substituting the function for V_z^S gives

$$\min_{0\le\xi^S\le 1}\int_{-Z}^{0} p^2\left(-\frac{3}{4\mu}\xi^S F\tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right)\sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right)-\frac{3}{4\mu}\xi^S F\sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right)\right)^2 dz. \tag{5.56}$$

Differentiating this expression to ξ^S and setting it equal to zero gives

$$2p^2\xi^S\int_{-Z}^{0}\left(\frac{3}{4\mu}F\tan\left(-\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}Z\right)\sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right)+\frac{3}{4\mu}F\sin\left(\frac{1}{2}\sqrt{\frac{3p\gamma_w}{K_s\mu}}z\right)\right)^2 dz=0, \tag{5.57}$$

which can only hold for $\xi^S = 0$. In other words, in the stationary situation the soil particles carry the full exerted stress. The functions for u_z^S and V_z^S are hence given by

$$V_z^S(z)=0, \tag{5.58}$$

$$u_z^S(z)=-\frac{F}{\alpha+\beta}(z+Z). \tag{5.59}$$

5.4. Numerical solution

For the numerical implementation, it is assumed that the parameter $\xi(t)$ is a known constant for every time step. The value of $\xi(t)$ at an arbitrary time $t = t_k$ is found with the Golden Section method, which is explained in Section 5.5. As always with a finite element approach, the first step is to find a weak formulation. Multiplying the partial differential equations with test functions η^{u_z} and η^{V_z} respectively and integrating over z from $z=-Z$ to $z=0$ gives

$$(1-\xi(t))\rho_p\int_{-Z}^{0}\eta^{u_z}\frac{\partial^2 u_z}{\partial t^2}dz-(\alpha+\beta)\int_{-Z}^{0}\eta^{u_z}\frac{\partial^2 u_z}{\partial z^2}dz-p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}V_z dz+p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}\frac{\partial u_z}{\partial t}dz=0, \tag{5.60}$$

$$\xi(t)\rho_w\int_{-Z}^{0}\eta^{V_z}\frac{\partial V_z}{\partial t}dz-\frac{4}{3}\mu\int_{-Z}^{0}\eta^{V_z}\frac{\partial^2 V_z}{\partial z^2}dz+p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{V_z}V_z dz-p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{V_z}\frac{\partial u_z}{\partial t}dz=0. \tag{5.61}$$

Applying Theorem 1 of Appendix A and Boundary conditions (5.36) and (5.37) gives

$$(1-\xi(t))\rho_p\int_{-Z}^{0}\eta^{u_z}\frac{\partial^2 u_z}{\partial t^2}dz+(\alpha+\beta)\int_{-Z}^{0}\frac{\partial\eta^{u_z}}{\partial z}\frac{\partial u_z}{\partial z}dz+(1-\xi(t))\eta^{u_z}(0)F_{zz}(t)-p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}V_z dz+p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{u_z}\frac{\partial u_z}{\partial t}dz=0, \tag{5.62}$$

$$\xi(t)\rho_w\int_{-Z}^{0}\eta^{V_z}\frac{\partial V_z}{\partial t}dz+\frac{4}{3}\mu\int_{-Z}^{0}\frac{\partial\eta^{V_z}}{\partial z}\frac{\partial V_z}{\partial z}dz+\xi(t)F_{zz}(t)\eta^{V_z}(0)+p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{V_z}V_z dz-p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta^{V_z}\frac{\partial u_z}{\partial t}dz=0, \tag{5.63}$$

Substituting the Galerkin approximations

$$u_z\approx\sum_{j=1}^{n}f_j(t)\eta_j(z), \tag{5.64}$$

$$V_z\approx\sum_{j=1}^{n}g_j(t)\eta_j(z), \tag{5.65}$$

and setting $\eta^{u_z}=\eta_i$ for some $i\in\{1,\dots,n\}$ and $\eta^{V_z}=\eta_i$ for some $i\in\{1,\dots,n\}$ gives the Galerkin equations

$$\sum_{j=1}^{n}\left\{\frac{d^2 f_j}{dt^2}(t)(1-\xi(t))\rho_p\int_{-Z}^{0}\eta_i\eta_j dz+f_j(t)(\alpha+\beta)\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}dz+(1-\xi(t))\eta_i(0)F_{zz}(t)-g_j(t)p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j{}_z dz+\frac{df_j}{dt}(t)p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j dz\right\}=0, \tag{5.66}$$

$$\sum_{j=1}^{n}\left\{\dot{g}_j(t)\xi(t)\rho_w\int_{-Z}^{0}\eta_i\eta_j dz+g_j(t)\frac{4}{3}\mu\int_{-Z}^{0}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}dz+\xi(t)\eta_i(0)F_{zz}(t)+g_j(t)p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j dz-\frac{df_j}{dt}(t)p\frac{\gamma_w}{K_s}\int_{-Z}^{0}\eta_i\eta_j dz\right\}=0. \tag{5.67}$$

