Overtopping failure in levees

Literature Report

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Photograph by Henri Cormont [3].

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by

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Nomenclature

$\bar{v}_i$	The expected value of the displacement of the pore water in the x_i-direction in a tube.
$\boldsymbol{n}$	The normal vector, pointing outwards.
δW_σ	Virtual work performed by body forces.
δW_g	Virtual work performed by body forces.
δW_g	Virtual work performed by internal and external forces.
ϵ_{ij}	The strain tensor for soil.
γ_w	The hydraulic conductivity.
μ	The dynamic viscosity of the pore water.
ω	Vorticity of the displacement field of the soil particles.
Ω_p	The part of the domain consisting of pore water.
Ω_p	The part of the domain consisting of soil particles.
ω_v	Vorticity of the displacement field of the pore water.
ρ_p	The density of the soil skeleton.
ρ_s	The density of the porous soil.
ρ_w	The density of the ground water.
σ_{ij}	The stress tensor for the soil skeleton.
σ_{ij}^w	The stress tensor for the pore water.
$\tilde{\sigma}_{ij}$	The stress tensor for an unsaturated soil skeleton.
$\tilde{\sigma}_{ij}^w$	The stress tensor for a fully saturated soil skeleton.
$\tilde{u}_i$	The local displacement of the soil particles in the x_i-direction in a tube.
$\tilde{v}_i$	The local displacement of the pore water in the x_i-direction in a tube.
G	The shear modulus.
g	The gravitational constant.
K	The compression modulus.
K_s	The calibration constant.
P	The water pressure.
p	The porosity of the soil.
q_i	The specific displacement in the x_i-direction.
S_p	The surface of domain Ω_p.
S_w	The surface of domain Ω_w.
u_i	The displacement of the soil particles in the x_i-direction.
v_i	The displacement of the pore water in the x_i-direction.

Contents

1 Introduction

Grass covers have shown to offer erosion protection to river levees and flood embankments. Overtopping waves can trigger erosion of the levee slope. In this thesis it is hypothesized that the hydrodynamic load that acts on a levee, induced by the overtopping flow, can result in deformation of the porous medium. This can lead to failure due to 'head-cut', 'roll-up' and 'collapse', that are thoroughly explained by Le et al. [8]. According to Steendam et al. [13], little research has been conducted with respect to the erosive effect of overtopping flow on dike slopes, mostly because of complications of scale models and the costs of overtopping tests. There have been some recent developments with respect on the other side of the spectrum. Van Bergeijk et al. [14] have made it possible to make a prediction for stresses that originate from a given wave. In other words, this model focuses on the external load on the levee. Moreover, with the development of the Wave overtopping simulator (Van der Meer et al. [17]), more and more practical field research is conducted. This is a simulator which is positioned on an isolated part of a flood embankment. For a given time, the simulator lets waves flow over the slope. At the same time, measurements are made that are used to provide a solid basis between the endured stresses and the reason of failure.

At the same time, theoretical models have been developed that predict water pressures in porous media. One of these methods is the PORO-WSSI model by Ye et al. [6] that is based on the dynamic response in porous seabeds. The idea is that harmonic waves on a porous seabed are somewhat equivalent to overtopping waves on a levee, and hence the same model can be applied on overtopping waves. This method requires the a priori assumption that the pore water pressures match the hydrodynamic pressures under the waves, with the result that the effective stresses are assumed to be 0. In other words, the assumption is made that the pore water instantaneously absorbs the full hydrodynamic surface load. Since the model results do not match the measured reality, the assumption was made that entrapped compressible air is present in the seabed. By accounting for the compressibility in tests, the model outcomes now fit the test results. In practice this is done by including a calibrated Skempton coefficient. This method has been the state of the art approach for some time now. However, this approach is rather questionable, since tests are being tweaked to match the model outcomes and vice versa.

In a new proposed model by Van Damme and Den Ouden-van der Horst [15], these a priori assumptions have been disregarded, in pursuit of a process based approach to more accurately determine the effect of overtopping flow. Instead of assuming that the pore water absorbs the full hydrodynamic surface load, momentum balance equations are taken as boundary conditions, to enforce that the momentum balance equation will be valid on the whole domain. This makes the system more complex, but also more similar to a real situation. Furthermore, the assumed compressibility of the pore water is questionable. It is not a reasonable assumption that pore water in a seabed contains air, since the air has had all the time to dissolve in the sea water over time.

In this thesis, a Finite Element solver is applied to the new model. In case this solver proves to be accurate, this model can provide significant insight in the way flood embankments fail. On top of that, the model can be implemented for other interesting applications related with flows over porous media, e.g. oxygen uptake in lung tissue. Being able to have a better understanding of the oxygen uptake in lung tissue can also provide more insight in processes that oxygen uptake is a result of, such as diffusion capacity, lung volume, breathing pattern, et cetera. (Lin et al. [9]). Another important application could be to analyze the effect of harmonic waves on the sand layers on top of buried offshore pipelines. As described by Martin et al. [10], the sand layer on top of these shallowly buried pipes can be affected by vibrations of the pipeline and by waves running over the layer. When the sand layer disappears, the pipe might experience an undesirable uplift. The overtopping waves model could potentially quantify the water pressures in the sand layer, therefor providing more insight in this phenomenon.

In Chapter 2, the model describing the physics in the levee will be derived, including the boundary- and initial conditions. In Chapter 3, a numerical approach will be chosen and applied to the model, which results in a linear system. Some time stepping methods will be discussed to solve this system. In Chapter 4, conclusions will be drawn and some possible extensions will be discussed for future work.

2

Physical Model

2.1. Notation

Firstly, some notation will be introduced to improve the understandability of this thesis. As is often done in soil mechanics, indices are replaced by the variable that belongs to the specific index. E.g. u_2 will be written as u_y, the component in the y-direction and **not** the partial derivative of u with respect to y. The same thing is done for tensors, e.g. σ_{12} is written as σ_{xy}. Partial derivatives will simply be written in the classical way, e.g. $\frac{\partial u_y}{\partial y}$. Einstein's notation is often used, to make expressions more concise. In short, a repeated index represents a summation over this index, i.e.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{2.1}$$

2.2. Assumptions

In order to derive at the system which describes the physics in the levee, some assumptions are made. It is possible that these assumptions will later be withdrawn, as an extension to the original goal of this thesis. The assumptions are:

- The densities of the soil particles and pore water are taken to be constant.
- The pore water is assumed incompressible.
- The advective acceleration of the soil particle matrix is taken to be zero.
- The acceleration of the pore water is taken to be zero.

2.3. Definitions

Normally the effective stresses of soil are defined to be positive for compression and water stresses are defined to be negative for tension. However to be consistent with the effective stress convention, the stress tensor for pore water, σ^w_{ij}, as defined by Falconer [4] has been converted. The stress tensor for the soil matrix σ_{ij} remains unchanged. The stress tensor is defined as:

$$\sigma_{ii} = -\left(\beta\frac{\partial u_j}{\partial x_j} + \alpha\frac{\partial u_i}{\partial x_i}\right), \qquad \sigma^w_{ii} = \mu\left(\frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t} - 2\frac{\partial^2 v_i}{\partial x_i \partial t}\right) + P, \tag{2.2}$$

$$\sigma_{ij}|_{i\neq j} = -\frac{\alpha}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \qquad \sigma^w_{ij}|_{i\neq j} = -\mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right), \tag{2.3}$$

where P denotes the water pressure, given by $P = \frac{1}{3}(\sigma^w_{xx} + \sigma^w_{yy} + \sigma^w_{zz})$. u_i represents the displacement of the soil particles in the x_i-direction, whereas v_i represents the displacement of the pore water in the x_i-direction.

The stress tensor for the soil particles is equivalent to the one defined by Verruijt [21] when we define the coefficients as

$$\alpha = 2G, \tag{2.4}$$

$$\beta = K - \frac{2}{3}G, \tag{2.5}$$

where K is the compression modulus and G is the shear modulus. The strain tensor, denoted by ϵ^p, is given by

$$\epsilon^p_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}. \tag{2.6}$$

Furthermore, note that the viscosity μ is the dynamic viscosity, otherwise the stresses $\sigma^w_{ij}|_{i \neq j}$ do not have the right dimensions. Now that the definitions are set, the system of equations that describe the physics in the levee, will be derived.

2.4. Mass balance equation

To arrive at the volume balance equations, some observations have been made. First of all, both the pore water and the soil particles are assumed incompressible. This means that a change in volume can only be induced by adding water or taking water out of the porous medium. The density formula for the porous medium is given by

$$\rho_s = \rho_p(1-p) + \rho_w p, \tag{2.7}$$

where ρ_s is the density of the porous medium, ρ_p is the density of the soil matrix, ρ_w is the density of the pore water and p is the porosity. The density ρ_s can only change when p changes, since the densities ρ_p and ρ_w are assumed constant. The change in porosity with respect to time is induced by the flux of the pore water; hence the volume balance equation for incompressible pore water, in Cartesian coordinates, is given by

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}\left(p\frac{\partial v_i}{\partial t}\right) = 0, \tag{2.8}$$

where v_i represent the displacements of the pore water in the three different directions. This is similar to the mass balance equations stated by Bui et al. [1], with the exception that in this expression the spatial gradient of the void fractions over the distribution is not considered to be negligible. In a similar manner, the volume balance equation for incompressible soil particles is given by

$$\frac{\partial(1-p)}{\partial t} + \frac{\partial}{\partial x_i}\left[(1-p)\frac{\partial u_i}{\partial t}\right] = 0, \tag{2.9}$$

where u_i represent the displacements of the soil particles in the three different directions. Now we impose that soil particles and pore water are mixed so well, that functions u_i and v_i are defined everywhere on the domain of interest. As a consequence, we can sum Equations (2.8) and (2.9) to arrive at the mass balance equation for the porous medium

$$\frac{\partial}{\partial x_i}\left\{p\left[\frac{\partial(v_i - u_i)}{\partial t}\right]\right\} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t}\right) = 0. \tag{2.10}$$

This mass balance equation will be used several times throughout this thesis. In order to derive useful partial differential equations, the expression for the virtual work will be derived.

2.5. Virtual work

The virtual work $\delta\hat{W}_g$ performed by body forces acting on both the soil matrix and the pore water, i.e. the virtual work due to gravitational force, is given by

$$\delta\hat{W}_g = \int_{\Omega_p} \rho_p g u_z^* d\Omega + \int_{\Omega_w} \rho_w g v_z^* d\Omega, \tag{2.11}$$

where Ω_p is the part of the domain consisting of soil particles, Ω_w is the part of the domain consisting of pore water, g is the gravitational constant and u_i^* and v_i^* are the virtual displacements of the soil particles and pore water respectively in the three different directions. The sum of the virtual work performed by internal and external forcing, denoted by δW_σ, is given by Van Damme and den Ouden-van der Horst [15] as

$$\delta W_\sigma = \oint_{S_p} u_i^* \tilde{\sigma}_{ij} n_j dS - \int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega + \oint_{S_w} v_i^* \tilde{\sigma}_{ij}^w n_j dS - \int_{\Omega_w} \epsilon_{ij}^{w*} \tilde{\sigma}_{ij}^w d\Omega, \tag{2.12}$$

where the unknown stress tensors $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}_{ij}^w$ are only defined on Ω_p and Ω_w respectively. Basically $\tilde{\sigma}_{ij}$ is a stress tensor for completely unsaturated soil, where $\tilde{\sigma}_{ij}^w$ is the stress tensor for fully saturated soil. However, the known stress tensors are only defined for the soil matrix as a whole, not for only the fraction containing soil particles or pore water. For the second integral of Equation (2.12), we have that:

$$\int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega = \frac{1}{2} \int_{\Omega_p} \left\{ 2\frac{\partial u_x^*}{\partial x} \tilde{\sigma}_{xx} + \left(\frac{\partial u_x^*}{\partial y} + \frac{\partial u_y^*}{\partial x} \right) \tilde{\sigma}_{xy} + \left(\frac{\partial u_x^*}{\partial z} + \frac{\partial u_z^*}{\partial x} \right) \tilde{\sigma}_{xz} \right\} d\Omega \tag{2.13}$$

$$+ \frac{1}{2} \int_{\Omega_p} \left\{ \left(\frac{\partial u_y^*}{\partial x} + \frac{\partial u_x^*}{\partial y} \right) \tilde{\sigma}_{yx} + 2\frac{\partial u_y^*}{\partial y} \tilde{\sigma}_{yy} + \left(\frac{\partial u_y^*}{\partial z} + \frac{\partial u_z^*}{\partial y} \right) \tilde{\sigma}_{yz} \right\} d\Omega \tag{2.14}$$

$$+ \frac{1}{2} \int_{\Omega_p} \left\{ \left(\frac{\partial u_z^*}{\partial x} + \frac{\partial u_x^*}{\partial z} \right) \tilde{\sigma}_{zx} + \left(\frac{\partial u_z^*}{\partial y} + \frac{\partial u_y^*}{\partial z} \right) \tilde{\sigma}_{zy} + 2\frac{\partial u_z^*}{\partial z} \tilde{\sigma}_{zz} \right\} d\Omega. \tag{2.15}$$

On every integral of this expression, Theorem 1 of Appendix A, will be applied, which is a corollary of Green's theorem. One term will be worked out explicitly, since the other terms can be done analogously. The second term can be rewritten as

$$\int_{\Omega_p} \left(\frac{\partial u_x^*}{\partial y} + \frac{\partial u_y^*}{\partial x} \right) \tilde{\sigma}_{xy} d\Omega = - \int_{\Omega_p} \nabla \cdot \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \tilde{\sigma}_{xy} d\Omega, \tag{2.16}$$

$$= \int_{\Omega_p} \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \cdot \nabla \tilde{\sigma}_{xy} d\Omega + \oint_{S_p} \tilde{\sigma}_{xy} \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \cdot \boldsymbol{n} dS. \tag{2.17}$$

