Acceleration of the 2D Helmholtz model HARES

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Outline

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HARES

- HARES $\rightarrow$ HArbour RESonance.

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- Determines wave penetration into harbours.

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- HARES $\rightarrow$ HArbour RESonance.
- Determines wave penetration into harbours.
- Uses the non-linear Mild-Slope equation.
- Developed by Svašek Hydraulics.
  - ◇ Consultant in coastal, harbour and river engineering.
  - ◇ Specialized in numerical fluid dynamics.

HARES
Example

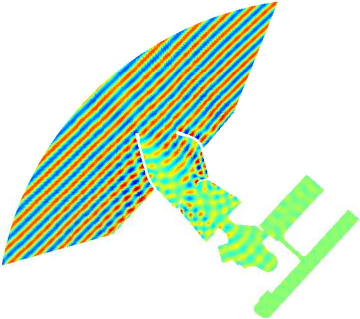

Figure: The harbour of Scheveningen

Project description

PROBLEM

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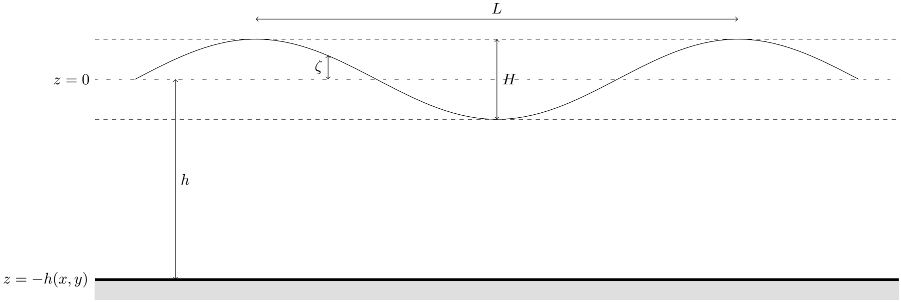
TASK

Accelerate HARES, decrease the computing time.

Outline

1. Introduction
2. Mild-Slope equation
3. Initial implementation
4. Proposed improvements
5. Numerical experiments
6. Conclusions & Recommendations
7. Future research

Wave motion



$h(x, y)$	Water depth	L	Wave length
H	Wave height	$\zeta(x, y, t)$	Elevation of the free surface

Wave motion transforming effects

$$\text{Objects in the domain} \implies \begin{cases} - & \text{Diffraction} \\ - & \text{Reflection} \end{cases}$$

$$\text{Decreasing water depth} \implies \begin{cases} - & \text{Refraction} \\ - & \text{Shoaling} \end{cases}$$

Wave motion transforming effects

$$\left.\begin{array}{ll} - & \text{Diffraction} \\ - & \text{Reflection} \\ & \\ - & \text{Refraction} \\ - & \text{Shoaling} \end{array}\right\} \Longrightarrow \text{Linear Mild-Slope equation}$$

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$$\left.\begin{array}{ll} - & \text{Wave breaking} \\ - & \text{Bottom friction} \end{array}\right\} \Longrightarrow \text{Non-linear term in the Mild-Slope equation}$$

Wave motion transforming effects

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- Wave motion is harmonic in time.
- Surface tension and Coriolis effect can be neglected.
- Changes in bottom topography are small.

Non-linear Mild-Slope equation

The non-linear Mild-Slope equation is given by

$$\nabla \cdot \left(\frac{n_0}{k_0^2} \nabla \tilde{\zeta} \right) + \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} = 0.$$

With

$n_0(x, y)$	Parameter $n_0 \in [\frac{1}{2}, 1]$
$k_0(x, y)$	Wave number
$\tilde{\zeta}(x, y)$	Elevation of the free surface
$W(x, y, \tilde{\zeta})$	Dissipation of wave energy

ω	Wave frequency
i	$= \sqrt{-1}$
∇	$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T$

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Non-linearity

$$W(x, y, \tilde{\zeta})\tilde{\zeta} = \left(\mathcal{A}|\tilde{\zeta}| + \frac{\mathcal{B}}{|\tilde{\zeta}|^2} \right) \tilde{\zeta}$$

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- The *open boundary* with an incoming wave from the exterior and an outgoing wave from the interior.
- The *closed boundary* where (partial) reflection occurs.

