Hamiltonian Discontinuous Galerkin Finite Element Method for Internal Gravity Waves

Delft University of Technology

Sander van Oers

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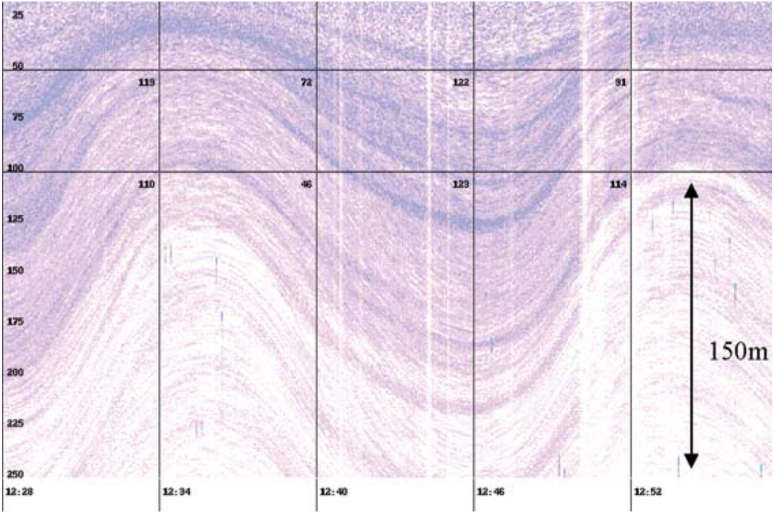
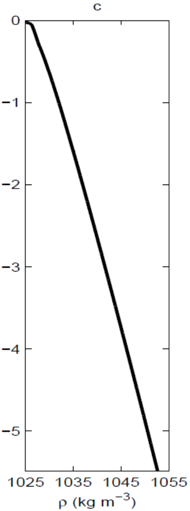
Observations

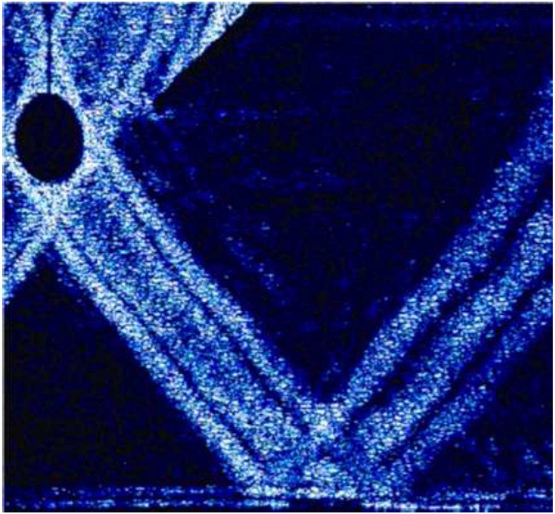


from R. SUSANTO, L. MITNIK, and Q.ZHENG (2005)

Stratification

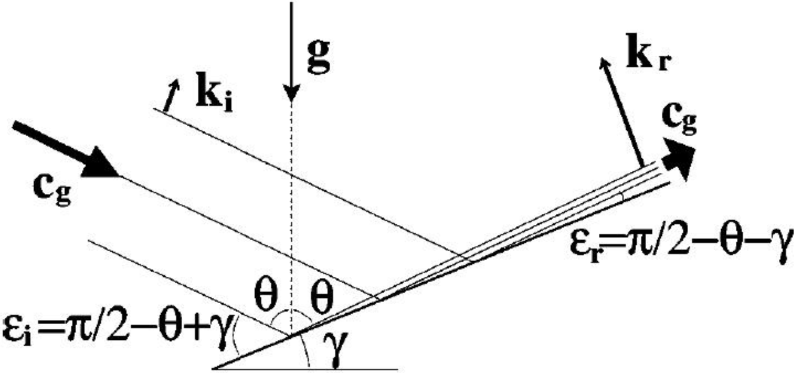


Reflection



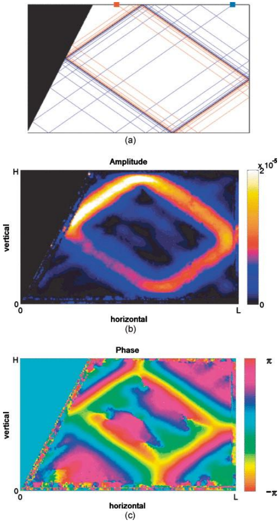
from Wave Attractors by L. Maas (2012)