Doing this for every term of Equation (2.13) gives

$$\int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega = -\frac{1}{2} \int_{\Omega_p} \left\{ \nabla \tilde{\sigma}_{xx} \cdot \begin{pmatrix} 2u_x^* \\ 0 \\ 0 \end{pmatrix} + \nabla \tilde{\sigma}_{xy} \cdot \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} + \nabla \tilde{\sigma}_{xz} \cdot \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \nabla \tilde{\sigma}_{yx} \cdot \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \right. \tag{2.18}$$

$$\left. + \nabla \tilde{\sigma}_{yy} \cdot \begin{pmatrix} 0 \\ 2u_y^* \\ 0 \end{pmatrix} + \nabla \tilde{\sigma}_{yz} \cdot \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \nabla \tilde{\sigma}_{zx} \cdot \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \nabla \tilde{\sigma}_{zy} \cdot \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \nabla \tilde{\sigma}_{zz} \cdot \begin{pmatrix} 0 \\ 0 \\ 2u_z^* \end{pmatrix} \cdot \boldsymbol{n} \right\} d\Omega \tag{2.19}$$

$$+ \frac{1}{2} \oint_{S_p} \left\{ \tilde{\sigma}_{xx} \begin{pmatrix} 2u_x^* \\ 0 \\ 0 \end{pmatrix} + \tilde{\sigma}_{xy} \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} + \tilde{\sigma}_{xz} \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \tilde{\sigma}_{yx} \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \right. \tag{2.20}$$

$$\left. + \tilde{\sigma}_{yy} \begin{pmatrix} 0 \\ 2u_y^* \\ 0 \end{pmatrix} + \tilde{\sigma}_{yz} \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \tilde{\sigma}_{zx} \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \tilde{\sigma}_{zy} \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \tilde{\sigma}_{zz} \begin{pmatrix} 0 \\ 0 \\ 2u_z^* \end{pmatrix} \cdot \boldsymbol{n} \right\} dS. \tag{2.21}$$

Substituting this expression in $\oint_{S_p} u_i^* \tilde{\sigma}_{ij} n_j dS - \int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega$ gives

$$\oint_{S_p} u_i^* \tilde{\sigma}_{ij} n_j dS - \int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega \tag{2.22}$$

$$= -\frac{1}{2}\int_{\Omega_p} \left\{ \nabla\tilde{\sigma}_{xx} \cdot \begin{pmatrix} 2u_x^* \\ 0 \\ 0 \end{pmatrix} + \nabla\tilde{\sigma}_{xy} \cdot \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} + \nabla\tilde{\sigma}_{xz} \cdot \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \nabla\tilde{\sigma}_{yx} \cdot \begin{pmatrix} u_y^* \\ u_x^* \\ 0 \end{pmatrix} \right. \tag{2.23}$$

$$\left. + \nabla\tilde{\sigma}_{yy} \cdot \begin{pmatrix} 0 \\ 2u_y^* \\ 0 \end{pmatrix} + \nabla\tilde{\sigma}_{yz} \cdot \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \nabla\tilde{\sigma}_{zx} \cdot \begin{pmatrix} u_z^* \\ 0 \\ u_x^* \end{pmatrix} + \nabla\tilde{\sigma}_{zy} \cdot \begin{pmatrix} 0 \\ u_z^* \\ u_y^* \end{pmatrix} + \nabla\tilde{\sigma}_{zz} \cdot \begin{pmatrix} 0 \\ 0 \\ 2u_z^* \end{pmatrix} \cdot \boldsymbol{n} \right\} d\Omega \tag{2.24}$$

$$+ \frac{1}{2}\oint_{S_p} \begin{pmatrix} -\tilde{\sigma}_{xy}u_y^* - \tilde{\sigma}_{xz}u_z^* + \tilde{\sigma}_{yx}u_y^* + \tilde{\sigma}_{zx}u_z^* \\ \tilde{\sigma}_{xy}u_x^* - \tilde{\sigma}_{yx}u_x^* - \tilde{\sigma}_{yz}u_z^* + \tilde{\sigma}_{zy}u_z^* \\ \tilde{\sigma}_{xz}u_x^* + \tilde{\sigma}_{yz}u_y^* - \tilde{\sigma}_{zx}u_x^* - \tilde{\sigma}_{zy}u_y^* \end{pmatrix} \cdot \boldsymbol{n} dS. \tag{2.25}$$

Applying the Divergence Theorem gives

$$\frac{1}{2}\oint_{S_p} \begin{pmatrix} -\tilde{\sigma}_{xy}u_y^* - \tilde{\sigma}_{xz}u_z^* + \tilde{\sigma}_{yx}u_y^* + \tilde{\sigma}_{zx}u_z^* \\ \tilde{\sigma}_{xy}u_x^* - \tilde{\sigma}_{yx}u_x^* - \tilde{\sigma}_{yz}u_z^* + \tilde{\sigma}_{zy}u_z^* \\ \tilde{\sigma}_{xz}u_x^* + \tilde{\sigma}_{yz}u_y^* - \tilde{\sigma}_{zx}u_x^* - \tilde{\sigma}_{zy}u_y^* \end{pmatrix} \cdot \boldsymbol{n} dS = \frac{1}{2}\int_{\Omega_p} \nabla \cdot \begin{pmatrix} -\tilde{\sigma}_{xy}u_y^* - \tilde{\sigma}_{xz}u_z^* + \tilde{\sigma}_{yx}u_y^* + \tilde{\sigma}_{zx}u_z^* \\ \tilde{\sigma}_{xy}u_x^* - \tilde{\sigma}_{yx}u_x^* - \tilde{\sigma}_{yz}u_z^* + \tilde{\sigma}_{zy}u_z^* \\ \tilde{\sigma}_{xz}u_x^* + \tilde{\sigma}_{yz}u_y^* - \tilde{\sigma}_{zx}u_x^* - \tilde{\sigma}_{zy}u_y^* \end{pmatrix} d\Omega. \tag{2.26}$$

Writing out the gradient operator results in

$$\oint_{S_p} u_i^* \tilde{\sigma}_{ij} n_j dS - \int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega \tag{2.27}$$

$$= \frac{1}{2}\int_{\Omega_p} (\tilde{\sigma}_{xy} - \tilde{\sigma}_{yx})\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) + (\tilde{\sigma}_{xz} - \tilde{\sigma}_{zx})\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + (\tilde{\sigma}_{yz} - \tilde{\sigma}_{zy})\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) d\Omega \tag{2.28}$$

$$+ \int_{\Omega_p} \left[u_x^* \left(\frac{\partial \tilde{\sigma}_{xx}}{\partial x} + \frac{\partial \tilde{\sigma}_{xy}}{\partial y} + \frac{\partial \tilde{\sigma}_{xz}}{\partial z} \right) + u_y^* \left(\frac{\partial \tilde{\sigma}_{yy}}{\partial y} + \frac{\partial \tilde{\sigma}_{yx}}{\partial x} + \frac{\partial \tilde{\sigma}_{yz}}{\partial z} \right) + u_z^* \left(\frac{\partial \tilde{\sigma}_{zz}}{\partial z} + \frac{\partial \tilde{\sigma}_{zx}}{\partial x} + \frac{\partial \tilde{\sigma}_{zy}}{\partial y} \right) \right] d\Omega. \tag{2.29}$$

Since for $i \neq j$, we have that $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$ and $\tilde{\sigma}_{ij}^w = \tilde{\sigma}_{ji}^w$, the first integral of the last expression drops out, so we are simply left with

$$\oint_{S_p} u_i^* \tilde{\sigma}_{ij} n_j dS - \int_{\Omega_p} \epsilon_{ij}^{p*} \tilde{\sigma}_{ij} d\Omega \tag{2.30}$$

$$= \int_{\Omega_p} \left[u_x^* \left(\frac{\partial \tilde{\sigma}_{xx}}{\partial x} + \frac{\partial \tilde{\sigma}_{xy}}{\partial y} + \frac{\partial \tilde{\sigma}_{xz}}{\partial z} \right) + u_y^* \left(\frac{\partial \tilde{\sigma}_{yy}}{\partial y} + \frac{\partial \tilde{\sigma}_{yx}}{\partial x} + \frac{\partial \tilde{\sigma}_{yz}}{\partial z} \right) + u_z^* \left(\frac{\partial \tilde{\sigma}_{zz}}{\partial z} + \frac{\partial \tilde{\sigma}_{zx}}{\partial x} + \frac{\partial \tilde{\sigma}_{zy}}{\partial y} \right) \right] d\Omega. \tag{2.31}$$

The same thing can be done for the integrals for the virtual work performed on the pore water. The virtual work of the inertial forces is given by:

$$\delta W_{\text{in}} = \int_{\Omega_p} u_i^* \left\{ \frac{\partial^2 \rho_p u_i}{\partial t^2} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p \left(\frac{\partial u_i}{\partial t} \right)^2 \right) \right\} d\Omega + \int_{\Omega_w} v_i^* \left\{ \frac{\partial^2 \rho_w v_i}{\partial t^2} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w \left(\frac{\partial v_i}{\partial t} \right)^2 \right) \right\} d\Omega, \tag{2.32}$$

also known as D'Alemberts principle. From the theory of virtual work it should hold that the total virtual work of the impressed forces should equal the total virtual work of the inertial forces, i.e.

$$\delta W_g + \delta W_\sigma = \delta W_{\text{in}}. \tag{2.33}$$

In other words, it has to hold that

$$\int_{\Omega_p} \left\{ u_i^* \left[\rho_p g_i + \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} - \frac{\partial^2 \rho_p u_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p \left(\frac{\partial u_i}{\partial t} \right)^2 \right) \right] \right\} d\Omega \tag{2.34}$$

$$+ \int_{\Omega_w} \left\{ v_i^* \left[\rho_w g_i + \frac{\partial \tilde{\sigma}_{ij}^w}{\partial x_j} - \frac{\partial^2 \rho_w v_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w \left(\frac{\partial v_i}{\partial t} \right)^2 \right) \right] \right\} d\Omega = 0. \tag{2.35}$$

An extension for unknown stress tensors $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}^w_{ij}$

In order to retrieve useful relations, an extension is needed for these stress tensors to an arbitrary domain Θ, containing both soil particles and pore water. $\tilde{\sigma}_{ij}$ should be defined such that the total energy of σ_{ij} on an arbitrary domain Θ is equivalent to the total energy of $\tilde{\sigma}_{ij}$ on $\Theta_p \subset \Theta$, i.e.

$$\int_{\Theta_p} \frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta = \int_{\Theta} \frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}d\Theta. \tag{2.36}$$

This makes sense because of the observation that σ_{ij} should theoretically only have a contribution on the fraction of Θ containing soil particles. Note that even though that is Einstein notated, this holds element wise, since the derivation should also hold in the one-dimensional case. In order to arrive at an extension for $\tilde{\sigma}_{ij}$, it makes sense to approximate the stress tensor with the use of averaging. Taking an infinitely small element Θ we use an averaging for the integrand $\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}$:

$$\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij} \approx \frac{1}{|\Theta|}\int_{\Theta} \frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij}d\Theta. \tag{2.37}$$

Using Requirement (2.36) gives

$$\frac{1}{2}\epsilon^{p*}_{ij}\sigma_{ij} \approx \frac{1}{|\Theta|}\int_{\Theta_p} \frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta. \tag{2.38}$$

Since Θ (and hence Θ_p) is an infinitely small domain, the assumption is made that the integrand $\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}^{p}_{ij}$ is constant on this small domain. This simplifies the integral to:

$$\begin{aligned}
\frac{1}{|\Theta|}\int_{\Theta_p} \frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}d\Theta &\approx \frac{|\Theta_p|}{|\Theta|}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}, && (2.39)\\
&= \frac{(1-p)|\Theta|}{|\Theta|}\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}, && (2.40)\\
&= (1-p)\frac{1}{2}\epsilon^{p*}_{ij}\tilde{\sigma}_{ij}. && (2.41)
\end{aligned}$$

Because Θ is an arbitrary small region, it can be concluded that

$$\sigma_{ij} \approx (1-p)\tilde{\sigma}_{ij}, \tag{2.42}$$

on the whole domain Ω, by combining Equations (2.38) and (2.39). The same thing can be done for σ^w_{ij}, resulting in the extension

$$\sigma^w_{ij} \approx p\tilde{\sigma}^w_{ij}, \tag{2.43}$$

on the whole domain Ω. These extensions can be utilized to say something about the partial derivative of the unknown stress tensors. Writing out this partial derivative gives

$$\begin{aligned}
\frac{\partial\tilde{\sigma}_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\frac{\sigma_{ij}}{1-p}\right), && (2.44)\\
&= \frac{(1-p)\frac{\partial\sigma_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_j}\sigma_{ij}}{(1-p)^2}, && (2.45)\\
&\approx \frac{(1-p)\frac{\partial\sigma_{ij}}{\partial x_j}}{(1-p)^2}, && (2.46)\\
&= \frac{1}{1-p}\frac{\partial\sigma_{ij}}{\partial x_j}, && (2.47)
\end{aligned}$$

where the approximation is justified by the fact that the partial derivatives in space of the porosity are nearly zero, while the porosity is somewhere around 0.4. A similar thing can be done for $\frac{\partial\tilde{\sigma}^w_{ij}}{\partial x_j}$. Since Ω_p and Ω_w are mixed well, the integral over Ω_p can be approximated by the integral over Ω multiplied by the factor $(1-p)$. The same thing can be done with the integral over Ω_w. Furthermore, since the velocities of the pore water are negligible, the assumption can be made that $\sigma^w_{ii} \approx P$. Note that this represents three elements (for $i = 1,2,3$) of the stress tensor and not Einstein notation. In other words, Equation (2.32) can be written as:

$$(1-p)\int_\Omega \left\{ u_i^* \left[\rho_p g_i + \frac{\partial \tilde{\sigma}^p_{ij}}{\partial x_j} - \frac{\partial^2 \rho_p u_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p \left(\frac{\partial u_i}{\partial t}\right)^2\right)\right]\right\} d\Omega \tag{2.48}$$

$$+p\int_\Omega \left\{ v_i^* \left[\rho_w g_i + \frac{\partial \tilde{\sigma}^w_{ij}}{\partial x_j} - \frac{\partial^2 \rho_w v_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w \left(\frac{\partial v_i}{\partial t}\right)^2\right)\right]\right\} d\Omega = 0. \tag{2.49}$$