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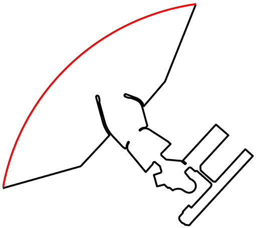

The condition for the *open boundary* is given by

$$\frac{\partial\tilde{\zeta}}{\partial n} = -i\left\{\hat{p}(\tilde{\zeta}-\tilde{\zeta}_{in}) + \frac{1}{2\hat{p}}\left(\frac{\partial^2\tilde{\zeta}}{\partial s^2} - \frac{\partial^2\tilde{\zeta}_{in}}{\partial s^2}\right) - \hat{p}(\boldsymbol{e}_{in}\cdot\boldsymbol{n})\tilde{\zeta}_{in}\right\}.$$

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$$\frac{\partial \tilde{\zeta}}{\partial n} = -i \left(\frac{1-R}{1+R} \right) \left\{ \hat{p}\tilde{\zeta} + \frac{1}{2\hat{p}} \frac{\partial^2 \tilde{\zeta}}{\partial s^2} \right\}.$$

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With

$\hat{p}(x, y, \tilde{\zeta})$	Modified wave number	R	Reflection coefficient
$\tilde{\zeta}_{in}$	Incoming wave	i	$= \sqrt{-1}$

Outline

Structure of HARES

HARES consist of three parts, i.e.

1. Outer loop to deal with the non-linearity of the equation.
   $\rightarrow$ Non-linear Mild-Slope equation is linearised.
2. Spatial discretization of the linearised Mild-Slope equation.
   $\rightarrow$ Results in a system of equations $\boldsymbol{S\zeta} = \boldsymbol{b}$.
3. Inner loop to determine the solution of $\boldsymbol{S\zeta} = \boldsymbol{b}$.

Initial implementation

The current programme has the following implementation:

1. Outer loop: Picard iteration.
2. Spatial discretization: Ritz-Galerkin finite element method.
3. Inner loop: ILU(0) - Bi-CGSTAB.

Linearising the non-linear equation

Picard iteration

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3. Repeat steps 1 & 2 until convergence is reached.

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The current programme repeats steps 1 & 2 25 times without knowing whether convergence has been reached.

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Spatial discretization

Ritz-Galerkin finite element method

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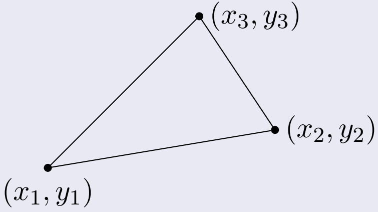
1. Divide the domain into linear triangular elements.



- Two types of elements:
  - Internal elements.
  - Boundary elements.
- Number of nodes N = Number of unknowns.

Spatial discretization

Ritz-Galerkin finite element method

The Ritz-Galerkin finite element method consist of the following steps:

1. Divide the domain into linear triangular elements.
2. Derive the weak formulation of the PDE.

Multiply the PDE by a test function $\eta(x, y)$, integrate it over the domain Ω and apply the boundary conditions.

Spatial discretization

Ritz-Galerkin finite element method

The Ritz-Galerkin finite element method consist of the following steps:

1. Divide the domain into linear triangular elements.
2. Derive the weak formulation of the PDE.
3. Approximate the solution by a linear combination of basis functions.

$$\tilde{\zeta}(x,y) \approx \sum_{j=1}^{N} \zeta_j \psi_j(x,y),$$

- $\psi_j(x,y)$ piecewise linear basis function.
- N unknown coefficients ζ_j.

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4. Replace the test function by each of the basis function separately.

$$\eta(x, y) \rightarrow \psi_m(x, y)$$

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5. Determine the element matrix $\boldsymbol{S}^e$ and element vector $\boldsymbol{b}^e$ for each element, with $\boldsymbol{S}^e \in \mathbb{C}^{3\times 3}$ and $\boldsymbol{b}^e \in \mathbb{C}^3$.