Focusing



from C. Staquet and J. Sommeria (2002)

Wave Attractor



from Hazewinkel et al. (2008)

Wave Attractor

Harmonic Oscillator

p = velocity

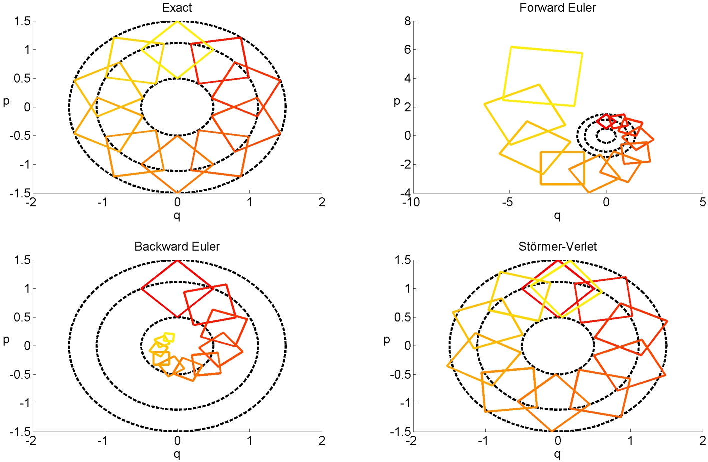
q = position

$$\frac{dq}{dt} = p$$
$$\frac{dp}{dt} = -q$$

Total Energy:

$$H = \frac{1}{2}(p^2 + q^2)$$

Harmonic Oscillator Solutions



Hamilton's equations

$$\boxed{\begin{aligned}\frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}\\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}\end{aligned}}$$

$$\frac{dq}{dt} = p$$
$$\frac{dp}{dt} = -q$$
$$H = \frac{1}{2}(p^2 + q^2)$$

with Hamiltonian H and phase space $\{(q_i, p_i)\}_{i=1,\dots,N}$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0$$

Noncanonical Representations

$$\boxed{\frac{du_i}{dt} = J_{ij}\frac{\partial H(u)}{\partial u_j}}$$

J_{ij} is skew-symmetric.

$$\boxed{\begin{gathered}[F,H] = \frac{\partial F}{\partial u_i} J_{ij} \frac{\partial H}{\partial u_j} \\ \frac{dF}{dt} = [F,H]\end{gathered}}$$

$[F,G]$ is skew-symmetric.

Harmonic Oscillator

$$\boxed{\frac{du_i}{dt} = J_{ij}\frac{\partial H(u)}{\partial u_j}}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and}$$

$$u = \begin{pmatrix} q \\ p \end{pmatrix}.$$

$$\frac{dq}{dt} = p$$

$$\frac{dp}{dt} = -q$$

$$H = \frac{1}{2}(p^2 + q^2)$$

$$[F, H] = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}$$

Noncanonical Representations - Casimirs

Two types:

- Canonical: J invertible
- Noncanonical: J non-invertible

Casimirs C are solutions of

$$J_{ij}\frac{\partial C}{\partial u_j} = 0 \qquad \text{or} \qquad [C, F] = 0 \quad \forall F$$

- Harmonic oscillator: canonical: no Casimirs
- Fluid dynamics: noncanonical: Casimirs (potential vorticity)

Direct Discretization

$$
\begin{tikzcd}
\text{Partial Differential Equations} \arrow[d, "\text{discretization}"] \\
\text{Discrete Equations}
\end{tikzcd}
$$

Discrete Hamiltonian Dynamics

$$\begin{tikzcd}
& \text{PDEs} \arrow[r, leftrightarrow] & \text{Hamiltonian Structure} \arrow[d, "\text{discretization}"] \\
& \text{Discrete Equations} & \text{Discrete Hamiltonian Structure} \arrow[l]
\end{tikzcd}$$

Overview Derivation

$$\begin{tikzcd}
\text{primitive Euler equations} & \text{compressible stratified Euler equations} & \text{incompressible stratified Euler equations} & \text{Euler-Boussinesq equations} \\
0 \arrow[r, "\text{linearization}"'] & 1 \arrow[r, "c_0^2 \to \infty"'] & 2 \arrow[r, "\rho_0 \underline{v} \to \rho^* \underline{v}"'] & 3
\end{tikzcd}$$