Substituting the derived expansions of the stress tensors to Ω results in

$$\int_\Omega \left\{ u_i^* \left[\frac{\partial \sigma_{ij}}{\partial x_j} + (1-p)\left(\rho_p g_i - \frac{\partial^2 \rho_p u_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p \left(\frac{\partial u_i}{\partial t}\right)^2\right)\right)\right]\right\} d\Omega \tag{2.50}$$

$$+\int_\Omega \left\{ v_i^* \left[\frac{\partial \sigma^w_{ij}}{\partial x_j} + p\left(\rho_w g_i - \frac{\partial^2 \rho_w v_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w \left(\frac{\partial v_i}{\partial t}\right)^2\right)\right)\right]\right\} d\Omega = 0. \tag{2.51}$$

Since this expression has to hold for any virtual displacement u_i^*, v_i^*, it follows that

$$\frac{\partial \sigma_{ij}}{\partial x_j} + (1-p)\left(\rho_p g_i - \frac{\partial^2 \rho_p u_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p \left(\frac{\partial u_i}{\partial t}\right)^2\right)\right) = 0, \tag{2.52}$$

$$\frac{\partial \sigma^w_{ij}}{\partial x_j} + p\left(\rho_w g_i - \frac{\partial^2 \rho_w v_i}{\partial t^2} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w \left(\frac{\partial v_i}{\partial t}\right)^2\right)\right) = 0. \tag{2.53}$$

If we consider the last term and assume that ρ_p is constant in space, we have that:

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_p\left(\frac{\partial u_i}{\partial t}\right)^2\right) = \frac{1}{2}\rho_p \frac{\partial}{\partial x_i}\left(\left(\frac{\partial u_i}{\partial t}\right)^2\right), \tag{2.54}$$

$$= \rho_p \frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial x_i \partial t}, \tag{2.55}$$

$$= \rho_p \frac{\partial x_i}{\partial t}\frac{\partial^2 u_i}{\partial x_i \partial t}, \tag{2.56}$$

$$\tag{2.57}$$

in which we see an advective acceleration term. Obviously the same thing can be done for $\frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho_w\left(\frac{\partial v_i}{\partial t}\right)^2\right)$.

Taking the curl in a three dimensional setting results in three non-trivial equations. However, only one of these equations can be easily put in a useful partial differential equation. Moreover, analysis of the intersection of a flood embankment would already provide lots of insight, so for the moment there is little incentive to implement a three dimensional model. This element of the curl only uses Equation (2.52) for $i = 1$ and $i = 3$, given by

$$\frac{\partial^2 \rho_p (1-p) u_x}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial z}\right)\right] - \rho_p(1-p)g_x \tag{2.58}$$

$$-\frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{2.59}$$

$$\frac{\partial^2 \rho_p (1-p) u_z}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial z}\right)\right] - \rho_p(1-p)g_z \tag{2.60}$$

$$+\frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0. \tag{2.61}$$

Darcy term

Note that the Darcy term $\frac{\gamma_w}{K_s}\frac{\partial p(v_i-u_i)}{\partial t}$ appears as a result of taking the partial derivative of the $\mu\left(\frac{\partial^2 v_i}{\partial x_j \partial t}+\frac{\partial^2 v_j}{\partial x_j \partial t}\right)$-term with $i=1$ and $j=3$. In order to understand this, we note that the pore water flows through small 'tubes'. The radii of these 'tubes' are randomly distributed, according to Van Damme [16]. Even though for a given radius the pore water velocity profile is known, the pore water velocity and hence the pore water displacements have stochastic values, because of the randomness of the tube geometry. Hence, locally the stochastic displacements are denoted by $\tilde{u}_i$ and $\tilde{v}_i$. If we observe one 'tube' in the soil, the velocity of the pore water $\dot{\tilde{v}}_x$ is parabolic, as can be seen in figure 2.1.

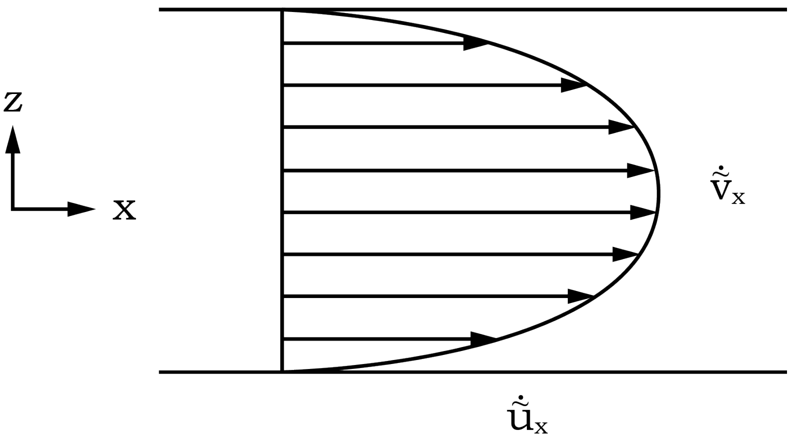

Figure 2.1: A horizontal 'tube' in the soil matrix

The previously stress tensors are based on an infinitely small element and subsequently averaged such that they hold for the whole soil matrix. The same thing will be done for the partial derivative of the stress tensor. Taking the partial derivative to z results in

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{v}_x}{\partial z^2}+\frac{\partial^2 \tilde{v}_z}{\partial x \partial z}\right)\approx C\frac{d\bar{v}_x}{dt}, \tag{2.62}$$

since there is barely any perpendicular acceleration (it is assumed that the 'tube' nearly has a constant width, so there will hardly be any pore water moving inward) and since the second derivative of a parabolic profile is a constant. In this expression $C\in\mathbb{R}$ and $\bar{v}_x$ is the average pore water velocity in the tube, i.e. a resulting pore water velocity of a random draw from the radius distribution. The soil contains a fraction p 'tubes', that all have a parabolic profile. This profile depends on the relative velocity with respect to the tube wall. To extrapolate this local constant to an expression that is valid for the whole soil matrix, we return back to the deterministic displacements. The constant is hence proportional to $p(v_x-u_x)$. Observe that this constant is only constant in space (within one tube), not in time. Note that v_x and u_x can be seen as the expected value of the horizontal velocities in a tube. The proportionality is made explicit by introducing a calibration constant K_s, which explains the Darcy term $\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}$ at the end of the expression. This calibration constant K_s has the units $\left[\frac{m}{s}\right]$ and equals the pressure gradient after multiplication by the specific density of the pore water. Obviously the same analysis can be done for $i=3$, $j=1$. Hence the momentum balance equations for the soil particles become

$$\frac{\partial^2 \rho_w p v_x}{\partial t^2}+\rho_w p\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 v_x}{\partial t \partial x}\right)+\frac{\partial z}{\partial t}\left(\frac{\partial^2 v_x}{\partial t \partial z}\right)\right]+\rho_w p g_x+\frac{\partial P}{\partial x}+\frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t}=0, \tag{2.63}$$

$$\frac{\partial^2 \rho_w p v_z}{\partial t^2}+\rho_w p\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 v_z}{\partial t \partial x}\right)+\frac{\partial z}{\partial t}\left(\frac{\partial^2 v_z}{\partial t \partial z}\right)\right]+\rho_w p g_z+\frac{\partial P}{\partial z}+\frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t}=0. \tag{2.64}$$

Similarly, the momentum balance equations for the pore water can be found. Note that the sign of the Darcy term should change sign, since action equals minus reaction. Often the advective accelerations are negligible compared with the contribution of the particle-particle and particle-water interaction. Hence the momentum balance equations for both the soil particles and pore water are reduced to

$$\rho_p(1-p)\frac{\partial^2 u_x}{\partial t^2} - \rho_p(1-p)g_x - \frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{2.65}$$

$$\rho_p(1-p)\frac{\partial^2 u_z}{\partial t^2} - \rho_p(1-p)g_z + \frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.66}$$

$$\rho_w p\frac{\partial^2 v_x}{\partial t^2} + \rho_w p g_x + \frac{\partial P}{\partial x} + \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{2.67}$$

$$\rho_w p\frac{\partial^2 v_z}{\partial t^2} + \rho_w p g_z + \frac{\partial P}{\partial z} + \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0. \tag{2.68}$$

From now on, the analysis has only been made for a two-dimensional setting, which makes it a suitable moment to introduce the domain of interest, which can be seen in figure 2.2. Note that Γ_1 is boundary $z = -Z$, Γ_2 is boundary $x = L$, Γ_3 is boundary $z = 0$ and Γ_4 is boundary $x = 0$. From now on this will be used interchangeably.

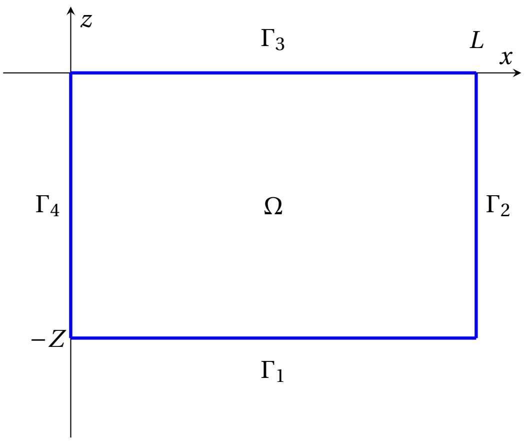

Figure 2.2: The domain of interest Ω.

2.6. Vorticity equation

Taking the curl of the momentum balance equations results in

$$\frac{\partial}{\partial z}\left(\rho_p(1-p)\frac{\partial^2 u_x}{\partial t^2} - \rho_p(1-p)g_x - \frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t}\right) \tag{2.69}$$

$$-\frac{\partial}{\partial x}\left(\rho_p(1-p)\frac{\partial^2 u_z}{\partial t^2} - \rho_p(1-p)g_z + \frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t}\right) = 0. \tag{2.70}$$

Working out the partial derivatives gives

$$\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial z}\right) - \frac{\partial}{\partial z}\left(\rho_p(1-p)g_x\right) - \frac{\alpha}{2}\frac{\partial^2}{\partial z^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial^2}{\partial z\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - \frac{\gamma_w}{K_s}\frac{\partial^2 p(v_x - u_x)}{\partial z\partial t} \tag{2.71}$$

$$-\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\rho_p(1-p)g_z\right) - \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + (\beta+\alpha)\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \frac{\gamma_w}{K_s}\frac{\partial^2 p(v_z - u_z)}{\partial x\partial t} = 0. \tag{2.72}$$

Re-arranging the terms gives the equation

$$\rho_p(1-p)\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - \frac{\alpha}{2}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\right] - \frac{\gamma_w p}{K_s}\left(\frac{\partial}{\partial t}\left[\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right]\right) \tag{2.73}$$

$$= \frac{\gamma_w p}{K_s}\left(\frac{\partial}{\partial t}\left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right]\right). \tag{2.74}$$

When the vorticity of the water displacement vector is defined as

$$w_v = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \tag{2.75}$$

and the vorticity for the soil matrix analogously, the constitutive equation for the vorticity can be written as

$$\rho_p(1-p)\frac{\partial^2 w}{\partial t^2} + \frac{\gamma_w p}{K_s}\frac{\partial w}{\partial t} - \frac{\alpha}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\gamma_w p}{K_s}\frac{\partial w_v}{\partial t}. \tag{2.76}$$

Analogously a constitutive relation for w_v can be found by taking the curl of the momentum balance Equations (2.67) and (2.68) of the pore water, given by:

$$\frac{\partial \rho_w p w_v}{\partial t^2} + \frac{\gamma_w p}{K_s}\frac{\partial w_v}{\partial t} = \frac{\gamma_w p}{K_s}\frac{\partial w}{\partial t}. \tag{2.77}$$

However, the momentum balance equation for soil states that $\omega = \omega_v$, so the only thing that remains is

$$\rho_p(1-p)\frac{\partial^2 \omega}{\partial t^2} - \frac{\alpha}{2}\Delta\omega = 0. \tag{2.78}$$

Now we have found a partial differential equation that describes the vorticity. In the next section, a partial differential equation for the volumetric strain will be derived.