Writing this in matrix form results in

$$M_{ff}\ddot{\boldsymbol{f}}(t) + W_{ff}\dot{\boldsymbol{f}}(t) + S_{ff}\boldsymbol{f}(t) + S_{fg}\boldsymbol{g}(t) = \tilde{\boldsymbol{f}}_f(t), \tag{5.68}$$

$$W_{gf}\dot{\boldsymbol{f}}(t) + W_{gg}\dot{\boldsymbol{g}}(t) + S_{gg}\boldsymbol{g}(t) = \tilde{\boldsymbol{f}}_g(t), \tag{5.69}$$

where the element matrices are given by

$$M_{ff}^{e_k} = \rho_p(1-\xi(t))\int_{e_k}\eta_i\eta_j\,dz, \tag{5.70}$$

$$W_{ff}^{e_k} = p\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j\,dz, \tag{5.71}$$

$$S_{ff}^{e_k} = (\alpha+\beta)\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}\,dz, \tag{5.72}$$

$$S_{fg}^{e_k} = -p\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j\,dz, \tag{5.73}$$

$$W_{gf}^{e_k} = -p\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j\,dz, \tag{5.74}$$

$$W_{gg}^{e_k} = \rho_w\xi(t)\int_{e_k}\eta_i\eta_j\,dz, \tag{5.75}$$

$$S_{gg}^{e_k} = \frac{4}{3}\mu\int_{e_k}\frac{\partial\eta_i}{\partial z}\frac{\partial\eta_j}{\partial z}\,dz + p\frac{\gamma_w}{K_s}\int_{e_k}\eta_i\eta_j\,dz, \tag{5.76}$$

or computed

$$M_{ff}^{e_k} = \rho_p(1-\xi(t))\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.77}$$

$$W_{ff}^{e_k} = p\frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.78}$$

$$S_{ff}^{e_k} = \frac{\alpha+\beta}{z_k - z_{k-1}}(-1+2\delta_{ij}), \tag{5.79}$$

$$S_{fg}^{e_k} = -p\frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.80}$$

$$W_{gf}^{e_k} = -p\frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.81}$$

$$W_{gg}^{e_k} = \rho_w\xi(t)\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.82}$$

$$S_{gg}^{e_k} = \frac{4}{3}\mu\frac{1}{z_k - z_{k-1}}(-1+2\delta_{ij}) + p\frac{\gamma_w}{K_s}\frac{z_k - z_{k-1}}{6}(1+\delta_{ij}), \tag{5.83}$$

There is no contribution for the element vector and for the boundary element matrices. The boundary element vector is given by

$$\tilde{f}_f^{be_k} = -(1-\xi(t))\eta_i(0)F_{zz}(t), \tag{5.84}$$

$$\tilde{f}_g^{be_k} = -\xi(t)\eta_i(0)F_{zz}(t), \tag{5.85}$$

or computed

$$\tilde{f}_f^{be_k} = -(1-\xi(t))F_{zz}(t), \tag{5.86}$$

$$\tilde{f}_g^{be_k} = -\xi(t)F_{zz}(t). \tag{5.87}$$

Introducing $\boldsymbol{\zeta} = \begin{pmatrix}\boldsymbol{f}\\ \dot{\boldsymbol{f}}\\ \boldsymbol{g}\end{pmatrix}$ gives the linear system

$$W_{\zeta\zeta}\dot{\boldsymbol{\zeta}} = S_{\zeta\zeta}\boldsymbol{\zeta} + \tilde{\boldsymbol{f}}_\zeta, \tag{5.88}$$

where

$$W_{\zeta\zeta} = \begin{pmatrix} I & \varnothing & \varnothing \\ W_{ff} & M_{ff} & \varnothing \\ \varnothing & W_{gf} & W_{gg} \end{pmatrix}, \tag{5.89}$$

$$S_{\zeta\zeta} = \begin{pmatrix} \varnothing & I & \varnothing \\ -S_{ff} & \varnothing & -S_{fg} \\ \varnothing & \varnothing & -S_{gg} \end{pmatrix}, \tag{5.90}$$

$$\tilde{\boldsymbol{f}}_\zeta = \begin{pmatrix} \mathbf{0} \\ \tilde{f}_f \\ \tilde{f}_g \end{pmatrix}. \tag{5.91}$$

The Backward Euler method is applied to this linear system, for which the value of $\xi(t)$ is needed on time $t = t_{n+1}$. Once this value is found, the value of ζ^{n+1} can be retrieved by solving

$$(W_{\zeta\zeta} - \Delta t S_{\zeta\zeta})\boldsymbol{\zeta}^{n+1} = W_{\zeta\zeta}\boldsymbol{\zeta}^n + \Delta t \tilde{\boldsymbol{f}}_\zeta^{n+1} \tag{5.92}$$

In the following section, it will be explained how the value of $\xi(t)$ will be found for a given time.

5.5. Golden-section search

In order to find a value for $\xi(t)$ for each time step, some sort of condition is needed. Since for this new model it is assumed that the volume balance does not hold, this equation can not be used. However, the volume balance expression can still be utilized to find a value for $\xi(t)$. It can be argued that even though the volume of the soil does not stay constant, the change of volume will be as little as possible, simply because the soil will pose resistance to any major changes. Hence, when the volume balance is squared and integrated over the whole domain, the value of $\xi(t)$ for any given time $t = t_{n+1}$ can be found by minimizing this function on $t = t_{n+1}$. This can be mathematically written down as

$$\min_{0\le\xi(t_{n+1})\le 1} \int_{-Z}^{0} \left(\frac{\partial}{\partial z}\left\{ p\left[V_z - \frac{\partial u_z}{\partial t} \right] \right\} + \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial t} \right) \right)^2 dz, \tag{5.93}$$

