Project: Domain Decomposition Techniques for the Helmholtz Equation

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Our group is world leading in the development of fast and robust solvers for the discretized Helmholtz problem. The Helmholtz equation is the time-harmonic equivalent of the wave equation and is used in a wide range of engineering practices. It mainly models electromagnetic waves and is used from scattering studies in medical imaging to seismology. The study of the Helmholtz equation and its correct simulation were and are, still to this day, a vital part of applied mathematics.

Several issues arise when trying to solve the Helmholtz equation numerically. The first issue is related to the accuracy of the numerical solution. Due to the heavy oscillatory nature of waves, the solution often contains numerical dispersion errors which translate to having phase differences. This is called the pollution error. In order to avoid this, the Helmholtz equation needs to be solved on very fine grids, leading to large linear systems.

The second issue is related to the efficiency of the numerical solvers. Due to the size of the linear system, numerical solutions are obtained by using iterative solvers. The challenge in designing efficient solvers is that the number of iterations to reach convergence grows with the wavenumber.

But all hope is not lost, as our group and other scientists from around the world are actively involved in finding new ways to accelerate and improve numerical solutions to the Helmholtz equation. In our group, recent progress has been made in bounding the number of iterations needed to reach convergence; from linear dependence on the wavenumber to almost wavenumber independent convergence using deflation techniques. One important feature of the deflation method is the use of higher-order parametric curves to construct the deflation vectors and coarse-spaces [2].

In this master project, the goal is to use domain decomposition techniques in order to construct an efficient and scalable solver for the Helmholtz equation, which exhibits robustness with respect to the wavenumber. Therefore, the combination of two-level overlapping Schwarz methods and the inclusion of these higher-order coarse-spaces from the deflation setting should be explored.

On the first level, the boundary value problem on the computational domain is decomposed into smaller subproblems on overlapping subdomains. These subproblems can be solved independently, making the setup and application of the first level very efficient and easy to parallelize. However, the resulting iterative scheme might require a large number of iterations since the subproblems are only coupled weakly, i.e., through their overlap. In order to facilitate scalability with respect to the number of subdomains and robustness with respect to the wavenumber, a second level (coarse level) has to be introduced.

Here, we are specifically interested in the use of a coarse level based on extension-based coarse spaces, such as GDSW (generalized Dryja–Smith–Widlund) type coarse spaces [1, 3], which are very flexible in adding additional coarse functions. An important research aspect will therefore focus on investigating the compatibility of (rational) parametric curves to be included as coarse functions in order to enable robustness with respect to the wavenumber. As the Schwarz framework allows for both an additive and multiplicative coupling of the coarse level, a part of the project can also be dedicated to investigate the use of a deflated Krylov method as the multiplicative coupling of the coarse level.

1 Research plan and schedule

In this section we outline the research plan and schedule.

- Literature study (approx. 3 months) with topics:
  - Helmholtz equation
  - Iterative methods and preconditioning
  - Deflation
  - Domain decomposition methods (in particular Schwarz methods and GDSW coarse spaces)
- Research phase I: Design of the new preconditioner + preliminary implementation for 1D/2D problems. Incorporate into thesis. (approx. 2-3 months).
- Research phase II: Testing phase and extension to 3D problems. Depending on the results, test problems with non-constant wave numbers. sIncorporate into thesis. (approx 2-3 months).
- Research phase III: Finalizing thesis. (1 month).

Contact

If you are interested in this project and/or have further questions please contact Vandana Dwarka, v.n.s.r.dwarka@tudelft.nl, and Alexander Heinlein, a.heinlein@tudelft.nl.

References

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