2.7. Volumetric strain equation

In vector notation, the momentum balance vector equation for the pore water is given by:

$$\begin{bmatrix} \frac{\partial^2 \rho_p(1-p)u_x}{\partial t^2} - \rho_p(1-p)g_x - \frac{\alpha}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x-u_x)}{\partial t} \\ \frac{\partial^2 \rho_p(1-p)u_z}{\partial t^2} - \rho_p(1-p)g_z + \frac{\alpha}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z-u_z)}{\partial t} \end{bmatrix} = \mathbf{0}. \tag{2.79}$$

Firstly, we define specific displacements q_x and q_z as:

$$q_x = p(v_x - u_x), \tag{2.80}$$

$$q_z = p(v_z - u_z). \tag{2.81}$$

Note that the volumetric strain, by definition, is equal to the divergence of the displacement vector [12], so in the two-dimensional case this becomes

$$\epsilon_{\text{vol}} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}. \tag{2.82}$$

Taking the divergence on both sides of Equation (2.79) gives the following equality:

$$\frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\frac{\partial u_x}{\partial x}\right) - \frac{\alpha}{2}\frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} - \frac{\gamma_w}{K_s}\frac{\partial^2 q_x}{\partial x \partial t} \tag{2.83}$$

$$+\frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\frac{\partial u_z}{\partial z}\right) + \frac{\alpha}{2}\frac{\partial^2}{\partial z \partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} - \frac{\gamma_w}{K_s}\frac{\partial^2 q_z}{\partial z \partial t} = 0, \tag{2.84}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\rho_p(1-p)\epsilon_{\text{vol}}\right) - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) - \frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z}\right) = 0, \tag{2.85}$$

$$\Rightarrow \rho_p(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) - \frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z}\right) = 0, \tag{2.86}$$

where it is assumed that u_x and u_z are sufficiently smooth, which enables changing the order of differentiation. Substituting mass balance Equation (2.10) results in

$$\rho_p(1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\beta+\alpha)\left(\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} + \frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2}\right) = 0. \tag{2.87}$$

This partial differential equation describes the course of the volumetric strain. Finally, a relation between the volumetric strain and the water pressure will be derived.

2.8. Pressure equation

A relation for the pressure needs to be derived as well. In order to do this, the momentum balance equation for the pore water will be used, given by

$$\begin{bmatrix} \rho_w p \frac{\partial^2 v_x}{\partial t^2} + \rho_w p g_x + \frac{\partial P}{\partial x} + \frac{\gamma_w}{K_s} \frac{\partial p(v_x - u_x)}{\partial t} \\ \rho_w p \frac{\partial^2 v_z}{\partial t^2} + \rho_w p g_z + \frac{\partial P}{\partial z} + \frac{\gamma_w}{K_s} \frac{\partial p(v_z - u_z)}{\partial t} \end{bmatrix} = \mathbf{0}. \tag{2.88}$$

Taking the divergence on both sides gives

$$\frac{\partial}{\partial x}\left(\rho_w p \frac{\partial^2 v_x}{\partial t^2}\right) + \frac{\partial}{\partial x}\rho_w p g_x + \frac{\partial^2 P}{\partial x^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x - u_x)}{\partial t} + \frac{\partial}{\partial z}\left(\rho_w p \frac{\partial^2 v_z}{\partial t^2}\right) + \frac{\partial}{\partial z}\rho_w p g_z + \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.89}$$

$$\Rightarrow \rho_w \frac{\partial}{\partial x}\left(\frac{\partial^2 p v_x}{\partial t^2}\right) + \frac{\partial^2 P}{\partial x^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x - u_x)}{\partial t} + \rho_w \frac{\partial}{\partial z}\left(\frac{\partial^2 p v_z}{\partial t^2}\right) + \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.90}$$

$$\Rightarrow \rho_w \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\frac{\partial p v_x}{\partial t} + \frac{\partial}{\partial z}\frac{\partial p v_z}{\partial t}\right) + \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x - u_x)}{\partial t} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.91}$$

$$\Rightarrow -\rho_w \frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x - u_x)}{\partial t} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.92}$$

$$\Rightarrow -\rho_w \frac{\partial}{\partial t}\left[(1-p)\frac{\partial \epsilon_{\text{vol}}}{\partial t}\right] + \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial x}\frac{\partial p(v_x - u_x)}{\partial t} + \frac{\gamma_w}{K_s}\frac{\partial}{\partial z}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{2.93}$$

$$\Rightarrow -\rho_w (1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} = 0. \tag{2.94}$$

The mass balance equation implies that

$$\frac{\partial p}{\partial t} = (1-p)\frac{\partial \epsilon_{\text{vol}}}{\partial t}, \tag{2.95}$$

with

$$\nabla \cdot \frac{\partial p v_i}{\partial t} = -\frac{\partial p}{\partial t}. \tag{2.96}$$

Equation (2.96) is used for Equality (2.92) and Equation (2.95) is used for Equality (2.93). For Equality (2.94) it is used that, when the functions are sufficiently smooth, Equation (2.10) can be rewritten as:

$$\frac{\partial}{\partial t}\left\{\left[\frac{\partial p(v_i - u_i)}{\partial x_i}\right]\right\} = -\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) = \frac{\partial \epsilon_{\text{vol}}}{\partial t}. \tag{2.97}$$

So we end up with expression

$$-\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = -\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - \rho_w (1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2}. \tag{2.98}$$

Hence, when the solution for ϵ_{vol} is known, the pressure P can easily be determined.

2.9. Relations for the displacement

Vorticity, volumetric strain and displacements can be related by working out

$$\begin{aligned} -\frac{\partial w}{\partial z} - \frac{\partial \epsilon_{\text{vol}}}{\partial x} &= -\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_z}{\partial x \partial z}, && (2.99) \\ &= -\frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2}, && (2.100) \end{aligned}$$

where we have assumed sufficiently smoothness.

This can be done analogously for $\frac{\partial w}{\partial x} - \frac{\epsilon_{\text{vol}}}{\partial z}$, assuming that u_x is sufficiently smooth, which results in the following set of equations:

$$\begin{cases} -\frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} &= -\frac{\partial w}{\partial z} - \frac{\partial \epsilon_{\text{vol}}}{\partial x}, \\ -\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} &= \frac{\partial w}{\partial x} - \frac{\partial \epsilon_{\text{vol}}}{\partial z}. \end{cases} \tag{2.101}$$

These relations are used as fourth and fifth equations of our system, since these expressions are more useful than the definitions of ω and ϵ_{vol}, for reasons that will become clear in Section 3.1. Now there are five partial differential equations that describe five parameters. The only thing that is left are stating the boundary conditions and initial conditions.

2.10. Boundary conditions

At $z = 0$, i.e. at the top of the levee, the boundary conditions are only related with $\tau_{xz} = \tau_{zx}$, which are equal due to the symmetry of the stress tensor, and σ_{zz}. Both of these functions are assumed to be known at $z = 0$ and represent the hydrodynamic loads, induced by the overtopping waves of interest. As we know, the total stress component is determined by the water pressure and the effective stress, i.e. $\sigma_{zz} = P + \sigma'_{zz}$. Substituting the expression for σ'_{zz} gives the boundary condition

$$\sigma_{zz}|_{z=0} = P|_{z=0} - \beta\epsilon_{\text{vol}}|_{z=0} - \alpha\frac{\partial u_z}{\partial z}|_{z=0}. \tag{2.102}$$

There is no Darcy friction term present on the surface of the domain [20]. Since the shear stress is a weighted average of the shear stresses experienced between all soil particles in two directions, we have that

$$\tau_{xz}|_{z=0} = \tau_{zx}|_{z=0} = \frac{\alpha}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)|_{z=0}, \tag{2.103}$$

due to symmetry of the stress tensor. It can easily be checked that boundary condition can equivalently be written as

$$\tau_{xz}|_{z=0} = \frac{\alpha}{2} w|_{z=0} - \alpha\frac{\partial u_x}{\partial z}|_{z=0}. \tag{2.104}$$

The momentum balance equation for the pore water in the vertical direction is given by:

$$\frac{\partial^2 \rho_p (1-p) u_z}{\partial t^2} + \frac{\partial^2 \rho_w p v_z}{\partial t^2} + \frac{\partial \tau_{xz}}{\partial x} = -\frac{\partial \sigma_{zz}}{\partial z}. \tag{2.105}$$

In the analytical approach by Van Damme and Den Ouden-van der Horst [15] the accelerations cannot be ignored since this would violate the existence of an analytical solution. However, numerical analysis provides more flexibility, so the accelerations will initially be neglected. This results in the simpler relation

$$\frac{\partial \tau_{xz}}{\partial x} = -\frac{\partial \sigma_{zz}}{\partial z}. \tag{2.106}$$

Since function τ_{xz} is known on the whole boundary $z = 0$, $\frac{\partial \tau_{xz}}{\partial x}$ is also known on this boundary. Substituting the expression for σ_{zz} yields another boundary condition:

$$\frac{\partial P}{\partial z}|_{z=0} - \beta\frac{\partial \epsilon_{\text{vol}}}{\partial z}|_{z=0} - \alpha\frac{\partial^2 u_z}{\partial z^2}|_{z=0} = -\frac{\partial \tau_{xz}}{\partial z}|_{z=0}. \tag{2.107}$$

On the sides of the levee, i.e. $x = 0$ and $x = L$, we impose the following boundary conditions:

$$u_x = 0, \tag{2.108}$$

$$\frac{\partial u_z}{\partial x} = 0, \tag{2.109}$$

$$\omega = 0, \tag{2.110}$$

$$\frac{\partial \epsilon_{\text{vol}}}{\partial x} = 0, \tag{2.111}$$

$$\frac{\partial P}{\partial x} = 0. \tag{2.112}$$

Following the boundary conditions of Van Damme and Den Ouden-van der Horst [15], it is assumed that the vertical displacement will smoothen out, which gives rise to the natural boundary conditions $\frac{\partial u_z}{\partial x}|_{x=0} = 0$ and $\frac{\partial u_z}{\partial x}|_{x=L} = 0$. Since the displacements on the sides of the domain are negligible when L is sufficiently large, it is imposed that $u_x|_{x=0} = 0$ and $u_x|_{x=L} = 0$. Hence the volumetric strain does not have a gradient on these boundaries, i.e. $\frac{\partial \epsilon_{\text{vol}}}{\partial x}|_{x=0,L} = 0$. Since the displacements of the pore water are also negligible on $x = 0$ and $x = L$, the water pressure does not have a gradient, i.e. $\frac{\partial P}{\partial x}|_{x=0,L} = 0$. Furthermore, the displacements flatten out, thus $\frac{\partial u_x}{\partial z}|_{x=0} = 0$ and $\frac{\partial u_x}{\partial z}|_{x=L} = 0$. Combined with the boundary conditions $\frac{\partial u_z}{\partial x}|_{x=0,L} = 0$, this also yields that $\omega|_{x=0,L} = 0$.

Following the boundary conditions for a porous seabed (Ye et. al [6]), the bottom of the levee, i.e. at $z = -Z$, is both rigid and impermeable. As a consequence the soil should not be allowed to sink here, hence the vertical displacement should be set to zero, i.e. $u_z = 0$. The same argument is made for the pore water, hence the displacement on $z = -Z$ is $v_z = 0$. As a consequence, there is no gradient of the water pressure at $z = -Z$, i.e. $\frac{\partial P}{\partial z} = 0$. To enforce that the effective stress gradient equals 0 as well, the shear stresses at $z = -Z$ have to be set to 0. Since the shear stresses can only be transferred by the soil particles, it is required that $\frac{\partial u_x}{\partial z} = 0$. Note that this boundary condition changes from the slightly stronger Dirichlet boundary condition in the analytical approach by Van Damme and Den Ouden-van der Horst [15].

An implicit boundary condition

On $z = 0$, we can differentiate the first boundary condition to x, obtaining

$$-\frac{\alpha}{2}\frac{\partial \omega}{\partial x} - \frac{\partial^2 u_z}{\partial x^2} = \frac{\partial \tau_{xz}}{\partial x}. \tag{2.113}$$

Substituting this value in the third boundary condition and using the definition of ϵ_{vol} results in an equivalent expression for the third boundary condition at $z = 0$, given by

$$-(\alpha+\beta)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\alpha}{2}\frac{\partial \omega}{\partial x} + \frac{\partial P}{\partial z} = 0. \tag{2.114}$$

This means that only first order derivatives in space appear in the system, something that will prove to be very useful later, since it enables the use of linear basis function in a finite element approach (linear basis functions would vanish when taking a second order derivative in space). Furthermore, the new momentum Equation (2.114) should, when acceleration terms are negligible, in theory be valid on the whole domain, so on $z = -Z$ as well. Since for $z = -Z$ we already have that $u_z = 0$ and hence $\frac{\partial u_z}{\partial x} = 0$, besides $\frac{\partial u_x}{\partial z} = 0$, it also holds that $\omega = 0$. This also implies that $\frac{\partial \omega}{\partial x} = 0$. Together with $\frac{\partial P}{\partial z} = 0$, this means it also has to hold that

$$\frac{\partial \epsilon_{\text{vol}}}{\partial z} = 0, \tag{2.115}$$

on Γ_1. Now that we have taken a more thorough look at the boundary conditions, the initial conditions will be analyzed.