where u_z and V_z should satisfy the system of partial differential equations and comply with the boundary- and initial conditions. Note that this is a constrained minimization problem, which could also be solved by introducing a penalty function or using Lagrange multipliers. However, it is chosen to use a more straightforward approach, making use of the Galerkin approximations. Substituting the Galerkin approximations gives

$$\min_{0\le\xi(t_{n+1})\le 1} \sum_{j=1}^{n} \int_{-Z}^{0} \left(p g_j(t) \frac{\partial \eta_j}{\partial z} + (1-p)\frac{d f_j}{dt}(t)\frac{\partial \eta_j}{\partial z} \right)^2 dz, \tag{5.94}$$

$$\Rightarrow \min_{0\le\xi(t_{n+1})\le 1} \sum_{k=1}^{K}\sum_{j=1}^{n} \int_{e_k} \left(p g_j(t) \frac{\partial \eta_j}{\partial z} + (1-p)\frac{d f_j}{dt}(t)\frac{\partial \eta_j}{\partial z} \right)^2 dz, \tag{5.95}$$

$$\Rightarrow \min_{0\le\xi(t_{n+1})\le 1} \sum_{k=1}^{K} \int_{e_k} \left(-\frac{p g_{k-1}(t)}{z_k - z_{k-1}} + \frac{p g_k(t)}{z_k - z_{k-1}} - \frac{(1-p)\frac{d f_{k-1}}{dt}(t)(t)}{z_k - z_{k-1}} + \frac{(1-p)\frac{d f_k}{dt}(t)(t)}{z_k - z_{k-1}} \right)^2 dz, \tag{5.96}$$

$$\Rightarrow \min_{0\le\xi(t_{n+1})\le 1} \sum_{k=1}^{K} \frac{1}{z_k - z_{k-1}} \left[p(g_k(t) - g_{k-1}(t)) + (1-p)(\frac{d f_k}{dt}(t)(t) - \frac{d f_{k-1}}{dt}(t)(t)) \right]^2, \tag{5.97}$$

For any given time $t = t_{n+1}$, the value of $\xi(t_{n+1})$ will be found through the means of a golden-section search. $\xi(t_{n+1})$ needs to be narrowed down in an efficient way. The golden-section search does exactly this, as will be explained with Figure 5.1. The initial triplet is $\{x_1, x_2, x_3\}$. Note that in our case, $x_1 = 0$ and $x_3 = 1$. The interval widths always have the same ratio, given by $2-\phi : 2\phi - 3 : 2 - \phi$, where ϕ is the golden ratio. The fourth point is chosen to be $x_4 = x_1 + (x_3 - x_2)$. For all these points, the function value is computed of the function that needs to be minimized. When the function value in point x_2, denoted by f_2 is smaller than the function value in point x_4, denoted by f_{4a}, the new triplet will be chosen as $\{x_1, x_2, x_4\}$. When the function value in point x_2, denoted by f_2 is bigger than the function value in point x_4, denoted by f_{4b}, the new triplet will be chosen as $\{x_2, x_4, x_3\}$. With the new triplet the same procedure will be carried out, until the interval is sufficiently small. For every time step this golden-section search is carried out. Implementing this yields some numerical results, that will be discussed in Section 5.7. However, before the numerical results are obtained it will be discussed what is expected of the solution.

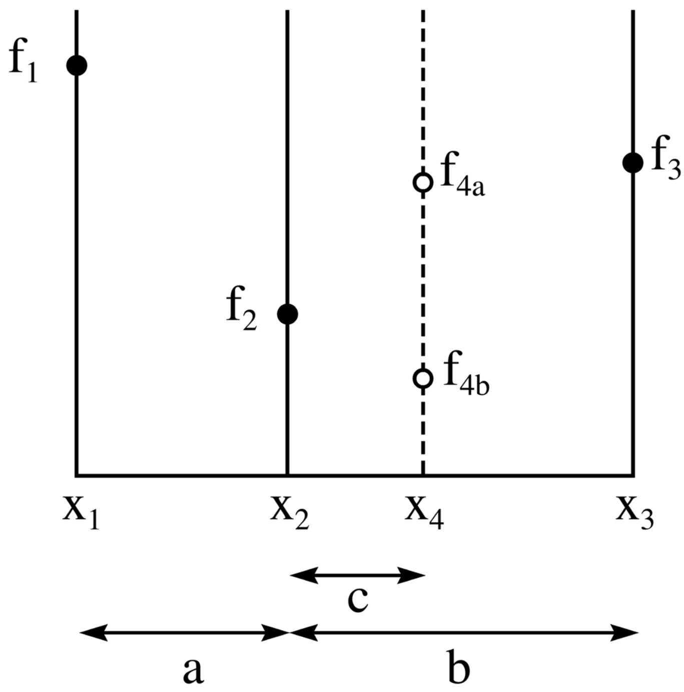


Figure 5.1: Diagram of Golden-Section search [21].

5.6. Expectations

As it has been mentioned several times in this thesis, several overtopping failure simulations have shown that the wave stress is initially being carried by the pore water, or at least for a major part. After some time this fraction decreases and the soil particles take over. When a load is being exerted for a larger amount of time, the soil particles completely take over the carrying of the load. This last phenomenon is also substantiated by the analytical stationary solution, in which $\xi(t) = 0$ for all $t > T$, for some $T > 0$. It is however important to realise what the physical explanation is of this occurrence and how the solutions are expected to behave. The soil can be seen as a spring, where the soil particles are being compressed by an exerted stress. The pore water serves as a damper in the soil matrix. Following this reasoning, it is logical that in a stationary situation in which the soil particles are compressed as far as the stress enforces, the soil matrix as a whole does not move anymore and as a result there are no damping terms and velocities of both water and soil present. But what happens exactly when the wave initially hits the soil?