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6. Obtain the global matrix $\boldsymbol{S}$ and global vector $\boldsymbol{b}$, with $\boldsymbol{S} \in \mathbb{C}^{N\times N}$ and $\boldsymbol{b} \in \mathbb{C}^N$.

$$\boldsymbol{S}^e \to \boldsymbol{S} \quad \text{and} \quad \boldsymbol{b}^e \to \boldsymbol{b}$$

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7. Compute the solution in each node by solving $\boldsymbol{S}\boldsymbol{\zeta} = \boldsymbol{b}$.

$$\nabla \cdot \left(\frac{n_0}{k_0^2} \nabla \tilde{\zeta} \right) + \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} = 0 \quad \text{and BC's}$$

Application of the Ritz-Galerkin finite element method results in element matrices of the following form:

Ritz-Galerkin finite element method

Non-linear Mild-Slope equation

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Application of the Ritz-Galerkin finite element method results in element matrices of the following form:

$$\boldsymbol{S}^e = -\frac{n_0}{k_0^2} \boldsymbol{L}^e + \left(n_0 - \frac{iW}{\omega} \right) \boldsymbol{M}^e - i \frac{n_0}{k_0^2} \left(\frac{1-R}{1+R} \right) \boldsymbol{C}^e.$$

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$$\left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} \quad \Longrightarrow \quad \left(n_0 - \frac{iW}{\omega} \right) \boldsymbol{M}^e$$

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Boundary conditions $\implies$ $-i \frac{n_0}{k_0^2} \left(\frac{1-R}{1+R} \right) \boldsymbol{C}^e$

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- Global matrix $\boldsymbol{S}$ is a symmetric, non-Hermitian, sparse matrix.
- Global vector $\boldsymbol{b}$ is completely determined by the incoming wave $\tilde{\zeta}_{in}$.

Solving a system of equations

After linearisation and spatial discretization we obtain the system of equations

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$\boldsymbol{S}$ is a general matrix $\Longrightarrow$ Krylov subspace methods

- Iterative solution method.

Starting vector ζ_0, iterations ζ_1, $\zeta_2, \ldots, \zeta_m$ until the stopping criterion is satisfied.

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- Iterative solution method.
- Krylov subspace of dimension m is given by

$$\mathcal{K}_m(\boldsymbol{S}; \boldsymbol{r}_0) = span\{\boldsymbol{r}_0, \boldsymbol{S}\boldsymbol{r}_0, \ldots, \boldsymbol{S}^{m-1}\boldsymbol{r}_0\},$$

with $\boldsymbol{r}_0 = \boldsymbol{b} - \boldsymbol{S}\boldsymbol{\zeta}_0$.

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- Number of matrix-vector products is an important measure.

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To accelerate the convergence we can apply a preconditioner $\boldsymbol{K}$ to the system of equations, i.e.

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- Preconditioner $\boldsymbol{K}$ is a good approximation of matrix $\boldsymbol{S}$
- Constructing the preconditioner $\boldsymbol{K}$ is not too expensive.

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- Stopping criterion for Bi-CGSTAB

$$\frac{\|\boldsymbol{b} - \boldsymbol{S}\zeta_m\|_2}{\|\boldsymbol{b} - \boldsymbol{S}\zeta_0\|_2} \leq \text{TOL}.$$

The system of equations is preconditioned with the incomplete LU decomposition of matrix S.

Solving a system of equations

Preconditioner - Incomplete LU decomposition

The system of equations is preconditioned with the incomplete LU decomposition of matrix $\boldsymbol{S}$.

- $\boldsymbol{S} = \boldsymbol{L}\boldsymbol{U} - \boldsymbol{R}$.
  - $\boldsymbol{L}$ lower triangular matrix.
  - $\boldsymbol{U}$ upper triangular matrix.
  - $\boldsymbol{R}$ residual matrix.

The system of equations is preconditioned with the incomplete LU decomposition of matrix $\boldsymbol{S}$.

