Modelling the interaction between the electromagnetic field and fluids

Literature review

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Introduction

- Electromagnetic field: Maxwell equations
- Fluid dynamics: Navier-Stokes equations
- Coupling through dependent terms:

  Induced currents and constitutive relations depend on velocity field;
  External force and source term depend on EM fields.

Maxwell equations

Maxwell equations in matter:

(1) $$-\nabla \times \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J}^{\mathsf{f}},$$

(2) $$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$

(3) $$\nabla \cdot \mathbf{D} = \rho_{\mathsf{f}},$$

(4) $$\nabla \cdot \mathbf{B} = 0.$$

Maxwell equations

Fields:

- $\mathbf{E}(\mathbf{x}, t)$, electric field;
- $\mathbf{H}(\mathbf{x}, t)$, magnetic field;
- $\mathbf{D}(\mathbf{x}, t)$, electric flux field;
- $\mathbf{B}(\mathbf{x}, t)$, magnetic flux field.

Other quantities:

- $\mathbf{J}^{\mathsf{f}}(\mathbf{x}, t)$, free current density,
- $\rho_{\mathsf{f}}(\mathbf{x}, t)$, free charge density.

Constitutive relations

We need relations between the different fields, constitutive relations.

For non-dispersive linear conductive media:

- $\mathbf{D} = \varepsilon \mathbf{E}$,
- $\mathbf{H} = \mu \mathbf{B}$.
- $\rho_{\mathsf{f}} = 0$,
- $\mathbf{J}^{\mathsf{f}} = \sigma \mathbf{E}$.

Constitutive relations

Up until now no flow. Assume velocity field $\mathbf{v}$, then using Lorentz invariance:

$$\mathbf{D} = \varepsilon\mathbf{E} + \varepsilon\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{H}, \tag{5}$$

$$\mathbf{B} = \mu\mathbf{H} - \mu\mathbf{v}\times\mathbf{D} + \frac{1}{c^2}\mathbf{v}\times\mathbf{E}. \tag{6}$$

Constitutive relations

Linearized and solved for $\mathbf{E}$ and $\mathbf{H}$:

(7) $$\mathbf{D} \approx \varepsilon\mathbf{E} + (\mu\varepsilon - \mu_0\varepsilon_0)\,\mathbf{v} \times \mathbf{H},$$

(8) $$\mathbf{B} \approx \mu\mathbf{H} - (\mu\varepsilon - \mu_0\varepsilon_0)\mathbf{v} \times \mathbf{E}.$$

Note:

$$\mu\varepsilon - \mu_0\varepsilon_0 = \frac{1}{c_m^2} - \frac{1}{c^2}.$$

Constitutive relations

For electric charge and current:

$$\begin{aligned} \rho_f &= \gamma \frac{1}{c^2} \sigma \mathbf{E} \cdot \mathbf{v}, \\ \mathbf{J}^f &= \gamma \sigma \mathbf{E} + \gamma \sigma \mathbf{v} \times \mathrm{B}. \end{aligned}$$

If $v \ll c$ and $\sigma \ll 1$:

$$\begin{aligned} \rho_f &\approx 0, \\ \mathbf{J}^f &\approx \sigma \mathbf{E} + \sigma \mathbf{v} \times \mathrm{B}. \end{aligned}$$

Maxwell equations

In terms of E and H, with external sources:

$$-\nabla \times \mathbf{H} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\mathbf{J}^{\text{ind}} - \mathbf{J}^{\text{ext}}, \qquad (9)$$

$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mathbf{K}^{\text{ind}} - \mathbf{K}^{\text{ext}}. \qquad (10)$$

- Induced currents are functions of the fields,
- External currents are independent of the field (e.g. the lab laser).

EM energy in vacuum

(11) $$\partial_t u_{em} = -\nabla \cdot \mathbf{S} - \mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K},$$

EM energy density

(12) $$u_{em} = \frac{1}{2}\left(\varepsilon_0 \|\mathbf{E}\|^2 + \mu_0 \|\mathbf{H}\|^2\right),$$