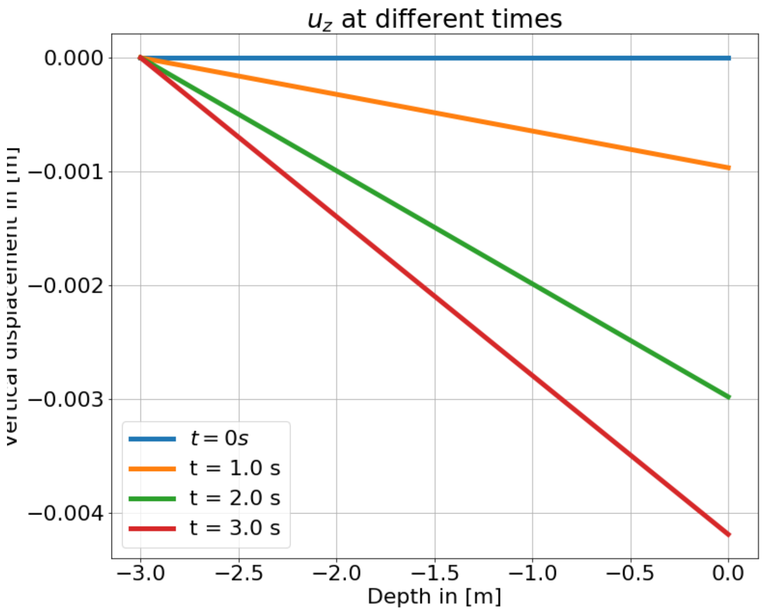
At the moment the wave hits the domain, the soil matrix wants to be compressed. However, the time derivative of the compression is limited by the pore water pressure in the soil. To relief the soil of this pressure, pore water needs to flow out. When pressure is relieved, the effective stresses in the soil increase. As a consequence, both the soil particles and the pore water are moving with a negative-, hence downward velocity. Since the soil matrix as a whole is moving with a negative velocity, the pore water will have an absolute negative velocity, but a relative positive velocity with respect to the soil particles. For the function $\xi(t)$, this means that $\xi(t)$ will have a peak close to $t = 0$, since the pore water is pushed downwards by the wave. After a while, when the pore water pressure has increased, the value of $\xi(t)$ will become smaller, since the pore water wants to escape the soil. When even more time has passed, $\xi(t)$ will tend to zero as the stationary solution will be reached. In the next section, the numerical results will be put to the test.

5.7. Numerical results

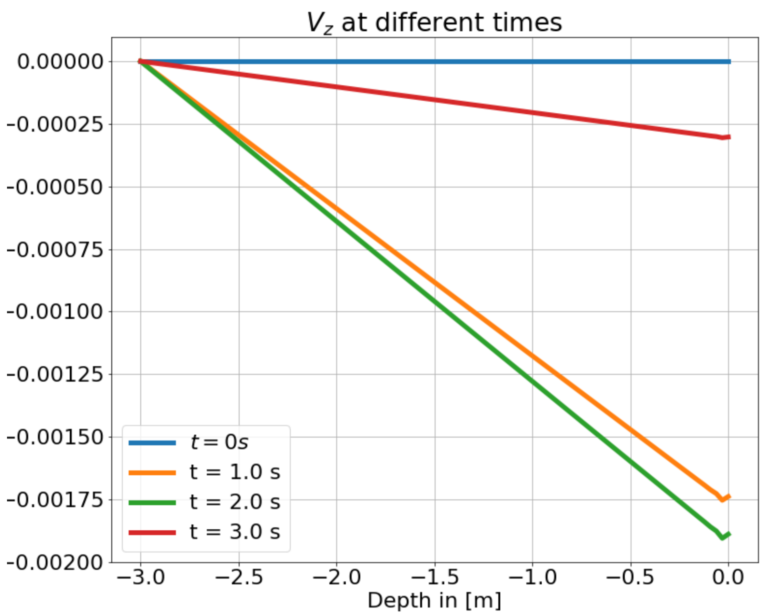
Three different scenarios are analysed, that only differ by the value of the calibration constant K_s, which is dependent on the type of the soil. It is expected that for lower values of this calibration constant, it will take longer to reach the stationary solution. The three different values of K_s are $K_s = 10^{-4}$, $K_s = 5 \cdot 10^{-5}$ and $K_s = 10^{-5}$. Furthermore, three different time step sizes are chosen: $\Delta t = 0.01, \Delta t = 0.001$ and $\Delta t = 0.0001$. In every scenario, the number of integration points is $n = 100$. First the results will be given for time step size $\Delta t = 0.01$.

5.7.1. Numerical results for $\Delta t = 0.01$

For $\Delta t = 0.01$ and $K_s = 10^{-4}$, the numerical results are given in Figures 5.2 and 5.3a.