- $\boldsymbol{S} = \boldsymbol{L}\boldsymbol{U} - \boldsymbol{R}$.
- The elements of matrices $\boldsymbol{L}$ and $\boldsymbol{U}$ are determined by
  - $\boldsymbol{L}$ and $\boldsymbol{U}$ have the same zero-pattern as $\boldsymbol{S}$, i.e. if $s_{i,j} = 0$ then $u_{i,j} = l_{i,j} = 0$ and if $s_{i,j} \neq 0$ then $u_{i,j} \neq 0$ and $l_{i,j} \neq 0$.
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  - diag$(\boldsymbol{L}) = 1$ and diag$(\boldsymbol{U})$ is determined by the algorithm.
- Preconditioning is done by $\boldsymbol{L}^{-1}\boldsymbol{S}\boldsymbol{U}^{-1}\boldsymbol{y} = \boldsymbol{L}^{-1}\boldsymbol{b}$ with $\boldsymbol{y} = \boldsymbol{U}\boldsymbol{x}$.

Outline

Proposed improvements

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1. Outer loop:
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1. Outer loop:
   - Implement a stopping criterion for Picard iteration.
   - Inexact Picard iteration.
2. Inner loop:
   - IDR(s) combined with the shifted Laplace preconditioner.
   - Direct method `MUMPS`.

Improvement of the outer loop

Stopping criterion for Picard iteration

- Current programme performs 25 outer iterations.
- A suitable stopping criterion is needed to determine when and whether the non-linear solution is obtained.

$$\frac{\|F(\zeta^k)\|_2}{\|F(\zeta^{\mathbf{0}})\|_2} \leq \mathsf{TOL}_{\mathsf{residual}}$$

- Value for $\mathsf{TOL}_{\mathsf{residual}}$ depends on the test case.

Each iteration of Picard iteration we need to determine the solution of the system of equations $\boldsymbol{S\zeta} = \boldsymbol{b}$. This can be done exactly.

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However, we can relax this condition with the following stopping criterion

$$\|\boldsymbol{S}\boldsymbol{\zeta}^k - \boldsymbol{b}\|_2 \leq \eta_k \|\boldsymbol{b}\|_2,$$

with

$$\eta_k = \mathsf{TOL} \cdot \frac{\|\boldsymbol{\zeta}^k - \boldsymbol{\zeta}^{k-1}\|_2}{\|\boldsymbol{\zeta}^0\|_2}.$$

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- Generate residuals $\boldsymbol{r}_n$ that are in the subspace $\mathcal{G}_j$ with decreasing dimension.

$$\mathcal{G}_j = (\boldsymbol{I} - \omega_j \boldsymbol{A}) \left(\mathcal{G}_{j-1} \cap \boldsymbol{P}^{\perp} \right),$$

with $\mathcal{G}_0 = \mathcal{K}^N(\boldsymbol{A}; \boldsymbol{v}_0)$ and $\boldsymbol{P} \in \mathbb{C}^{N \times s}$.

- IDR is proposed by P. Sonneveld in 1980.
- Krylov subspace method.
- Generate residuals $\boldsymbol{r}_n$ that are in the subspace $\mathcal{G}_j$ with decreasing dimension.
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(i) $\mathcal{G}_j \subset \mathcal{G}_{j-1}$ for all $\mathcal{G}_{j-1} \neq \{\mathbf{0}\}, j > 0$,

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$\Longrightarrow$ Finite method, requires at most $N + \frac{N}{s}$ matrix-vector multiplications.

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- Stopping criterion implemented in IDR(s)

$$\frac{\|\boldsymbol{b} - \boldsymbol{S}\boldsymbol{\zeta}_m\|_2}{\|\boldsymbol{b}\|_2} \leq \text{TOL}.$$

For each element the shifted Laplace preconditioner is given by

$$\boldsymbol{K}^e = -\frac{n_0}{k_0^2}\boldsymbol{L}^e - \xi^2\boldsymbol{M}^e - i\frac{n_0}{k_0^2}\left(\frac{1-R}{1+R}\right)\boldsymbol{C}^e$$

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Very similar to the element matrices

$$\boldsymbol{S}^e = -\frac{n_0}{k_0^2}\boldsymbol{L}^2 + \left(n_0 - \frac{iW}{\omega}\right)\boldsymbol{M}^e - i\frac{n_0}{k_0^2}\left(\frac{1-R}{1+R}\right)\boldsymbol{C}^e.$$

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Solving a system of equations
Direct method `MUMPS`

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- Available in a sequential and parallel version.