2.11. Initial conditions

Since the partial differential equations of the model are of the second order in time, two initial conditions for both the vorticity and volumetric strain are necessary. We make the assumption that at $t = 0$ no hydrodynamic load is present on the soil. In other words, there will not be a shear stress on the surface, so the initial vorticiy will be equal to zero as well, i.e. $w|_{t=0} = 0$. Furthermore, in order to create a second initial condition to the vorticity, it is assumed that the first overtopping wave will only arrive after some time. Hence the vorticity will initially not change over time, so it can be imposed that $\frac{\partial w}{\partial t}|_{t=0} = 0$. Furthermore, it is assumed that at $t = 0$ effective stresses are absent and as a consequence the volumetric strain will be zero, i.e. $\epsilon_{\text{vol}} = 0$. Since the water pressure P is also assumed to be zero at $t = 0$, it has to hold that $\frac{\partial \epsilon_{\text{vol}}}{\partial t}|_{t=0} = 0$. In summary, the initial conditions are

$$w|_{t=0} = \frac{\partial w}{\partial t}|_{t=0} = 0, \tag{2.116}$$

$$\epsilon_{\text{vol}}|_{t=0} = \frac{\partial \epsilon_{\text{vol}}}{\partial t}|_{t=0} = 0. \tag{2.117}$$

Implicit initial conditions

Apart from the initial conditions that are elaborated on, more initial conditions are needed for solving the linear system that follows from a numerical scheme. Luckily, this is pretty straightforward. Note that, since the overtopping waves are only arriving after some time, it holds that any order of time derivatives of ϵ_{vol} and ω will be equal to zero. Applying this to the partial differential equations, evaluated in $t = 0$, results in

$$\Delta P = 0, \tag{2.118}$$
$$\Delta u_x = 0, \tag{2.119}$$
$$\Delta u_z = 0, \tag{2.120}$$

with boundary conditions

$$\text{on } \Gamma_1 \begin{cases} \frac{\partial P}{\partial z} & = 0, \\ u_z & = 0, \\ \frac{\partial u_x}{\partial z} & = 0, \end{cases} \tag{2.121}$$
$$\text{on } \Gamma_2 \text{ and } \Gamma_4 \begin{cases} \frac{\partial P}{\partial x} & = 0, \\ u_x & = 0, \\ \frac{\partial u_z}{\partial x} & = 0, \end{cases} \tag{2.122}$$
$$\text{on } \Gamma_3 \begin{cases} \frac{\partial P}{\partial z} & = 0, \\ \alpha \frac{\partial u_x}{\partial x} + P & = 0, \\ \frac{\partial u_z}{\partial x} & = 0. \end{cases} \tag{2.123}$$
$$\tag{2.124}$$

Note that $\frac{\partial u_z}{\partial x} = 0$ implies a Dirichlet boundary condition on Γ_3. By using separation of variables the homogeneous Poisson equation for u_x can be found, which is the trivial solution. This is because of the homogeneous Dirichlet boundary conditions on Γ_2 and Γ_4. This implies a homogeneous Dirichlet boundary condition for P on Γ_3. The two other decoupled Poisson equations both only have the trivial solution as well. As a consequence

$$P = 0, \tag{2.125}$$
$$u_x = 0, \tag{2.126}$$
$$u_z = 0, \tag{2.127}$$

on $t = 0$ for $\boldsymbol{x} \in \Omega$. This is perfectly in line with our assumptions, since without overtopping waves no dynamic pressure should be present in the levee and there will not be any displacements either. If $t_1 > 0$ is defined as the time where the first overtopping wave flows over the levee, the situation at $t = 0$ will initially remain the same. Hence it also holds that:

$$\frac{\partial P}{\partial t} = 0, \tag{2.128}$$
$$\frac{\partial u_x}{\partial t} = 0, \tag{2.129}$$
$$\frac{\partial u_z}{\partial t} = 0. \tag{2.130}$$

This completes the derivation of the total system.

2.12. Complete system

In this chapter, a full derivation of the complete system that describes the physics in the levee is given. In conclusion, the system can be written as

$$\text{for } \boldsymbol{x} \in \Omega \begin{cases} (1-p)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{\rho_p K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - \frac{3\beta-2\alpha}{\rho_p}\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} - \frac{3\beta-2\alpha}{\rho_p}\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} & = 0, \\ \rho_p(1-p)\frac{\partial^2 w}{\partial t^2} - \frac{\alpha}{2}\frac{\partial^2 w}{\partial x^2} - \frac{\alpha}{2}\frac{\partial^2 w}{\partial z^2} & = 0, \\ \rho_w \frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} + \frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} & = 0, \\ \frac{\partial \omega}{\partial z} + \frac{\partial \epsilon_{\text{vol}}}{\partial x} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial z^2} & = 0, \\ -\frac{\partial \omega}{\partial x} + \frac{\partial \epsilon_{\text{vol}}}{\partial z} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} & = 0, \end{cases} \tag{2.131}$$

With the boundary conditions

$$\text{for } x = 0 \text{ and } x = L\text{: } \begin{cases} u_x & = 0, \\ \frac{\partial u_z}{\partial x} & = 0, \\ w & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial x} & = 0, \\ \frac{\partial P}{\partial x} & = 0. \end{cases} \tag{2.132}$$

$$\text{For } z = 0\text{: } \begin{cases} \frac{\alpha}{2}\omega - \alpha \frac{\partial u_x}{\partial z} & = \tau_{xz}, \\ -\beta \epsilon_{\text{vol}} - \alpha \frac{\partial u_z}{\partial z} + P & = \sigma_{zz}, \\ -(\alpha+\beta)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\alpha}{2}\frac{\partial \omega}{\partial x} + \frac{\partial P}{\partial z} & = 0. \end{cases} \tag{2.133}$$

$$\text{For } z = -Z\text{: } \begin{cases} u_z & = 0, \\ \frac{\partial u_x}{\partial z} & = 0, \\ \frac{\partial P}{\partial z} & = 0, \\ \omega & = 0, \\ \frac{\partial \epsilon_{\text{vol}}}{\partial z} & = 0. \end{cases} \tag{2.134}$$

Finally, we have the initial conditions:

$$\begin{cases} \dot{\epsilon}_{\text{vol}}|_{t=0} = \epsilon_{\text{vol}}|_{t=0} & = 0, \\ \dot{\omega}|_{t=0} = \omega|_{t=0} & = 0, \\ \dot{P}|_{t=0} = P|_{t=0} & = 0, \\ \dot{u}_x|_{t=0} = u_x|_{t=0} & = 0, \\ \dot{u}_z|_{t=0} = u_z|_{t=0} & = 0. \end{cases} \tag{2.135}$$

In the following section, a numerical approach will be extensively worked out in order to solve the system.

3

Numerical approach

In this section, a numerical method will be chosen to try and solve the system. Several options are available and will now be discussed.

3.1. Different approaches

Since the system is extensive, several approaches are possible in order to make a numerical approximation. A Finite Difference Method (something similar to Chorin's projection method [2]) does not provide a framework to take the boundary conditions implicitly into account. This means that the boundary conditions have to be treated explicitly, by solving a system with constraints. The system given by Equations (2.131)-(2.135) is almost fully decoupled by introducing the displacement relations instead of the definitions of ϵ_{vol} and ω (there is only a one sided coupling instead of a two sided coupling for the last two partial differential equations of the system). A Finite Element Analysis can reduce the order of the partial differential equations and enables treating the boundary conditions in a natural way, such that they become implicit in the weak formulation of the system. A similar thing can be done with a Finite Volume Method, which have the benefit of local conservation in grid cells; a Finite Elements Method on the other hand only has global conservation. Moreover, a Finite Volume Method controls the local fluxes, such that unphysical oscillations will not appear. A finite element approach on the other hand, enables improving the accuracy of the system quite easily, by simply choosing basis functions of a higher order. Moreover, despite the system being described for a rectangular domain, it is very likely that other applications of the model might be connected to a more complex domain. Finite Elements Methods are known to be very flexible when it comes to more difficult domains. Using a finite elements approach hence makes the research more valuable for potential extensions in other fields of research, i.e. more widely applicable. In conclusion a finite element method is used.

In this thesis, linear triangular elements will be used but an extension to quadrilateral- or other elements is possible. Initially, the definitions of ϵ_{vol} and ω were used as the fourth- and fifth partial differential equation of the system. However the definitions for ϵ_{vol} and ω provide some difficulties, since the test functions connected to u_x and u_z do not have boundary conditions that are able to get rid- or simplify the boundary integrals in the final weak form. Furthermore it is not clear which equation should be multiplied with which test function. On top of that, there are natural boundary conditions that cannot be incorporated in the weak formulation. Whether or not this would result in the solution not converging is not known. Furthermore, an attempt was made to substitute the definitions of ϵ_{vol} and ω into the system, resulting in a system consisting of three partial differential equations with three unknown parameters. However, by doing this the boundary conditions and partial differential equations become heavily coupled. Furthermore a lot of boundary conditions cannot be incorporated into the weak formulation and would act as a constraint to the problem, so this was not a suitable way to proceed. In conclusion, using the system given by Equations (2.131)-(2.135) is definitely the more natural choice, since all natural boundary conditions can be substituted into the weak form. In this section, the weak formulation of the system and the resulting Galerkin equations will be derived in this section. Finally, two suitable time integration methods will be treated, one explicit and one implicit. It is hard to say whether the known stress functions $(\tau_{xz}(x,0), \sigma_{zz}(x,0))$ are smooth enough, and hence whether an explicit method will show spurious oscillations. Because of this, both time stepping methods will be explained, implemented and compared.

3.2. Weak formulation

Every partial differential equation of Equation (2.131) will be multiplied by test functions $\eta^{\epsilon_{\text{vol}}}, \eta^{\omega}, \eta^{P}, \eta^{u_x}$ and η^{u_z} respectively and integrated over domain Ω. We will define the set Σ_η by all test function vectors that satisfy the boundary conditions given by:

$$\text{for } x = 0 \text{ and } x = L\text{:} \begin{cases} \eta^{u_x} & = 0, \\ \eta^{w} & = 0. \end{cases} \tag{3.1}$$

$$\text{For } z = -Z\text{:} \begin{cases} \eta^{u_z} & = 0, \\ \eta^{\omega} & = 0. \end{cases} \tag{3.2}$$

Analogously, Σ_Γ is defined as the set of all test function vectors that satisfy the boundary conditions (2.132), (2.133) and (2.134). The requirements

$$\begin{pmatrix} \eta^{\epsilon_{\text{vol}}} \\ \eta^{\omega} \\ \eta^{P} \\ \eta^{u_x} \\ \eta^{u_z} \end{pmatrix} \in H^5_{\Sigma_\eta}(\Omega), \tag{3.3}$$

and

$$\begin{pmatrix} \epsilon_{\text{vol}} \\ \omega \\ P \\ u_x \\ u_z \end{pmatrix} \in H^5_{\Sigma_\Gamma}(\Omega), \tag{3.4}$$

are imposed, in order to derive a weak formulation. These sets are defined as:

$$H^5_{\Sigma_\eta}(\Omega) := \left\{ \eta^{\epsilon_{\text{vol}}}, \eta^{\omega}, \eta^{P}, \eta^{u_x}, \eta^{u_z} \in H^1(\Omega) \,\middle|\, \begin{pmatrix} \eta^{\epsilon_{\text{vol}}} \\ \eta^{\omega} \\ \eta^{P} \\ \eta^{u_x} \\ \eta^{u_z} \end{pmatrix} \in \Sigma_\eta \right\}, \tag{3.5}$$

$$H^5_{\Sigma_\Gamma}(\Omega) := \left\{ \epsilon_{\text{vol}}, \omega, P, u_x, u_z \in H^1(\Omega) \,\middle|\, \begin{pmatrix} \epsilon_{\text{vol}} \\ \omega \\ P \\ u_x \\ u_z \end{pmatrix} \in \Sigma_\Gamma \right\}, \tag{3.6}$$

where

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) \,\middle|\, \int_\Omega \|\nabla u\|^2 d\Omega < \infty \right\}, \tag{3.7}$$

$$L^2(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \,\middle|\, \int_\Omega u^2 d\Omega < \infty \right\}. \tag{3.8}$$

For the first partial differential equation, we have that:

$$(1-p)\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial^2\epsilon_{\text{vol}}}{\partial t^2}d\Omega+\frac{\gamma_w}{\rho_p K_s}\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial\epsilon_{\text{vol}}}{\partial t}d\Omega+\frac{2\alpha-3\beta}{\rho_p}\int_\Omega \eta^{\epsilon_{\text{vol}}}\Delta\epsilon_{\text{vol}}d\Omega=0, \quad (3.9)$$

$$\Rightarrow (1-p)\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial^2\epsilon_{\text{vol}}}{\partial t^2}d\Omega+\frac{\gamma_w}{\rho_p K_s}\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial\epsilon_{\text{vol}}}{\partial t}d\Omega+\frac{2\alpha-3\beta}{\rho_p}\left(\int_\Gamma \eta^{\epsilon_{\text{vol}}}\nabla\epsilon_{\text{vol}}\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^{\epsilon_{\text{vol}}}\cdot\nabla\epsilon_{\text{vol}}d\Omega\right)=0, \quad (3.10)$$

$$\Rightarrow (1-p)\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial^2\epsilon_{\text{vol}}}{\partial t^2}d\Omega+\frac{\gamma_w}{\rho_p K_s}\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial\epsilon_{\text{vol}}}{\partial t}d\Omega+\frac{2\alpha-3\beta}{\rho_p}\left(\int_{\Gamma_3} \eta^{\epsilon_{\text{vol}}}\nabla\epsilon_{\text{vol}}\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^{\epsilon_{\text{vol}}}\cdot\nabla\epsilon_{\text{vol}}d\Omega\right)=0, \quad (3.11)$$

$$\Rightarrow (1-p)\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial^2\epsilon_{\text{vol}}}{\partial t^2}d\Omega+\frac{\gamma_w}{\rho_p K_s}\int_\Omega \eta^{\epsilon_{\text{vol}}}\frac{\partial\epsilon_{\text{vol}}}{\partial t}d\Omega+\frac{2\alpha-3\beta}{\rho_p}\left(\int_{\Gamma_3} \eta^{\epsilon_{\text{vol}}}\left(\frac{\alpha}{2\alpha+2\beta}\frac{\partial\omega}{\partial x}+\frac{1}{\alpha+\beta}\frac{\partial P}{\partial z}\right)d\Gamma-\int_\Omega\nabla\eta^{\epsilon_{\text{vol}}}\cdot\nabla\epsilon_{\text{vol}}d\Omega\right)=0, \quad (3.12)$$

where for the second equality Theorem 1 of Appendix A is used and for the final equality the boundary conditions and the expression for the normal vector are used.