Poynting vector

(13) $$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Lorentz force

Regular form for point charges:

$$\mathbf{F} = q\left[\mathbf{E} + \mu_0 \mathbf{v} \times \mathbf{H}\right].$$

Continuous form

$$\mathbf{f} = \rho_e \mathbf{E} + \mu_0 \mathbf{J} \times \mathbf{H},$$

with electric charge density

$$\rho_e = \varepsilon_0 \nabla \cdot \mathbf{E}.$$

Lorentz force

Generalization with magnetic currents:

$$\mathbf{f} = \rho_e \mathbf{E} + \rho_m \mathbf{H} + \mu_0 \mathbf{J} \times \mathbf{H} - \varepsilon_0 \mathbf{K} \times \mathbf{E}.$$

with magnetic charge density

$$\rho_m = \mu_0 \nabla \cdot \mathbf{H}.$$

From Maxwell equations we can derive:

$$\mathbf{f} = \nabla \cdot \mathbf{T} - \frac{\partial \mathbf{S}}{\partial t},$$

Stress tensor

Stress tensor in components:

$$T_{ij} = \mu_0 H_i H_j + \varepsilon_0 E_i E_j - \frac{1}{2}\delta_{ij}\left[\varepsilon_0 E_i E_i + \mu_0 H_i H_i\right]. \quad (14)$$

In general we need 12 quantities.

Currents without velocity

For linear conducting media:

$$\mathbf{J}^{\text{ind}} = (\varepsilon - \varepsilon_0)\frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}, \tag{15}$$

$$\mathbf{K}^{\text{ind}} = (\mu - \mu_0)\frac{\partial \mathbf{H}}{\partial t}. \tag{16}$$

- External magnetic currents zero.
- External electric current depend on situation.

EM energy in (linear) matter

(17) $$\partial_t u_{em} = -\nabla \cdot \mathbf{S} - \sigma \|\mathbf{E}\|^2 - \mathbf{E} \cdot \mathbf{J}^{ext},$$

EM energy density

(18) $$u_{em} = \frac{1}{2} \left(\varepsilon \|\mathbf{E}\|^2 + \mu_0 \|\mathbf{H}\|^2 \right),$$

Poynting vector

(19) $$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Currents with velocity

For linear conducting media:

$$\begin{aligned}\mathbf{J}^{\text{ind}} &= (\varepsilon-\varepsilon_0)\frac{\partial \mathbf{E}}{\partial t} + (\mu\varepsilon-\mu_0\varepsilon_0)\,\frac{\partial}{\partial t}\,(\mathbf{v}\times\mathbf{H}) \\ &\quad +\sigma\,(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H})\,, \\ \mathbf{K}^{\text{ind}} &= (\mu-\mu_0)\,\frac{\partial \mathbf{H}}{\partial t} - (\mu\varepsilon-\mu_0\varepsilon_0)\frac{\partial}{\partial t}\,(\mathbf{v}\times\mathbf{E})\,.\end{aligned}$$

EM energy in fluid

$$\partial_t u_{em} = -\nabla\cdot\mathbf{S} - \sigma\mathbf{E}\cdot(\mathbf{E} + \mu_0\mathbf{v}\times\mathbf{H}) - \mathbf{v}\cdot\frac{\partial\mathbf{S}}{\partial t} - \mathbf{E}\cdot\mathbf{J}^{ext}, \tag{20}$$

EM energy density

$$u_{em} = \frac{1}{2}\left(\varepsilon\|\mathbf{E}\|^2 + \mu_0\|\mathbf{H}\|^2\right) + 2\mathbf{v}\cdot\mathbf{S}, \tag{21}$$

Poynting vector

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{22}$$

Time averaging

For periodic sources, the fields are periodic. We have

(23) $$E_l(\mathbf{x}, t) = \mathrm{Re}\left\{\hat{E}_l(\mathbf{x})e^{i\omega t}\right\},$$

angular time frequency ω.

Time averaging:

$$\langle f(t)\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt.$$

Time averaging

For fields:

$$\langle \mathbf{E} \rangle = \langle \mathbf{H} \rangle = 0,$$

but:

$$\begin{aligned} \langle f_l \rangle &= \partial_j \left[\mu_0 \text{Re} \left\{ \hat{H}_l \bar{H}_j \right\} + \varepsilon_0 \text{Re} \left\{ \hat{E}_l \bar{E}_j \right\} \right. \\ &\quad \left. - \frac{1}{2} \delta_{lj} \left(\mu_0 |\hat{H}_l|^2 - \varepsilon_0 |\hat{E}_l|^2 \right) \right]. \end{aligned}$$

EM simulations: Meep

- Open source program, developed at MIT;
- Time domain simulations;
- Ability for frequency domain;
- Relatively easy C++ interface, easy to couple with other programs.
- Uses dimensionless Maxwell equations.