Outline

1. Introduction
2. Mild-Slope equation
3. Initial implementation
4. Proposed improvements
5. Numerical experiments
6. Conclusions & Recommendations
7. Future research

Four test cases are considered:

1. Harbour of Scheveningen
    - 63,253 unknowns

Numerical experiments

Test cases

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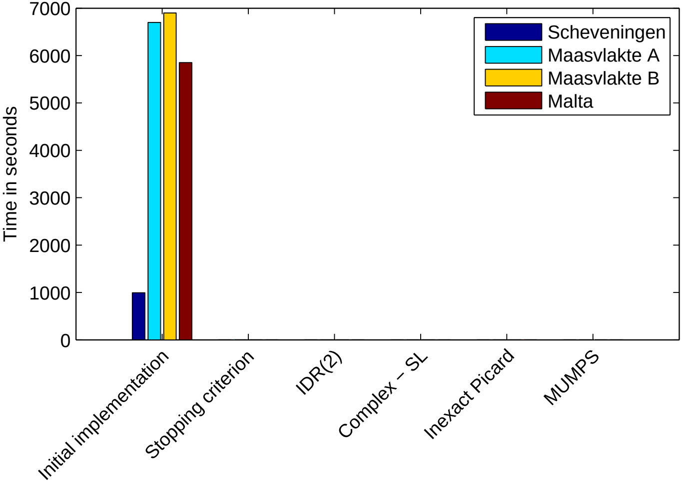
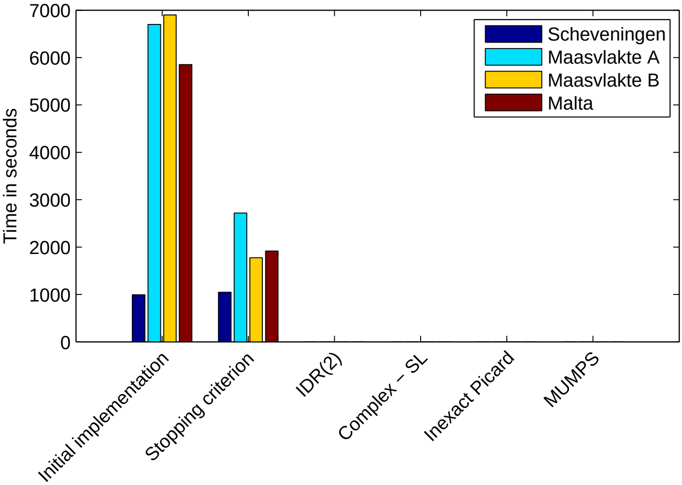
4. Harbour of Marsaxlokk - Malta
   - 170,423 unknowns

Numerical experiments

Results - computing time




7000
6000
5000
4000
3000
2000
1000
0
Time in seconds
Scheveningen
Maasvlakte A
Maasvlakte B
Malta
Initial implementation
Stopping criterion
IDR(2)
Complex – SL
Inexact Picard
MUMPS

Numerical experiments
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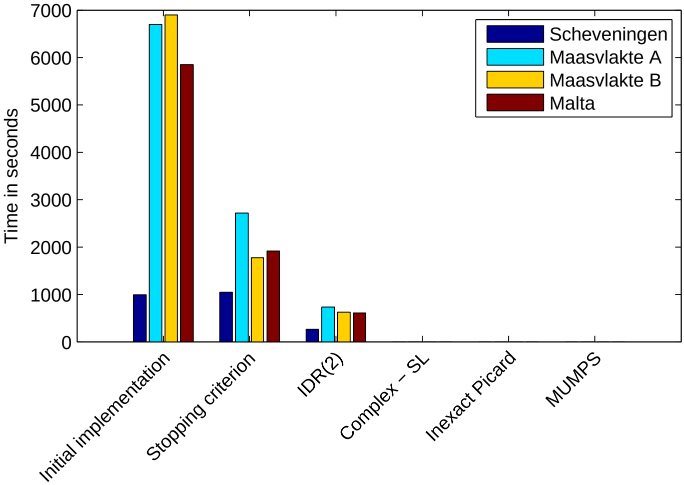