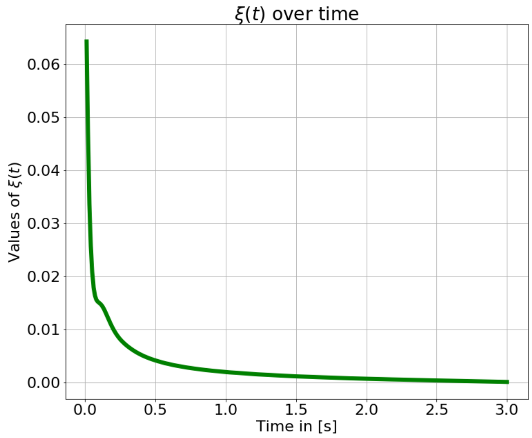


(a) u_z over time

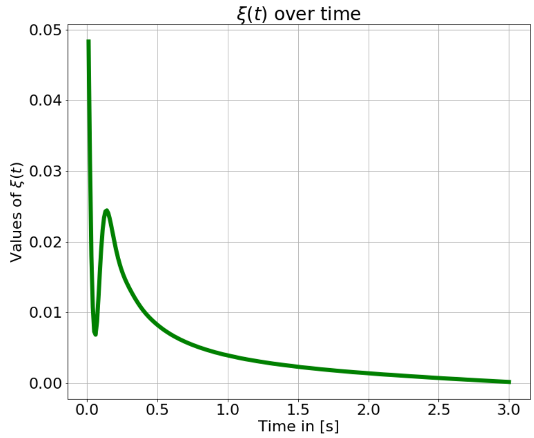


(b) V_z over time

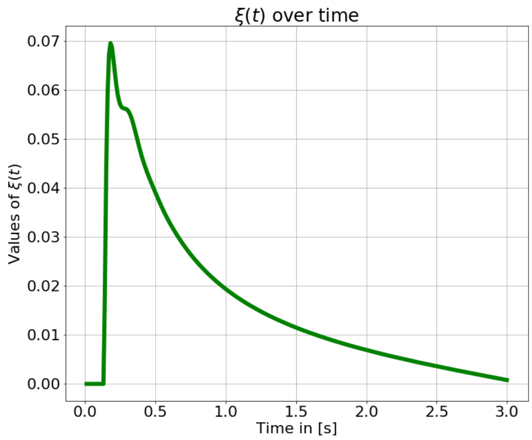
Figure 5.2: Solutions of u_z and V_z for different times with $K_s = 10^{-4}$ and $\Delta t = 0.01$.



(a) Value of $\xi(t)$ over time with $K_s = 10^{-4}$ and $\Delta t = 0.01$.



(b) Value of $\xi(t)$ over time with $K_s = 5 \cdot 10^{-5}$ and $\Delta t = 0.01$.



(c) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.01$.

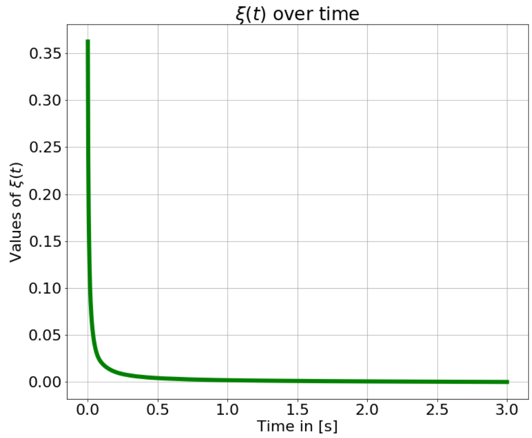
Figure 5.3: Value of $\xi(t)$ over time with $\Delta t = 0.01$ and different values for K_s

It can be seen that the solutions of u_z and V_z tend to the stationary solution that was described in Section 5.3. Furthermore, the value of $\xi(t)$ initially is above 0, but tends to zero over time. This is also something that was expected, since the pore water initially carries more load than the soil particles, but this shifts once the load is applied over a longer period of time. However, the peak value of $\xi(t)$ is still very low, around 0.06. This means that the pore water only carries 6% of the load on its peak, which is not very little. Nevertheless, this value increases once the resolution of the solution is higher, as can be seen in Sections 5.7.2 and 5.7.3. When the value of K_s is decreased to $K_s = 5 \cdot 10^{-5}$, the solutions of u_z and V_z for the given times are not visibly different, hence only Figure 5.3b is given as a result. Decreasing the value even further to $K_s = 10^{-5}$ gives Figure 5.3c.

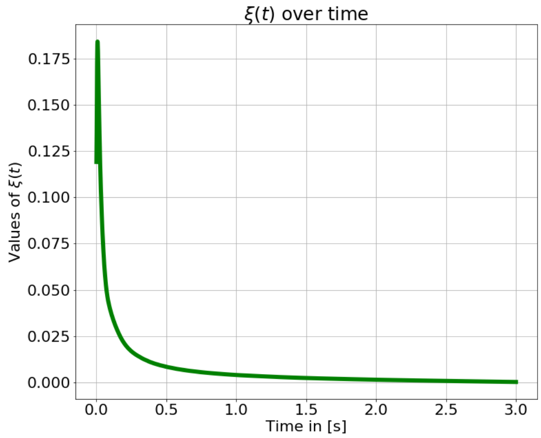
As stated before, decreasing the value of K_s results in very similar numerical results for u_z and V_z for the given times. However, locally there could be differences, induced by the different functions for $\xi(t)$, depicted in Figures 5.3a, 5.3b and 5.3c. These different functions of $\xi(t)$ mostly differ close to $t = 0$, the moment where the stress starts being exerted. It is however important to note that, since Backward Euler is used with time step size $\Delta t = 0.01$, these differences could occur due to numerical errors. In the next section, the time step size is made ten times smaller.