Similarly, for the second partial differential equation, we have that

$$\rho_p(1-p)\int_\Omega \eta^\omega\frac{\partial^2 w}{\partial t^2}d\Omega-\frac{\alpha}{2}\int_\Omega\eta^\omega\Delta w d\Omega=0, \quad (3.13)$$

$$\Rightarrow \rho_p(1-p)\int_\Omega \eta^\omega\frac{\partial^2 w}{\partial t^2}d\Omega-\frac{\alpha}{2}\left(\int_\Gamma\eta^\omega\nabla w\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^\omega\cdot\nabla w d\Omega\right)=0, \quad (3.14)$$

$$\Rightarrow \rho_p(1-p)\int_\Omega \eta^\omega\frac{\partial^2 w}{\partial t^2}d\Omega-\frac{\alpha}{2}\left(\int_{\Gamma_3}\eta^\omega\frac{\partial w}{\partial z}d\Gamma-\int_\Omega\nabla\eta^\omega\cdot\nabla w d\Omega\right)=0, \quad (3.15)$$

since $\eta^\omega=0$ for $x=0, x=L$ and $z=-Z$. Again, in a similar manner, it holds for the third partial differential equation that

$$\rho_w\int_\Omega\eta^P\frac{\partial^2\epsilon}{\partial t^2}d\Omega+\frac{\gamma_w}{K_s}\int_\Omega\eta^P\frac{\partial\epsilon}{\partial t}d\Omega-\int_\Omega\eta^P\Delta P d\Omega=0, \quad (3.16)$$

$$\Rightarrow \rho_w\int_\Omega\eta^P\frac{\partial^2\epsilon}{\partial t^2}d\Omega+\frac{\gamma_w}{K_s}\int_\Omega\eta^P\frac{\partial\epsilon}{\partial t}d\Omega-\left(\int_\Gamma\eta^P\nabla P\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^P\cdot\nabla P d\Omega\right)=0, \quad (3.17)$$

$$\Rightarrow \rho_w\int_\Omega\eta^P\frac{\partial^2\epsilon}{\partial t^2}d\Omega+\frac{\gamma_w}{K_s}\int_\Omega\eta^P\frac{\partial\epsilon}{\partial t}d\Omega-\int_{\Gamma_3}\eta^P\frac{\partial P}{\partial z}d\Gamma+\int_\Omega\nabla\eta^P\cdot\nabla P d\Omega=0. \quad (3.18)$$

For the displacement relations we have that:

$$\text{for } \boldsymbol{x}\in\Omega\begin{cases}\int_\Omega\eta^{u_x}\left(\frac{\partial\omega}{\partial z}+\frac{\partial\epsilon_{\text{vol}}}{\partial x}\right)d\Omega-\int_\Omega\eta^{u_x}\Delta u_x d\Omega &=0,\\ \int_\Omega\eta^{u_z}\left(\frac{\partial\epsilon_{\text{vol}}}{\partial z}-\frac{\partial\omega}{\partial x}\right)d\Omega-\int_\Omega\eta^{u_z}\Delta u_z d\Omega &=0,\end{cases} \quad (3.19)$$

and after using Theorem 1 of Appendix A

$$\text{for } \boldsymbol{x}\in\Omega\begin{cases}\int_\Gamma\eta^{u_x}\begin{pmatrix}\epsilon_{\text{vol}}\\ \omega\end{pmatrix}\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^{u_x}\cdot\begin{pmatrix}\epsilon_{\text{vol}}\\ \omega\end{pmatrix}d\Omega-\int_\Gamma\eta^{u_x}\frac{\partial u_x}{\partial n}d\Gamma+\int_\Omega\nabla\eta^{u_x}\cdot\nabla u_x d\Omega &=0,\\ \int_\Gamma\eta^{u_z}\begin{pmatrix}-\omega\\ \epsilon_{\text{vol}}\end{pmatrix}\cdot\boldsymbol{n}d\Gamma-\int_\Omega\nabla\eta^{u_z}\cdot\begin{pmatrix}-\omega\\ \epsilon_{\text{vol}}\end{pmatrix}d\Omega-\int_\Gamma\eta^{u_z}\frac{\partial u_z}{\partial n}d\Gamma+\int_\Omega\nabla\eta^{u_z}\cdot\nabla u_z d\Omega &=0.\end{cases} \quad (3.20)$$

Since we have that $u_x=0$ on $x=0$ and $x=L$, it also has to hold that $\eta^{u_x}=0$ on these boundaries. Furthermore $\frac{\partial u_x}{\partial z}=0$ for $z=-Z$. So the first equation of Equation (3.20) is equivalent to

$$-\int_{\Gamma_1}\eta^{u_x}\omega d\Gamma-\int_\Omega\nabla\eta^{u_x}\cdot\begin{pmatrix}\epsilon_{\text{vol}}\\ \omega\end{pmatrix}d\Omega-\int_{\Gamma_3}\eta^{u_x}\frac{\partial u_x}{\partial z}d\Gamma+\int_\Omega\nabla\eta^{u_x}\cdot\nabla u_x d\Omega=0. \quad (3.21)$$

Furthermore, it can be used that for $z=0$ it holds that $\frac{\partial u_x}{\partial z}=\frac{1}{2}\omega-\frac{1}{\alpha}\tau_{xz}$, which results in the following weak form of the first equality of Equation (3.20):

$$\int_{\Gamma_1} \eta^{u_x} \omega d\Gamma + \int_\Omega \nabla \eta^{u_x} \cdot \begin{pmatrix} \epsilon_{\text{vol}} \\ \omega \end{pmatrix} d\Omega + \frac{1}{2} \int_{\Gamma_3} \eta^{u_x} \omega d\Gamma - \int_\Omega \nabla \eta^{u_x} \cdot \nabla u_x d\Omega = \frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_x} \tau_{xz} d\Gamma. \tag{3.22}$$

For the second equality of Equation (3.20), we have that $\frac{\partial u_z}{\partial x} = 0$ and $\omega = 0$ for $x = 0$ and $x = L$, $u_z = 0$, so $\eta^{u_z} = 0$ for $z = -Z$ and $\frac{\partial u_z}{\partial z} = \frac{1}{\alpha}\left(P - \beta \epsilon_{\text{vol}} - \sigma\right)$ for $z = 0$. The weak formulation becomes

$$\left(\frac{\beta}{\alpha} - 1\right) \int_{\Gamma_3} \eta^{u_z} \epsilon_{\text{vol}} d\Gamma - \frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_z} P d\Gamma + \int_\Omega \nabla \eta^{u_z} \cdot \nabla u_z d\Omega + \int_\Omega \nabla \eta^{u_z} \cdot \begin{pmatrix} -\omega \\ \epsilon_{\text{vol}} \end{pmatrix} d\Omega = -\frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_z} \sigma_{zz} d\Gamma. \tag{3.23}$$

Hence the weak formulation of the system given by Equations (2.131)-(2.135) becomes:

$$\begin{cases}
(1-p) \int_\Omega \eta^{\epsilon_{\text{vol}}} \frac{\partial^2 \epsilon_{\text{vol}}}{\partial t^2} d\Omega + \frac{\gamma_w}{\rho_p K_s} \int_\Omega \eta^{\epsilon_{\text{vol}}} \frac{\partial \epsilon_{\text{vol}}}{\partial t} d\Omega + \frac{2\alpha - 3\beta}{\rho_p} \left(\int_{\Gamma_3} \eta^{\epsilon_{\text{vol}}} \left(\frac{\alpha}{2\alpha + 2\beta} \frac{\partial \omega}{\partial x} + \frac{1}{\alpha + \beta} \frac{\partial P}{\partial z} \right) d\Gamma - \int_\Omega \nabla \eta^{\epsilon_{\text{vol}}} \cdot \nabla \epsilon_{\text{vol}} d\Omega \right) & = 0, \\
\rho_p (1-p) \int_\Omega \eta^\omega \frac{\partial^2 w}{\partial t^2} d\Omega - \frac{\alpha}{2} \left(\int_{\Gamma_3} \eta^\omega \frac{\partial w}{\partial z} d\Gamma - \int_\Omega \nabla \eta^\omega \cdot \nabla w d\Omega \right) & = 0, \\
\rho_w \int_\Omega \eta^P \frac{\partial^2 \epsilon}{\partial t^2} d\Omega + \frac{\gamma_w}{K_s} \int_\Omega \eta^P \frac{\partial \epsilon}{\partial t} d\Omega - \int_{\Gamma_3} \eta^P \frac{\partial P}{\partial z} d\Gamma + \int_\Omega \nabla \eta^P \cdot \nabla P d\Omega & = 0, \\
\int_{\Gamma_1} \eta^{u_x} \omega d\Gamma + \int_\Omega \nabla \eta^{u_x} \cdot \begin{pmatrix} \epsilon_{\text{vol}} \\ \omega \end{pmatrix} d\Omega + \frac{1}{2} \int_{\Gamma_3} \eta^{u_x} \omega d\Gamma - \int_\Omega \nabla \eta^{u_x} \cdot \nabla u_x d\Omega & = \frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_x} \tau_{xz} d\Gamma, \\
\left(\frac{\beta}{\alpha} - 1\right) \int_{\Gamma_3} \eta^{u_z} \epsilon_{\text{vol}} d\Gamma - \frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_z} P d\Gamma + \int_\Omega \nabla \eta^{u_z} \cdot \nabla u_z d\Omega + \int_\Omega \nabla \eta^{u_z} \cdot \begin{pmatrix} -\omega \\ \epsilon_{\text{vol}} \end{pmatrix} d\Omega & = -\frac{1}{\alpha} \int_{\Gamma_3} \eta^{u_z} \sigma_{zz} d\Gamma.
\end{cases} \tag{3.24}$$

This weak formulation will be used to derive the Galerkin equations.

3.3. Galerkin equations

Following Galerkin's method, the parameters are approximated by a linear combination of a fixed set of basis functions, where the coefficients of the linear combination are dependent on time.

$$\epsilon_{\text{vol}}(\boldsymbol{x}, t) \approx \epsilon_{\text{vol}}^n(\boldsymbol{x}, t) = \sum_{j=1}^n a_j(t) \eta_j(\boldsymbol{x}), \tag{3.25}$$

$$w(\boldsymbol{x}, t) \approx w^n(\boldsymbol{x}, t) = \sum_{j=1}^n b_j(t) \eta_j(\boldsymbol{x}), \tag{3.26}$$

$$P(\boldsymbol{x}, t) \approx P^n(\boldsymbol{x}, t) = \sum_{j=1}^n c_j(t) \eta_j(\boldsymbol{x}), \tag{3.27}$$

$$u_x(\boldsymbol{x}, t) \approx u_x^n(\boldsymbol{x}, t) = \sum_{j=1}^n d_j(t) \eta_j(\boldsymbol{x}), \tag{3.28}$$

$$u_z(\boldsymbol{x}, t) \approx u_z^n(\boldsymbol{x}, t) = \sum_{j=1}^n e_j(t) \eta_j(\boldsymbol{x}). \tag{3.29}$$

Furthermore, every test function will be replaced by $\eta_i(\boldsymbol{x})$ for some $i \in \{1, \ldots, n\}$. Plugging in the expressions (3.25) gives us:

$$\sum_{j=1}^{n}\left\{(1-p)\frac{d^2a_j}{dt^2}\int_\Omega \eta_i\eta_j d\Omega + \frac{\gamma_w}{\rho_p K_s}\frac{da_j}{dt}\int_\Omega \eta_i\eta_j d\Omega - \frac{2\alpha-3\beta}{\rho_p}a_j\int_\Omega \nabla\eta_i\cdot\nabla\eta_j d\Omega + \frac{\alpha(2\alpha-3\beta)}{\rho_p(2\alpha+2\beta)}b_j\int_{\Gamma_3}\eta_i\frac{\partial\eta_j}{\partial x}d\Gamma + \frac{2\alpha-3\beta}{\rho_p(\alpha+\beta)}c_j\int_{\Gamma_3}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma\right\} = 0, \tag{3.30}$$

$$\sum_{j=1}^{n}\left\{\rho_p(1-p)\frac{d^2b_j}{dt^2}\int_\Omega \eta_i\eta_j d\Omega - \frac{\alpha}{2}b_j\left(\int_{\Gamma_3}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma - \int_\Omega \nabla\eta_i\cdot\nabla\eta_j d\Omega\right)\right\} = 0, \tag{3.31}$$

$$\sum_{j=1}^{n}\left\{\rho_w\frac{d^2a_j}{dt^2}\int_\Omega \eta_i\eta_j d\Omega + \frac{\gamma_w}{K_s}\frac{da_j}{dt}\int_\Omega \eta_i\eta_j d\Omega + c_j\left(\int_\Omega \nabla\eta_i\cdot\nabla\eta_j d\Omega - \int_{\Gamma_3}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma\right)\right\} = 0, \tag{3.32}$$

$$\sum_{j=1}^{n}\left\{a_j\int_\Omega \frac{\partial\eta_i}{\partial x}\eta_j d\Omega + b_j\left(\int_{\Gamma_1}\eta_i\eta_j d\Gamma + \int_\Omega \frac{\partial\eta_i}{\partial z}\eta_j d\Omega + \frac{1}{2}\int_{\Gamma_3}\eta_i\eta_j d\Gamma\right) - d_j\int_\Omega \nabla\eta_i\cdot\nabla\eta_j d\Omega\right\} = \frac{1}{\alpha}\int_{\Gamma_3}\eta_i\tau_{xz}d\Gamma, \tag{3.33}$$

$$\sum_{j=1}^{n}\left\{a_j\left(\left(\frac{\beta}{\alpha}-1\right)\int_{\Gamma_3}\eta_i\eta_j d\Gamma + \int_\Omega \frac{\partial\eta_i}{\partial z}\eta_j d\Omega\right) - b_j\int_\Omega \frac{\partial\eta_i}{\partial x}\eta_j d\Omega - \frac{1}{\alpha}c_j\int_{\Gamma_3}\eta_i\eta_j d\Gamma + e_j\int_\Omega \nabla\eta_i\cdot\nabla\eta_j d\Omega\right\} = -\frac{1}{\alpha}\int_{\Gamma_3}\eta_i\sigma_{zz}d\Gamma, \tag{3.34}$$

for $i \in \{1,\ldots,n\}$. Instead of evaluating the integrals on the whole domain and boundaries, the domain will be split up in n_e triangular linear elements, i.e. $\bigcup_{k=1}^{n_e}\Omega_{e_k} = \Omega$, where $|\Omega_{e_k}|$ is the area of linear triangular element e_k. The aim is to find a linear system

$$M\ddot{\boldsymbol{a}}(t) + W\dot{\boldsymbol{a}}(t) + S\boldsymbol{a}(t) = \boldsymbol{f}, \tag{3.35}$$

where

$$\boldsymbol{a}(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \\ b_1(t) \\ \vdots \\ b_n(t) \\ c_1(t) \\ \vdots \\ c_n(t) \\ d_1(t) \\ \vdots \\ d_n(t) \\ e_1(t) \\ \vdots \\ e_n(t) \end{pmatrix}. \tag{3.36}$$

By introducing linear triangular elements, the matrix S can be defined by

$$S_{ij} := \sum_{k=1}^{n_e}(S^{e_k})_{ij} + \sum_{k\in\Lambda_{\Gamma_1}}(S^{be_k})_{ij} + \sum_{k\in\Lambda_{\Gamma_2}}(S^{be_k})_{ij} + \sum_{k\in\Lambda_{\Gamma_3}}(S^{be_k})_{ij} + \sum_{k\in\Lambda_{\Gamma_4}}(S^{be_k})_{ij}, \tag{3.37}$$

where n_e is the number of internal elements, Λ_{Γ_1} the set of boundary elements on boundary Γ_1, Λ_{Γ_2} the set of boundary elements on boundary Γ_2, Λ_{Γ_3} the set of boundary elements on boundary Γ_3 and Λ_{Γ_4} the set of boundary elements on boundary Γ_4. In this expression S^{e_k} denotes an internal element matrix and S_{be_k} denotes a boundary element matrix. Matrices M and W and vector $\boldsymbol{f}$ are defined analogously. In order to compute these element matrices, the basis functions have to be chosen.