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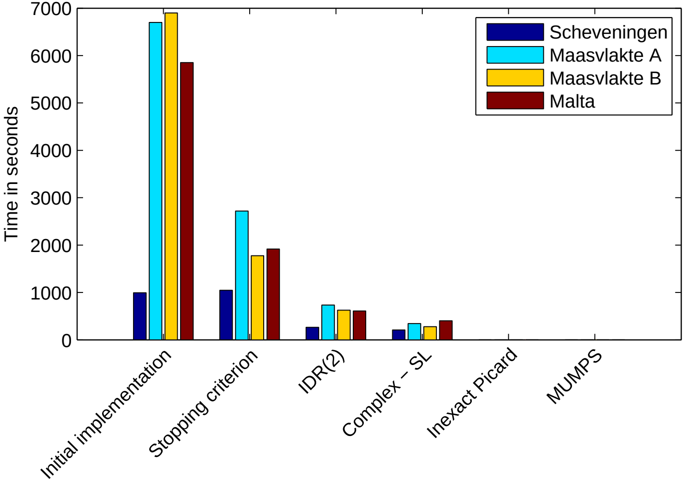

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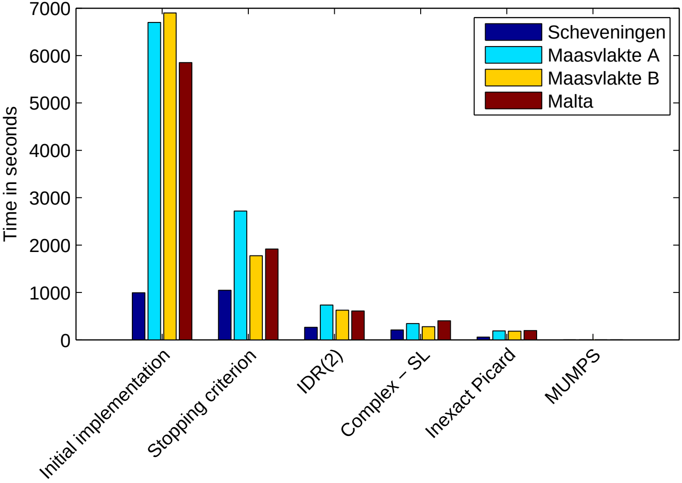

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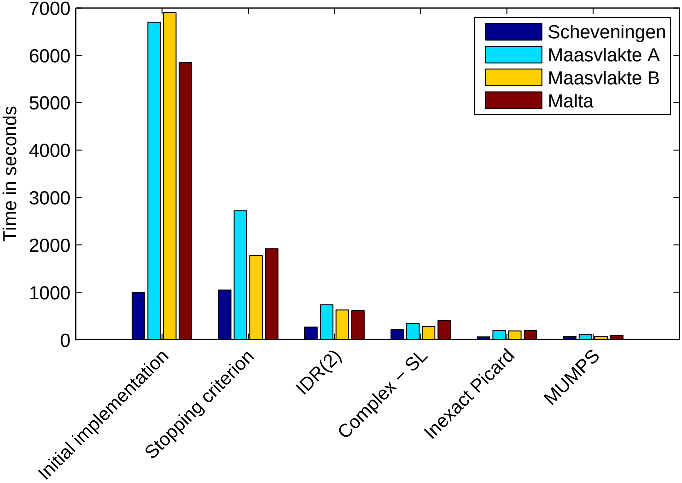

After implementing the proposed improvements we need the following percentages of the computing time of the initial implementation.

	Scheveningen	Maasvlakte A	Maasvlakte B	Malta
Iterative	5.8 %	2.8 %	2.7 %	3.4 %
Direct	7.0 %	1.6 %	1.0 %	1.5 %

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- Use a direct method, e.g. `MUMPS`, to determine the solution of the system of equations.
- If the dimension of the sparse matrix is considerably larger we propose inexact Picard iteration, where the system of equations is solved using IDR(s) preconditioned with the shifted Laplace preconditioner.

Outline

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- Parallel computation of the global matrix $\boldsymbol{S}$.
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- Inexact Picard iteration based on a different forcing sequence.

Numerical experiments

Computing time - logarithmic scale