5.7.2. Numerical results for $\Delta t = 0.001$

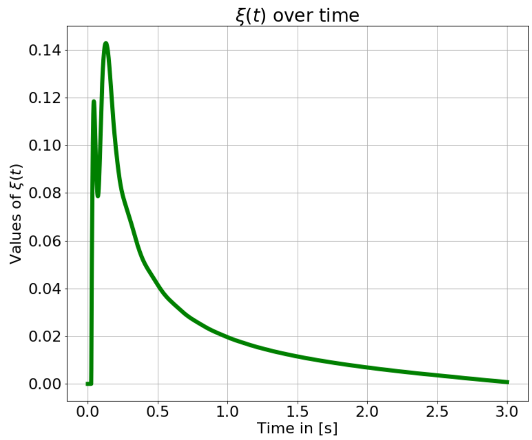
For this smaller time step size, the functions $u_z(z,t)$ and $V_z(z,t)$ for the given times remain basically unchanged with respect to the ones found with time step size $\Delta t = 0.01$. Hence, only the results for function $\xi(t)$ will be given, depicted in Figures 5.4a, 5.4b and 5.4c.



(a) Value of $\xi(t)$ over time with $K_s = 10^{-4}$ and $\Delta t = 0.001$.



(b) Value of $\xi(t)$ over time with $K_s = 5 \cdot 10^{-5}$ and $\Delta t = 0.001$.



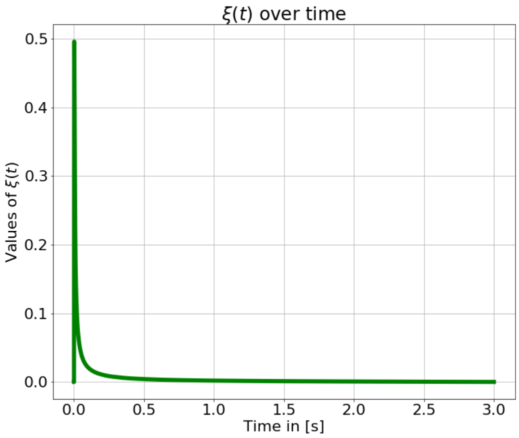
(c) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.001$.

Figure 5.4: Value of $\xi(t)$ over time with $\Delta t = 0.001$ and different values for K_s

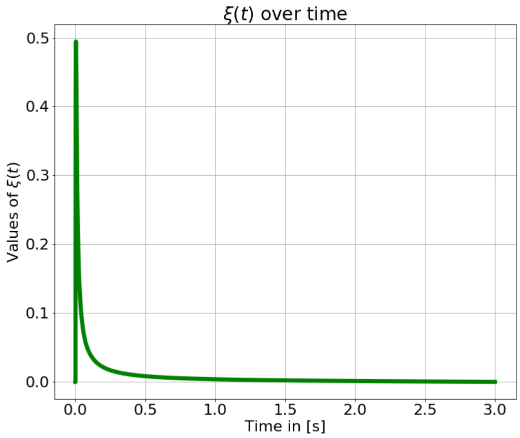
The functions for $\xi(t)$ seem to be a bit smoother, which could be due to reduced numerical errors. Also note that the peak values of the $\xi(t)$ functions are higher. The Figures 5.4b and 5.4c nevertheless still show some strange oscillations. To ensure that this is because of the chosen numerical scheme, the time step will be made smaller one more time.

5.7.3. Numerical results for $\Delta t = 0.0001$

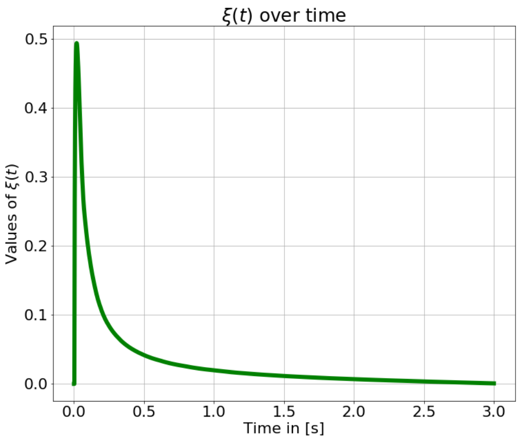
Again, even for this small time step size, the functions $u_z(z,t)$ and $V_z(z,t)$ remain approximately the same. Hence, only the results for function $\xi(t)$ will be given, depicted in Figures 5.5a, 5.5b and 5.5c. It can clearly be seen that no strange oscillations occur anymore with this higher resolution. To illustrate this observation in one view, the ξ-functions corresponding to $K_s = 10^{-5}$ have been depicted in one figure, Figure 5.6. Moreover, the peak values of $\xi(t)$ for the different values of K_s are all located around 0.5, which is a value that physically makes sense, since initially the pore water and soil particles would contribute more or less equally. The only major difference that can be seen is that the smaller K_s, the more time it takes for $\xi(t)$ to be close to zero. This is something that was expected as well, since with a smaller K_s, the soil poses more resistance to deformation and hence the whole process is slowed down. With these results with a higher resolution, our hypotheses are confirmed, which concludes the proof of concept of this new method.