Linear basis functions

Because in the Galerkin equations there are only first order derivatives present in space, the basis functions can chosen to be linear per triangle, to keep the implementation simple. If we take a closer look to an arbitrary linear triangular element with vertices $((x_1,z_1),(x_2,z_2),(x_3,z_3))$, every basis function can be written as:

$$\eta_i = \psi_i + \theta_i x + \zeta_i z, \tag{3.38}$$

where $\psi_i, \theta_i, \zeta_i \in \mathbb{R}$. According to Van Kan et al. [18] the coefficients can be determined for each $i \in \{1,\ldots,n\}$ and are given by

$$\theta_1 = \frac{1}{\Delta}(z_2 - z_3), \qquad \theta_2 = \frac{1}{\Delta}(z_3 - z_1), \qquad \theta_3 = \frac{1}{\Delta}(z_1 - z_2), \tag{3.39}$$

$$\zeta_1 = \frac{1}{\Delta}(x_3 - x_2), \qquad \zeta_2 = \frac{1}{\Delta}(x_1 - x_3), \qquad \zeta_3 = \frac{1}{\Delta}(x_2 - x_1), \tag{3.40}$$

$$\psi_i = 1 - \theta_i x_i - \zeta_i z_i. \tag{3.41}$$

Δ is the coefficient determinant defined by:

$$\Delta = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, \tag{3.42}$$

$$= (x_2 - x_1)(z_3 - z_1) - (z_2 - z_1)(x_3 - x_1), \tag{3.43}$$

with $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ the vertices of triangle e_k. Van Kan et. al [18] show that this value $|\Delta|$ equals twice the area of triangle e_k. For the computations of the element matrices and vectors, the Theorems 2 and 3 in Appendix A will repeatedly be used.

Internal element matrices

For an internal element, i.e. $k \in \{1,\ldots,n_e\}$ with vertices $\boldsymbol{x}_{k_1}, \boldsymbol{x}_{k_2}, \boldsymbol{x}_{k_3}$ we have that the internal element matrix is given by:

$$S^{e_k} = \begin{pmatrix} S^{e_k}_{aa} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & S^{e_k}_{bb} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & S^{e_k}_{cc} & \varnothing & \varnothing \\ S^{e_k}_{da} & S^{e_k}_{db} & \varnothing & S^{e_k}_{dd} & \varnothing \\ S^{e_k}_{eb} & S^{e_k}_{eb} & \varnothing & \varnothing & S^{e_k}_{ee} \end{pmatrix}, \tag{3.44}$$

where the non-empty submatrices are given by:

$$(S^{e_k}_{aa})_{ij} = -\frac{2\alpha - 3\beta}{\rho_p}\int_{e_k} \nabla\eta_i \cdot \nabla\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.45}$$

$$(S^{e_k}_{bb})_{ij} = \frac{\alpha}{2}\int_{e_k} \nabla\eta_i \cdot \nabla\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.46}$$

$$(S^{e_k}_{cc})_{ij} = \int_{e_k} \nabla\eta_i \cdot \nabla\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.47}$$

$$(S^{e_k}_{da})_{ij} = \int_{e_k} \frac{\partial\eta_i}{\partial x}\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.48}$$

$$(S^{e_k}_{db})_{ij} = \int_{e_k} \frac{\partial\eta_i}{\partial z}\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.49}$$

$$(S^{e_k}_{dd})_{ij} = -\int_{e_k} \nabla\eta_i \cdot \nabla\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.50}$$

$$(S^{e_k}_{ea})_{ij} = \int_{e_k} \frac{\partial\eta_i}{\partial z}\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.51}$$

$$(S^{e_k}_{eb})_{ij} = -\int_{e_k} \frac{\partial\eta_i}{\partial x}\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2, \tag{3.52}$$

$$(S^{e_k}_{ee})_{ij} = \int_{e_k} \nabla\eta_i \cdot \nabla\eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1,k_2,k_3\}^2. \tag{3.53}$$

Computing these values with the use of Theorem 3 of Appendix A gives the expressions

$$(S^{e_k}_{aa})_{ij} = -\frac{(2\alpha-3\beta)(\theta_i\theta_j+\zeta_i\zeta_j)|\Delta_{e_k}|}{2\rho_p} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.54}$$

$$(S^{e_k}_{bb})_{ij} = \frac{\alpha(\theta_i\theta_j+\zeta_i\zeta_j)|\Delta_{e_k}|}{4} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.55}$$

$$(S^{e_k}_{cc})_{ij} = \frac{(\theta_i\theta_j+\zeta_i\zeta_j)|\Delta_{e_k}|}{2} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.56}$$

$$(S^{e_k}_{da})_{ij} = \theta_i\frac{|\Delta_{e_k}|}{6} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.57}$$

$$(S^{e_k}_{db})_{ij} = \zeta_i\frac{|\Delta_{e_k}|}{6} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.58}$$

$$(S^{e_k}_{dd})_{ij} = -\frac{(\theta_i\theta_j+\zeta_i\zeta_j)|\Delta_{e_k}|}{2} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.59}$$

$$(S^{e_k}_{ea})_{ij} = \zeta_i\frac{|\Delta_{e_k}|}{6} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.60}$$

$$(S^{e_k}_{eb})_{ij} = -\theta_i\frac{|\Delta_{e_k}|}{6} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2, \tag{3.61}$$

$$(S^{e_k}_{ee})_{ij} = \frac{(\theta_i\theta_j+\zeta_i\zeta_j)|\Delta_{e_k}|}{2} \quad \text{for } (i,j)\in\{k_1,k_2,k_3\}^2. \tag{3.62}$$

The element vector for an internal element equals zero. Now the boundary elements need to be evaluated.

Boundary element matrices

If we take $\boldsymbol{x}_{l_1}$ and $\boldsymbol{x}_{l_2}$ as the end points of an arbitrary boundary element on Γ_1, i.e. be_k with $k\in\Lambda_{\Gamma_1}$ the boundary element matrix will be

$$S^{be_k} = \begin{pmatrix} S^{be_k}_{aa} & S^{be_k}_{ab} & S^{be_k}_{ac} & S^{be_k}_{ad} & S^{be_k}_{ae} \\ S^{be_k}_{ba} & S^{be_k}_{bb} & S^{be_k}_{bc} & S^{be_k}_{bd} & S^{be_k}_{be} \\ S^{be_k}_{ca} & S^{be_k}_{cb} & S^{be_k}_{cc} & S^{be_k}_{cd} & S^{be_k}_{ce} \\ S^{be_k}_{da} & S^{be_k}_{db} & S^{be_k}_{dc} & S^{be_k}_{dd} & S^{be_k}_{de} \\ S^{be_k}_{ea} & S^{be_k}_{eb} & S^{be_k}_{ec} & S^{be_k}_{ed} & S^{be_k}_{ee} \end{pmatrix}, \tag{3.63}$$

for $k\in\Lambda_{\Gamma_1}$, where the non-empty submatrix is given by:

$$(S^{be_k}_{db})_{ij} = \int_{be_k}\eta_i\eta_j\,d\Gamma \qquad \text{for } (i,j)\in\{l_1,l_2\}^2. \tag{3.64}$$

The integral can be computed with Theorem 2 of Appendix A, which results in

$$(S^{be_k}_{db})_{ij} = \frac{\|\boldsymbol{x}_{l_1}-\boldsymbol{x}_{l_2}\|}{6}(1+\delta_{ij}) \qquad \text{for } (i,j)\in\{l_1,l_2\}^2. \tag{3.65}$$

There is no contribution for the boundary element vector. If we take $\boldsymbol{x}_{o_1}$ and $\boldsymbol{x}_{o_2}$ as the end points of an arbitrary boundary element on Γ_3, the boundary element matrix is given by expression (3.63) with $k\in\Lambda_{\Gamma_3}$, with non-empty submatrices

$$(S_{ab}^{be_k})_{ij} = \frac{\alpha(2\alpha-3\beta)}{\rho_p(2\alpha+2\beta)}\int_{be_k}\eta_i\frac{\partial\eta_j}{\partial x}d\Gamma \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.66}$$

$$(S_{ac}^{be_k})_{ij} = \frac{2\alpha-3\beta}{\rho_p(\alpha+\beta)}\int_{be_k}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.67}$$

$$(S_{bb}^{be_k})_{ij} = -\frac{\alpha}{2}\int_{be_k}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.68}$$

$$(S_{cc}^{be_k})_{ij} = -\int_{be_k}\eta_i\frac{\partial\eta_j}{\partial z}d\Gamma \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.69}$$

$$(S_{db}^{be_k})_{ij} = \frac{1}{2}\int_{be_k}\eta_i\eta_j d\Gamma \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.70}$$

$$(S_{ea}^{be_k})_{ij} = \left(\frac{\beta}{\alpha}-1\right)\int_{be_k}\eta_i\eta_j d\Gamma \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.71}$$

$$(S_{ec}^{be_k})_{ij} = -\frac{1}{\alpha}\int_{be_k}\eta_i\eta_j d\Gamma \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.72}$$

Calculating these expressions with Theorem 2 of Appendix A gives

$$(S_{ab}^{be_k})_{ij} = \frac{\alpha(2\alpha-3\beta)}{\rho_p(2\alpha+2\beta)}\frac{\theta_j\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{2} \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.73}$$

$$(S_{ac}^{be_k})_{ij} = \frac{2\alpha-3\beta}{\rho_p(\alpha+\beta)}\frac{\zeta_j\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{2} \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.74}$$

$$(S_{bb}^{be_k})_{ij} = -\frac{\alpha\zeta_j\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{4} \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.75}$$

$$(S_{cc}^{be_k})_{ij} = -\frac{\zeta_j\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{2} \qquad \text{for } i\in\{o_1,o_2\}, j\in\{o_1,o_2,o_3\}, \tag{3.76}$$

$$(S_{db}^{be_k})_{ij} = \frac{\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{12}(1+\delta_{ij}) \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.77}$$

$$(S_{ea}^{be_k})_{ij} = \left(\frac{\beta}{\alpha}-1\right)\frac{\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{6}(1+\delta_{ij}) \qquad \text{for } (i,j)\in\{o_1,o_2\}^2, \tag{3.78}$$

$$(S_{ec}^{be_k})_{ij} = -\frac{\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{6\alpha}(1+\delta_{ij}) \qquad \text{for } (i,j)\in\{o_1,o_2\}^2. \tag{3.79}$$

On Γ_3 there is a contribution for the boundary element vector. The boundary element vector is given by:

$$\boldsymbol{f}^{be_k} = \begin{bmatrix} \boldsymbol{f}_a^{be_k} \\ \boldsymbol{f}_b^{be_k} \\ \boldsymbol{f}_c^{be_k} \\ \boldsymbol{f}_d^{be_k} \\ \boldsymbol{f}_e^{be_k} \end{bmatrix}, \tag{3.80}$$

with $k\in\Lambda_{\Gamma_3}$. The non-empty vectors are given by:

$$(\boldsymbol{f}_d^{be_k})_i = \frac{1}{\alpha}\int_{be_k}\eta_i\tau_{xz}d\Gamma \qquad \text{for } i\in\{o_1,o_2\}, \tag{3.81}$$

$$(\boldsymbol{f}_e^{be_k})_i = -\frac{1}{\alpha}\int_{be_k}\eta_i\sigma_{zz}d\Gamma \qquad \text{for } i\in\{o_1,o_2\}. \tag{3.82}$$

Application of the Newton-Cotes rule for integration gives the approximation:

$$(\boldsymbol{f}_d^{be_k})_i = \frac{\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{\alpha}\tau_{xz}(\boldsymbol{x}_i) \qquad \text{for } i\in\{o_1,o_2\}, \tag{3.83}$$

$$(\boldsymbol{f}_e^{be_k})_i = -\frac{\|\boldsymbol{x}_{o_1}-\boldsymbol{x}_{o_2}\|}{\alpha}\sigma_{zz}(\boldsymbol{x}_i) \qquad \text{for } i\in\{o_1,o_2\}. \tag{3.84}$$