Stratified Compressible Euler Equations

$$
\begin{aligned}
\frac{\partial(\rho_0 \underline{v}')}{\partial t} &= -\rho' g \underline{\hat{z}} - \nabla p', \\
\frac{\partial \rho'}{\partial t} &= -\nabla.(\rho_0 \underline{v}'), \\
\frac{\partial p'}{\partial t} &= \rho_0 g w' - c_0^2(z) \rho_0 \nabla . \underline{v}'
\end{aligned}
$$

Hamiltonian Structure

Poisson Bracket:

$$\{\mathcal{F},\mathcal{G}\} = \int_\Omega \Bigg[\frac{\delta \mathcal{G}}{\delta \rho'} \nabla . \left(\rho_0 \frac{\delta \mathcal{F}}{\delta(\rho_0 \underline{v}')} \right) - \frac{\delta \mathcal{F}}{\delta \rho'} \nabla . \left(\rho_0 \frac{\delta \mathcal{G}}{\delta(\rho_0 \underline{v}')} \right) + \frac{\delta \mathcal{F}}{\delta p'} \left(\rho_0 g \frac{\delta \mathcal{G}}{\delta(\rho_0 w')} - c_0^2(z) \rho_0 \nabla . \frac{\delta \mathcal{G}}{\delta(\rho_0 \underline{v}')} \right) - \frac{\delta \mathcal{G}}{\delta p'} \left(\rho_0 g \frac{\delta \mathcal{F}}{\delta(\rho_0 w')} - c_0^2(z) \rho_0 \nabla . \frac{\delta \mathcal{F}}{\delta(\rho_0 \underline{v}')} \right) \Bigg] \, \mathrm{d}\underline{x}$$

Hamiltonian:

$$\mathcal{A} = \int_\Omega \left[\frac{1}{2} \rho_0 |\underline{v}'|^2 + \frac{1}{2} \frac{g^2}{\rho_0 N^2} \left(\rho' - \frac{p'}{c_0^2} \right)^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2} \right] \mathrm{d}\underline{x}$$

Primitive Euler Equations - Equations from Bracket

$$\frac{\delta \mathcal{A}}{\delta(\rho_0 \underline{v}')} = \underline{v}'$$

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{A}\} = \int_\Omega -\frac{\delta \mathcal{F}}{\delta \rho'} \nabla.(\rho_0 \underline{v}') + \dots \, \mathrm{d}\underline{x}$$

Discrete Poisson Bracket

Discrete Poisson Bracket

$$[F,G] = \frac{\partial F}{\partial R_i}\left(\frac{\partial G}{\partial \underline{U}_j}.\underline{DIV}_{kl}M_{ik}^{-1}M_{jl}^{-1} - \frac{\partial G}{\partial W_j}{}^1N_{kl}M_{ik}^{-1}M_{jl}^{-1}\right)$$
$$- \frac{\partial G}{\partial R_i}\left(\frac{\partial F}{\partial \underline{U}_j}.\underline{DIV}_{kl}M_{ik}^{-1}M_{jl}^{-1} - \frac{\partial F}{\partial W_j}{}^1N_{kl}M_{ik}^{-1}M_{jl}^{-1}\right)$$
$$+ \frac{\partial F}{\partial P_i}\left(\frac{\partial G}{\partial W_j}N_{kl}M_{ik}^{-1}M_{jl}^{-1} + c_0^2\frac{\partial G}{\partial \underline{U}_j}.\underline{DIV}_{kl}M_{ik}^{-1}M_{jl}^{-1}\right)$$
$$- \frac{\partial G}{\partial P_i}\left(\frac{\partial F}{\partial W_j}N_{kl}M_{ik}^{-1}M_{jl}^{-1} + c_0^2\frac{\partial F}{\partial \underline{U}_j}.\underline{DIV}_{kl}M_{ik}^{-1}M_{jl}^{-1}\right)$$

Discrete Hamiltonian

$$H = \frac{1}{2}{}^1M_{ij}\underline{U}_i.\underline{U}_j + \frac{1}{2}{}^2M_{ij}R_iR_j - \frac{1}{c_0^2}{}^2M_{ij}R_iP_j + \frac{1}{2}\frac{1}{c_0^4}{}^2M_{ij}P_iP_j$$
$$+ \frac{1}{2}\frac{1}{c_0^2}{}^1M_{ij}P_iP_j$$