(a) Value of $\xi(t)$ over time with $K_s = 10^{-4}$ and $\Delta t = 0.0001$.



(b) Value of $\xi(t)$ over time with $K_s = 5 \cdot 10^{-5}$ and $\Delta t = 0.0001$.

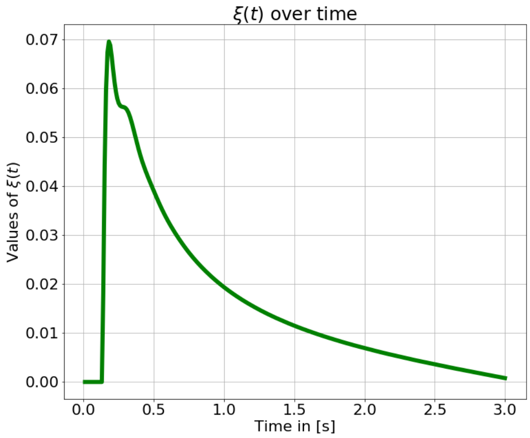


(c) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.0001$.

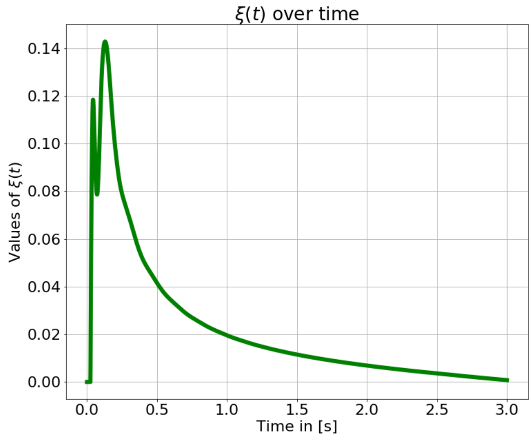
Figure 5.5: Value of $\xi(t)$ over time with $\Delta t = 0.0001$ and different values for K_s

5.8. Extensions and remarks

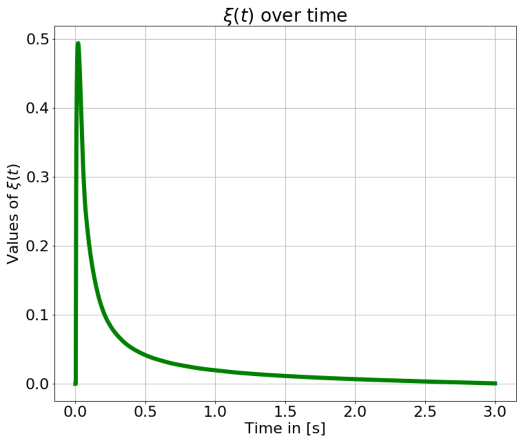
As we have seen in the previous section, the new model imitates the findings in overtopping simulations: initially the pore water plays a role in carrying the load exerted by the wave, but over time this contribution diminishes and all that is left is a compressed domain of soil, with no pore water velocity. This was all based on the assumption that the volume balance does not hold, but is minimized over $\xi(t)$. Whether this assumption holds should be thoroughly analysed in soil experiments, if possible. However, the results leading from this assumption are hopeful. Obviously the applicability of the results in practice can only be assessed by a comparison with an overtopping simulation. Therefore a two-dimensional extension of this model is essential, since shear stresses are inherent to overtopping flow. It is important to realize that this extension could theoretically be made straightforwardly. This can be done by relaxing some assumptions, such as imposing a non-zero function for the frictional stresses $F_{xz}(t)$ (and $F_{yz}(t)$) exerted by the wave, not neglecting the variables in the $x-$ (and $y-$) direction and not ignoring the spatial derivatives to x (and y). In order to solve the system for the new variables, the momentum balance equations in the x- (and y-) direction need to be taken into consideration and implemented in the linear system. Furthermore, Assumptions (5.1) and (5.2) should hold, which is now only verified for a one-dimensional domain. When these assumptions are incorrect, thus shear- and axial stresses are distributed in a different manner, additional equations are needed. For these additional equations, new empirical relations are essential, which can only find their origin in soil experiments. However, the invalidity of these assumptions is doubtful, since it is assumed that pore water and soil particles are mixed perfectly and are seen as loose particles in the domain, not having interactive mechanisms that



(a) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.01$.



(b) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.001$.



(c) Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and $\Delta t = 0.0001$.

Figure 5.6: Value of $\xi(t)$ over time with $K_s = 10^{-5}$ and different time step sizes

strengthen the soil in either axial or shear direction.

In the case that this model would be falsified in a two- or three-dimensional setting, a remaining possibility could be to not differentiate between soil particles and pore water, but as an alternative define a stress tensor that is valid for the soil matrix as a whole. A candidate could be a weighted average of the stress tensors of soil particles and pore water, where $w_i = u_i = v_i$, where w_i is the displacement of the soil matrix in x_i-direction.

6 Conclusion and discussion

The aim of this thesis has been to provide a mathematical framework that describes the physics in a flood embankment in a two-dimensional setting. Nowadays decisions are usually based on overtopping simulations, but these simulations are expensive, only test one specific kind of wave stress and one specific type of soil. There is a mathematical status quo approach [5], based on a porous seabed, which can be used to describe the dynamic pressure in levees. However, the outcome of this method does not match the findings in overtopping simulations. This difference is explained by the questionable assumption that the pore water is considered to be compressible. Furthermore it makes use of the assumption that the stresses resulting from waves are solely being absorbed by the pore water. Physically it makes more sense that these stresses are endured by both the soil particles and the pore water. In order to retrieve a more accurate model for the computation of the water pressure, these assumptions were abandoned in this thesis.