Mass matrix

For an arbitrary linear triangular internal element (with vertices $\boldsymbol{x}_{k_1}, \boldsymbol{x}_{k_2}$ and $\boldsymbol{x}_{k_3}$), the mass element matrix M^{e_k}, connected to the second derivative in time, is given by:

$$M^{e_k} = \begin{pmatrix} M^{e_k}_{aa} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & M^{e_k}_{bb} & \varnothing & \varnothing & \varnothing \\ M^{e_k}_{ca} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}, \tag{3.85}$$

with $k = 1, \ldots, n_e$, where the non-empty submatrices are given by:

$$(M^{e_k}_{aa})_{ij} = (1-p)\int_{e_k} \eta_i \eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.86}$$

$$(M^{e_k}_{bb})_{ij} = \rho_p (1-p)\int_{e_k} \eta_i \eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.87}$$

$$(M^{e_k}_{ca})_{ij} = \rho_w \int_{e_k} \eta_i \eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.88}$$

or computed (again with the use of Theorem 3 of Appendix A)

$$(M^{e_k}_{aa})_{ij} = (1-p)\frac{|\Delta_{e_k}|}{24}(1+\delta_{ij}) \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.89}$$

$$(M^{e_k}_{bb})_{ij} = \rho_p (1-p)\frac{|\Delta_{e_k}|}{24}(1+\delta_{ij}) \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.90}$$

$$(M^{e_k}_{ca})_{ij} = \rho_w \frac{|\Delta_{e_k}|}{24}(1+\delta_{ij}) \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2. \tag{3.91}$$

Damping matrix

Similarly, the damping matrix W^{e_k}, connected to the first derivative in time, is given by

$$W^{e_k} = \begin{pmatrix} W^{e_k}_{aa} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ W^{e_k}_{ca} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}, \tag{3.92}$$

where the non-empty submatrices are given by:

$$(W^{e_k}_{aa})_{ij} = \frac{\gamma_w}{\rho_p K_s}\int_{e_k} \eta_i \eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.93}$$

$$(W^{e_k}_{ca})_{ij} = \frac{\gamma_w}{K_s}\int_{e_k} \eta_i \eta_j \, d\Omega \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.94}$$

or computed, using Theorem 3 of Appendix A

$$(W^{e_k}_{aa})_{ij} = \frac{\gamma_w}{\rho_p K_s}\frac{|\Delta_{e_k}|}{24}(1+\delta_{ij}) \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2, \tag{3.95}$$

$$(W^{e_k}_{ca})_{ij} = \frac{\gamma_w}{K_s}\frac{|\Delta_{e_k}|}{24}(1+\delta_{ij}) \qquad \text{for } (i,j) \in \{k_1, k_2, k_3\}^2. \tag{3.96}$$

This concludes the expressions for matrices M, W and S and vector $\boldsymbol{f}$ of linear System 3.35. Note that there is not a unique way to construct this linear system. Another option is introducing new parameters, like

$$q^p = \frac{\partial P}{\partial z}, \tag{3.97}$$

to get rid of the normal derivatives in the boundary integrals of weak formulation (3.30). This means that for each additional parameter an extra weak formulation needs to be derived and implemented. By doing this, the term $\frac{\partial P}{\partial z}$ will be approximated by a set of linear functions, instead of a constant. Since it is expected that close to the boundary Γ_3 potential problems will occur, since this is the location where the waves flow over the domain, introducing these additional parameters might prove to be a useful tool in case the just-described system runs into problems, especially regarding continuity of the parameters close to boundary Γ_3. In order to solve this system, it needs to be manipulated a bit and time stepping methods need to be introduced.

3.4. Time stepping method

Defining the vector $\boldsymbol{\delta}(t) = \begin{pmatrix} \boldsymbol{a}(t) \\ \dot{\boldsymbol{a}}(t) \end{pmatrix}$ enables writing the Equation (3.35) in a form with a reduced order in time, namely as

$$\tilde{M}\dot{\boldsymbol{\delta}} = \tilde{S}\boldsymbol{\delta} + \tilde{\boldsymbol{f}}, \tag{3.98}$$

where

$$\tilde{M} = \begin{pmatrix} M & \varnothing \\ \varnothing & M \end{pmatrix}, \tag{3.99}$$

$$\tilde{S} = \begin{pmatrix} \varnothing & M \\ -S & -W \end{pmatrix}, \tag{3.100}$$

$$\tilde{\boldsymbol{f}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix}. \tag{3.101}$$

Since every parameter is equal to zero on the whole domain Ω at $t = 0$, this means that we require the approximations to be zero as well. If we take $\epsilon^n_{\text{vol}}(\boldsymbol{x}, 0)$ for example, it has to hold that:

$$\sum_{j=1}^{n} a_j(0)\eta_j(\boldsymbol{x}) = 0, \tag{3.102}$$

which can only be the case if $a_j(0) = 0$ for all $j \in \{1, \dots, n\}$. To prove this, simply evaluate the series in a node of a linear triangular element, where all basis functions have to be zero, except one. This means that the coefficient belonging to this basis function has to equal 0. Since the node was arbitrary, it holds for all $j \in \{1, \dots, n\}$. Using the same procedure, it holds that:

$$a_j(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.103}$$

$$\frac{da_j}{dt}(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.104}$$

$$b_j(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.105}$$

$$\frac{db_j}{dt}(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.106}$$

$$c_j(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.107}$$

$$\frac{dc_j}{dt}(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.108}$$

$$d_j(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.109}$$

$$\frac{dd_j}{dt}(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.110}$$

$$e_j(0) = 0 \qquad \forall j = 1, \dots, n, \tag{3.111}$$

$$\frac{de_j}{dt}(0) = 0 \qquad \forall j = 1, \dots, n. \tag{3.112}$$

As a consequence, the initial condition for the linear system (3.98) is

$$\boldsymbol{\delta}(0) = \boldsymbol{0}. \tag{3.113}$$

For the time integration, a time stepping method has to be chosen. When we define the mesh size as $h = \max_k |\Omega_{e_k}|$, where $|\Omega_{e_k}|$ is the area of triangle e_k, the CFL number can be defined as:

$$CFL = \frac{\Delta t}{h}. \tag{3.114}$$

For explicit methods, there is a maximum value for CFL, since otherwise the method does not converge. This maximum value will be found by looking at the eigenvalues of the system. In the next section two time integration methods will be treated.

Singularity of matrix

Since the matrix $\tilde{M}$ is singular, the system needs to be manipulated first before applying a time stepping method. As described by Van Ophem et al. [19] , matrix $\tilde{M}$ can be decomposed as

$$\tilde{M} = XY^T. \tag{3.115}$$

Assuming that $\tilde{S}$ is invertible and introducing the new state variable

$$\boldsymbol{\psi}(t) = Y^T\boldsymbol{\delta}(t), \tag{3.116}$$

results in the linear system

$$\tilde{E}\dot{\boldsymbol{\psi}}(t) = \boldsymbol{\psi}(t) + \tilde{g}, \tag{3.117}$$

where

$$\begin{cases} \tilde{E} & = Y^T\tilde{S}^{-1}X, \\ \tilde{g} & = Y^T\tilde{S}^{-1}\tilde{f}, \\ \boldsymbol{\psi}(0) & = \mathbf{0}. \end{cases} \tag{3.118}$$

Now that we have successfully removed the singular matrix $\tilde{M}$, we can apply a time integration method.

Crank-Nicolson Method

The Crank-Nicolson Method is an implicit time integration scheme of second order. Applied to the system given by Equations (3.117) and (3.118), it results in the linear equation

$$\left(\tilde{E} - \frac{1}{2}\Delta t I\right)\boldsymbol{\psi}^{n+1} = \left(\tilde{E} + \frac{1}{2}\Delta t I\right)\boldsymbol{\psi}^{n} + \frac{1}{2}\Delta t\left(\tilde{\boldsymbol{g}}^{n+1} + \tilde{\boldsymbol{g}}^{n}\right). \tag{3.119}$$

Obviously this implicit time scheme is quite expensive to solve per time step. However, the Crank-Nicolson Method is known to be unconditionally stable, so will theoretically converge for any $\Delta t > 0$ [22]. However, since the stresses resulting from overtopping waves are heavily fluctuating with time, it is likely that in order to accurately capture the stresses, Δt has to be chosen small anyways. This gives rise to the idea that an explicit method can be used as well, to reduce the computational time per time step. A commonly used second order explicit scheme is the Modified Euler Method.

Modified Euler Method

When the Modified Euler Method is applied to system given by Equations (3.117) and (3.118), the predictor vector $\boldsymbol{\psi}_*^{n+1}$ is implicitly given by:

$$\tilde{E}\boldsymbol{\psi}_*^{n+1} = (\tilde{E} + \Delta t\tilde{I})\boldsymbol{\psi}^{n} + \tilde{\boldsymbol{g}}^{n}, \tag{3.120}$$

which enables computation of the value of $\boldsymbol{\psi}^{n+1}$ by solving the system

$$\tilde{E}\boldsymbol{\psi}^{n+1} = \left(\tilde{E} + \frac{\Delta t}{2}I\right)\boldsymbol{\psi}^{n} + \frac{\Delta t}{2}I\boldsymbol{\psi}_*^{n+1} + \frac{\Delta t}{2}\left(\tilde{\boldsymbol{g}}^{n} + \tilde{\boldsymbol{g}}^{n+1}\right). \tag{3.121}$$

Equation (3.121) is very easy to implement and will require little solving time. Both the Crank-Nicolson Method and the Modified Euler Method will be implemented and compared.

Spurious oscillations

It could happen that both time schemes show spurious oscillations as a result of non-smooth data or simply the characteristics of the time stepping scheme, which means that the system should be solved with more advanced techniques. A possibility is to introduce a flux correction [7] or the method described by Mirbagheri et al. [11]. These methods will be elaborated more thoroughly in case they prove to be necessary after obtaining the first results.

4 Conclusion

The goal of this literature research has been to provide a mathematical framework that describes the physics in a flood embankment in a two-dimensional setting. There are some status quo methods, based on a porous seabed, that are currently being used to describe the dynamic pressure in levees. However, the current method makes use of the questionable assumption that the pore water is considered to be compressible. Furthermore it makes use of the assumption that the stresses resulting from waves are solely being absorbed by the pore water. Physically it makes more sense that these stresses are endured by both the soil particles and the pore water.In order to retrieve a more accurate model for the computation of the water pressure, these assumptions are abandoned.

In Chapter 2 the new mathematical framework is derived by making several assumptions, such as incompressibility of the pore water, fixed densities for the pore water and soil particles and neglecting the advective acceleration of the soil particles and the acceleration of the pore water. Furthermore, in some cases, terms are considered to be negligible because of the small order of magnitude they have. The trade off between the absorption of the stresses between the soil particles and the pore water is enforced by putting the momentum balance equation as a boundary condition on two boundaries.

In Chapter 3 a numerical method is chosen to make the system numerically solvable. The Finite Element approach will be used to solve the system, since it is flexible when it comes to a changing domain and it provides a good framework to deal with the complicated boundary conditions. For the Finite Element approach, a weak formulation is derived. By approximating the unknown parameters with a linear combination of basis functions, the Galerkin equations are derived. These Galerkin equations can be solved using a time stepping method. To keep the second order accuracy, two time stepping methods are proposed: one explicit and one implicit. Both of these time stepping methods will be implemented.

In the remainder of this project the numerical system will be implemented and compared with an analytical solution that is yet to be published by Myron van Damme. In case the numerical approach gives a good estimate of the dynamic water pressure, several extensions can be made. One obvious extension could be simply applying the same equations on a more complex domain, something that can be very helpful for potential other applications. Another extension could be adding the advective acceleration into the numerical system. It is not unimaginable that this advective accelerations plays a big role, since there does not exist an analytical solution of the model without it. Another extension could be making the porosity an unknown parameter as well. Lastly, the model could always be extended to a three-dimensional setting.

A
Theorems

The following theorem is a corollary of the Divergence theorem.

Theorem 1 *Let $\boldsymbol{F}$ be a continuously differentiable vector field, g be a scalar function and $\Omega \subset \mathbb{R}^3$ a volume in three-dimensional space which is compact and has a piecewise smooth boundary S. Then it holds that:*

$$\int_\Omega \left[\boldsymbol{F} \cdot \nabla g + g(\nabla \cdot \boldsymbol{F})\right] d\Omega = \oint_S g\boldsymbol{F} \cdot \boldsymbol{n} dS. \tag{A.1}$$

The following theorems are by Holand et al. [5].

Theorem 2 *Let e be the line segment between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, let λ_1 and λ_2 be linear on e such that $\lambda_i(\boldsymbol{x}_j) = \delta_{ij}$, and let $m_1, m_2 \in \mathbf{N}_0 = \{1, 2, \ldots\}$. Then:*

$$\int_e \lambda_1^{m_1} \lambda_2^{m_2} d\Gamma = \frac{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| m_1! m_2!}{(1 + m_1 + m_2)!}. \tag{A.2}$$

Theorem 3 *Suppose that e is a triangle with vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Let λ_1, λ_2 and λ_3 be linear functions on e subject to $\lambda_i(\boldsymbol{x}_j) = \delta_{ij}$ and let $m_1, m_2, m_3 \in \mathbb{N}$. Then:*

$$\int_e \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} d\Omega = \frac{|\Delta_e| m_1! m_2! m_3!}{(2 + m_1 + m_2 + m_3)!}, \tag{A.3}$$

with

$$|\Delta_e| = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} = \|(\boldsymbol{x}_2 - \boldsymbol{x}_1) \times (\boldsymbol{x}_3 - \boldsymbol{x}_1)\|, \tag{A.4}$$

the area of the parallelogram, which is twice the area of the triangle.

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