Maxwell equations: Dimensionless

Classical EM has four basic units:

- Electric current: I_0 in ampere,
- Distance: a in meter,
- Velocity: c in meter per second,
- Permittivity: ε in farad per meter.

Maxwell equations: Dimensionless

Dimensionless quantities:

$$
\begin{aligned}
E &= \frac{I_0}{a\varepsilon c}E', & D &= \frac{I_0}{ac}D', \\
H &= \frac{I_0}{a}H', & B &= \frac{I_0}{ac^2\varepsilon}B', \\
J &= \frac{I_0}{a^2}J', & K &= \frac{I_0}{a^2c\varepsilon}K', \\
\sigma &= \frac{\varepsilon c}{a}\sigma', & \sigma_D &= \frac{c}{a}\sigma'_D,
\end{aligned}
$$

Furthermore:

$$x = ax', \; t = \frac{a}{c}t'.$$

Maxwell equations: Dimensionless

Dimensionless equations:

$$
\begin{aligned}
-\epsilon_{ijk}\partial_{j'}H'_k + \partial_{t'}E'_i &= -J_i'^{,\text{ind}} - J_i'^{,\text{ext}}, \\
\epsilon_{ijk}\partial_{j'}E'_k + \partial_{t'}H'_i &= -K_i'^{,\text{ind}} - K_i'^{,\text{ext}}.
\end{aligned}
$$

Meep uses this form.

Navier Stokes equations

Conservation laws:

- Continuity equation: mass,
- Navier-Stokes: (linear) momentum,
- Energy equation: energy.

Continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

- $\rho(\mathbf{x}, t)$ is the mass density,
- $\mathbf{v}(\mathbf{x}, t)$ is the fluid velocity.

Navier-Stokes equations

$$\rho\left(\partial_t v_i + v_j \partial_j v_i\right) = -\partial_i p + \partial_j T'_{ij} + f_i^{\mathsf{b}}.$$

- $p(\mathbf{x}, t)$ is the pressure,
- T'_{ij} is the deviatoric stress tensor,
- f_i^{b} are the body forces.

Stress tensor

For Newtonian fluids we can write:

$$T'_{ij} = 2\mu \left(e_{ij} - \frac{1}{3}\Delta\delta_{ij} \right),$$

where

- μ is the viscosity,
- $e_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$,
- $\Delta = e_{kk} = \partial_k v_k$

Incompressible

When flow is incompressible:

$$\nabla \cdot \mathbf{v} = 0.$$

Navier-Stokes simplifies:

$$\rho \left(\partial_t v_i + v_j \partial_j v_i \right) = -\partial_i p + \mu \partial_j^2 v_i + f_i^{\mathsf{b}}.$$

Energy equation

Conservation of energy:

$$\partial_t E + \partial_i (E v_i) = f_i^{\text{b}} v_i + \partial_j (T_{ij} v_i) + k \partial_i^2 T + q,$$

with

- E energy density,
- k thermal conductivity,
- q heat source density.

Energy density

For the energy density we have:

$$E = \rho\left(e + \frac{1}{2}v_i v_i\right),$$

with e the specific internal energy,

$$e = c_p T.$$

c_p is the specific heat (by constant pressure).

Free convection

If ΔT small,

$$T = T_0 + \Delta T, \;\; \rho = \rho_0 + \Delta\rho.$$

Using linearisation we have:

$$\rho' = \left(\frac{\partial \rho_0}{\partial T}\right)_p T' = -\rho_0 \beta T',$$

So for the density:

$$\rho = \rho_0 \left(1 - \beta(T - T_0)\right).$$

Overview of equations

$$
\begin{aligned}
\partial_i v_i &= 0, \\
\rho\left(\partial_t v_i + v_j \partial_j v_i\right) &= -\partial_i p + \mu \partial_j^2 v_i + f_i^{\mathrm{b}}, \\
\rho c_p\left(\partial_t T + v_i \partial_i T\right) &= T_{ij}\partial_j v_i + k\partial_i^2 T + q, \\
\rho &= \rho_0\left(1 - \beta(T - T_0)\right).
\end{aligned}
$$

Six equations, in six unknowns.

Solving the equations

TODO, first approach:

- Using OpenFoam for fluid simulations,
- Using Meep for EM simulations,
- Couple both programs.

Second approach:

Investigate if COMSOL can be of use.

Any suggestions?

Research questions

Once everything runs:

- Are the effects of the EM field significant?
- Which effect dominates, Lorentz force or heat convection?
- See if the model can be validated by experimental data.
- Investigate different materials.

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