In Chapter 2 a mathematical framework was derived by applying the curl- and divergence operators to the momentum balance equations of the pore water and soil particles and assuming a balance of volume. This system of equations was solved with the use of finite elements, which was described in the antecedent literature research [11]. However, the numerical approach did not yield any results. After a short analysis of a one-dimensional simplification, carried out in Chapter 3, the stationary version of the system turned out to be ill-posed, with an infinite amount of solutions. Changing the boundary condition on the bottom of the domain, i.e. $z = -Z$, resulted in numerical solutions. However these solutions did not agree with the findings of experiments, since in these results the pore water hardly contributed to the absorption of the stresses.

In Chapter 4, another approach has been attempted. Instead of applying the divergence- and curl operators on the momentum balance equations, the momentum balance equations are solved directly, both analytically and numerically. In order to do this, again balance of volume has been assumed. The analytical- and numerical solutions both exploded, hence it was clear that some assumptions had to be modified. In Chapter 5, new extensions for the stress tensors and a new assumptions for the stress distribution have been derived with the use of a new parameter $\xi(t)$, which represents the fraction of the load that is carried by the pore water. This $\xi(t)$ is dependent on time, and was solved using a Golden-Section search, during every time step. The resulting solutions for u_z and V_z coincided with the analytical findings for the stationary solution. Moreover, the function $\xi(t)$ became smoother for smaller time step sizes and had the expected shape: initially being relatively close to 1, but gradually tending to 0 over time. When the value K_s was decreased, it took longer to reach the stationary solution, which also agrees with the situation in practice. The proof of concept of this new method in a one-dimensional setting is hence concluded.

Nevertheless, there are some important aspects that need to be touched upon in further research. Of course, this research is purely theoretical and need to be tested against actual overtopping simulations. It could be seen that the model follows the general trend of a situation as in an experiment, but ideally the numerical model should be able to replace overtopping simulations and hence has to be sufficiently accurate. The final model is based on the assumption that the volume balance does not hold, but is in fact minimized. The theoretical substantiation of this argument is solid, but should be tested by experiments. In order to make a valid comparison, the numerical model should at least be extended to two dimensions. How this should be done is explained in this thesis, but not yet worked out. This would work under the assumption that axial- and shear stresses are distributed similarly. If this assumption proves to be incorrect, an additional parameter needs to be introduced, that represents the fraction of the shear stress that is carried by the pore water. An additional equation needs to be found that enables solving the system for these new parameters, comparable to the volume balance. Moreover, in an ideal situation, the assumption of negligible advective accelerations can be withdrawn, although this is expected to make very little difference. If the model still does not agree with the experiments, some adjustments can still be made, e.g. changing the domain from being fixed to a moving domain. The assumption that the domain is fixed works for small displacements, but when we are interested in stresses with a big order of magnitude and a long running time, it might pose problems with respect to the accuracy of the results.

In the case that the model would still not agree with findings in practice, one last completely different approach can be tried in which no distinction is made between soil particles and the pore water. In other words, $u_i = v_i$ and a stress tensor for the soil as a whole should be defined. A logical option would be to choose for a weighted average of the stress tensors of the pore water and -soil particles. The weights in the expression of the new tensor can serve as an answer to the question of what the dominating damping factor is in the soil.

A

Theorems

The following theorem is a corollary of the Divergence theorem.

Theorem 1 *Let $\boldsymbol{F}$ be a continuously differentiable vector field, g be a scalar function and $\Omega \subset \mathbb{R}^3$ a volume in three-dimensional space which is compact and has a piecewise smooth boundary S. Then it holds that:*

$$\int_\Omega \left[\boldsymbol{F}\cdot\nabla g + g(\nabla\cdot\boldsymbol{F})\right] d\Omega = \oint_S g\boldsymbol{F}\cdot\boldsymbol{n} dS. \tag{A.1}$$

The following theorems are by Holand et al. [4].

Theorem 2 *Let e be the line segment between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, let λ_1 and λ_2 be linear on e such that $\lambda_i(\boldsymbol{x}_j) = \delta_{ij}$, and let $m_1, m_2 \in \mathbf{N}_0 = \{1, 2, \ldots\}$. Then:*

$$\int_e \lambda_1^{m_1}\lambda_2^{m_2} d\Gamma = \frac{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| m_1! m_2!}{(1 + m_1 + m_2)!}. \tag{A.2}$$

Theorem 3 *Suppose that e is a triangle with vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Let λ_1, λ_2 and λ_3 be linear functions on e subject to $\lambda_i(\boldsymbol{x}_j) = \delta_{ij}$ and let $m_1, m_2, m_3 \in \mathbb{N}$. Then:*

$$\int_e \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3} d\Omega = \frac{|\Delta_e| m_1! m_2! m_3!}{(2 + m_1 + m_2 + m_3)!}, \tag{A.3}$$

with

$$|\Delta_e| = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} = \|(\boldsymbol{x}_2 - \boldsymbol{x}_1) \times (\boldsymbol{x}_3 - \boldsymbol{x}_1)\|, \tag{A.4}$$

the area of the parallelogram, which is twice the area of the triangle.

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