Discrete Equation of Motion

$$\frac{\underline{U}_j^{n+1} - \underline{U}_j^{n}}{\Delta t} = \left({}^2M_{il}\,{}^1N_{kl} + \frac{1}{c_0^2}\,{}^2M_{il}N_{kl} \right) \underline{\hat{z}}\, \frac{R_l^{n+1} + R_l^{n}}{2} M_{ik}^{-1} M_{jl}^{-1}$$

$$+ \left(-\frac{1}{c_0^2}\,{}^2M_{il}\,{}^1N_{kl}\underline{\hat{z}} - \frac{1}{c_0^4}\,{}^2M_{il}N_{kl}\underline{\hat{z}} - \frac{1}{c_0^2}\,{}^1M_{il}N_{kl}\underline{\hat{z}} - {}^1M_{il}\,\underline{DIV}_{kl} \right) \frac{P_l^{n+1} + P_l^{n}}{2} M_{ik}^{-1} M_{jl}^{-1}$$

$$\frac{R_l^{n+1} - R_l^{n}}{\Delta t} = \frac{\underline{U}_j^{n+1} + \underline{U}_j^{n}}{2} . \underline{DIV}_{jk}\,{}^1M_{ij} M_{lj}^{-1} M_{ik}^{-1} - \frac{W_j^{n+1} + W_j^{n}}{2} .\,{}^1N_{jk}\,{}^1M_{ij} M_{lj}^{-1} M_{ik}^{-1}$$

$$\frac{P_l^{n+1} - P_l^{n}}{\Delta t} = \frac{W_j^{n+1} + W_j^{n}}{2}\,{}^1M_{ij} N_{jk} M_{lj}^{-1} M_{ik}^{-1} + c_0^2 \frac{\underline{U}_j^{n+1} + \underline{U}_j^{n}}{2} . \underline{DIV}_{jk}\,{}^1M_{ij} M_{lj}^{-1} M_{ik}^{-1}$$

Discrete Incompressible Equations

Zero perturbation pressure and divergence free velocity constraints forced in the discrete compressible Hamiltonian via Lagrange multipliers

Discrete Incompressible Equations

Replace Pressure by Lagrange multiplier Θ and let speed of sound approach infinity:

$$\frac{\underline{U}_j^{n+1} - \underline{U}_j^n}{\Delta t} = \hat{\underline{z}}^2 M_{il}{}^1 N_{kl} \frac{R_l^{n+1} + R_l^n}{2} M_{ik}^{-1} M_{jl}^{-1} - {}^1 M_{il} \Theta_l^{n+1} M_{il}^{-1} \underline{DIV}_{ml} M_{jm}^{-1}$$

$$\frac{R_l^{n+1} - R_l^n}{\Delta t} = -{}^1 M_{ij}{}^1 N_{jk} \frac{W_j^{n+1} + W_j^n}{2} M_{lj}^{-1} M_{ik}^{-1}$$

$$\begin{aligned} {}^1 M_{ij} \underline{DIV}_{jk} M_{ij}^{-1} . \hat{\underline{z}}^2 M_{il}{}^1 N_{kl} \frac{R_l^{n+1} + R_l^n}{2} M_{ik}^{-1} M_{jl}^{-1} &= \\ = {}^1 M_{ij} \underline{DIV}_{jk} M_{ij}^{-1}{}^1 M_{il} \Theta_l^{n+1} M_{il}^{-1} M_{jm}^{-1} . \underline{DIV}_{ml} \end{aligned}$$

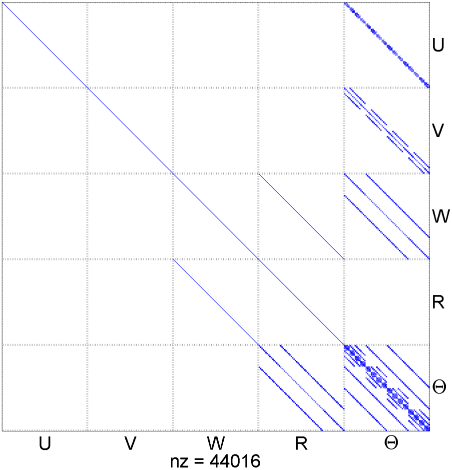
Global Formulation

$$PX^{n+1} = QX^n$$

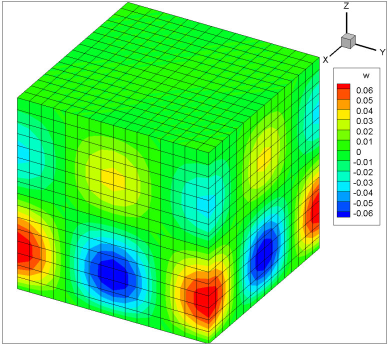
Implemented using:

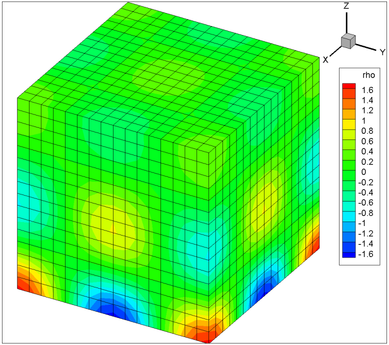
- hpGEM
- PETSc

Sparse Matrix - Incompressible

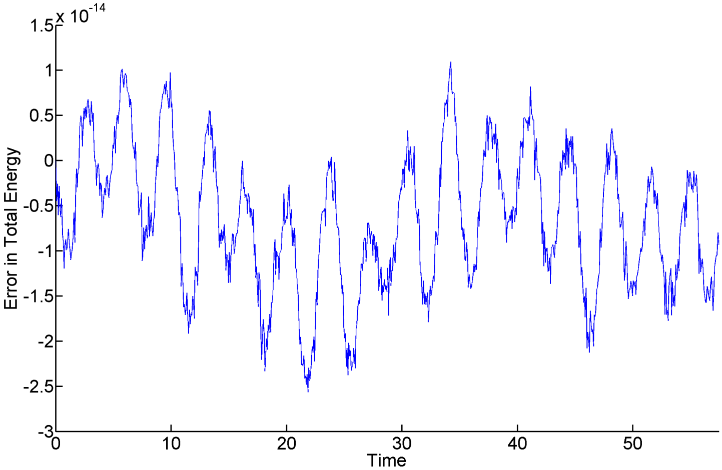


Verification-Compressible

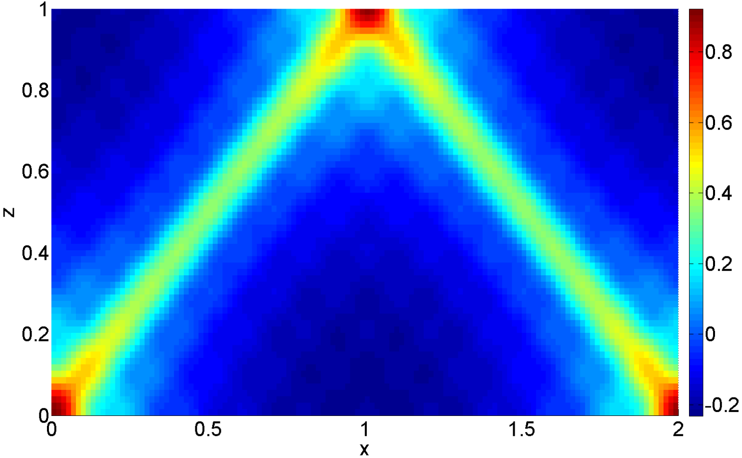




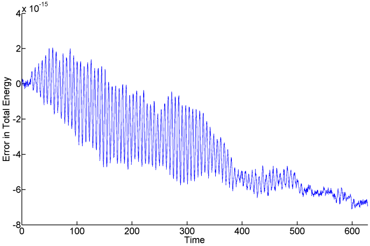
Verification-Compressible

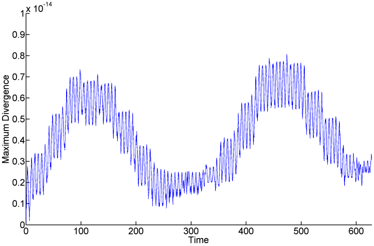


Verification - Incompressible



Verification - Incompressible





Wave Attractor

Conclusion

- Find Hamiltonian Structure Governing Internal Gravity Waves
- Discretize Hamiltonion Structure
- Obtain Discrete Equations of Motion

Results:

- Preservation of Hamiltonian structure ensures stability of discretization
- Exact Conservation of Energy, Divergence, Mass, Momentum