Literature Presentation

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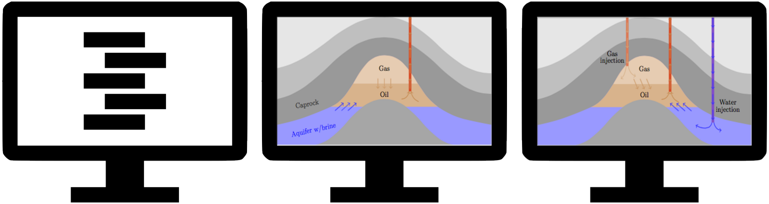
27 February 2018

Introduction



Structure

Problem

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

Properties of **A**

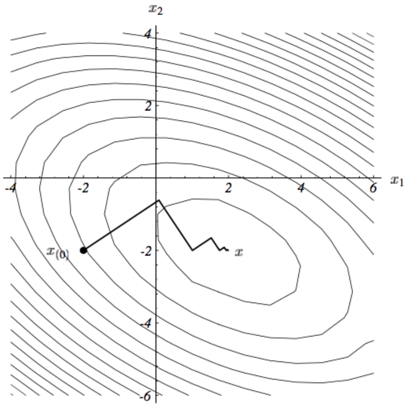
- Large
- SPD (symmetric positive definite)
- Sparse

Condition number

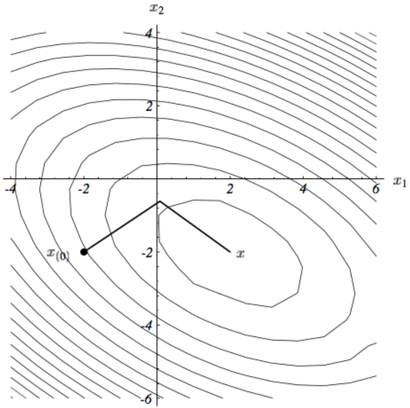
$$\kappa_2(\mathbf{A}) = \frac{\lambda_{\mathrm{max}}(\mathbf{A})}{\lambda_{\mathrm{min}}(\mathbf{A})} \tag{2}$$

Iterative Numerical Methods

$$\mathbf{Ax} = \mathbf{b} \tag{3}$$



Conjugate Gradient



Convergence

$$\|\mathbf{x}-\mathbf{x}^k\|_{\mathbf{A}} \leq 2\|\mathbf{x}-\mathbf{x}^0\|_{\mathbf{A}}\left(\frac{\sqrt{\kappa_2(\mathbf{A})}-1}{\sqrt{\kappa_2(\mathbf{A})}+1}\right)^k. \tag{4}$$

Conjugate Gradient

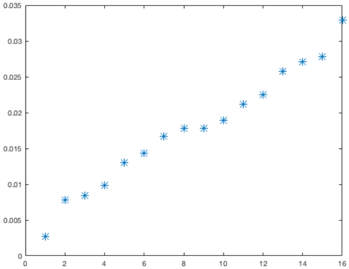


Figure: Eigenvalues of **A**

$$\frac{\sqrt{\kappa_2(\mathbf{A})} - 1}{\sqrt{\kappa_2(\mathbf{A})} + 1} \approx 0.5971 \tag{5}$$

Preconditioner

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b}, \tag{6}$$

$\mathbf{M}$: preconditioner

Requirements for $\mathbf{M}$

- SPD
- $\mathbf{M}^{-\frac{1}{2}}$ exists and symmetric

Preconditioner

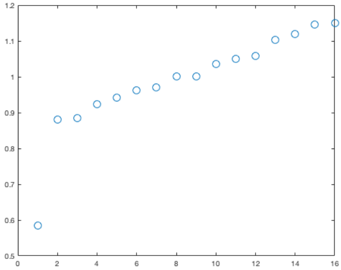


Figure: Eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$

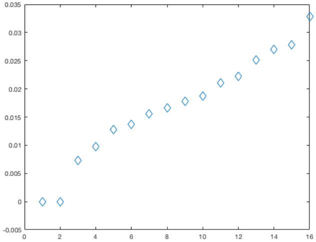
$$\frac{\sqrt{\kappa_2(\mathbf{M}^{-1}\mathbf{A})}-1}{\sqrt{\kappa_2(\mathbf{M}^{-1}\mathbf{A})}+1} \approx 0.2663 \tag{7}$$

Deflated Method

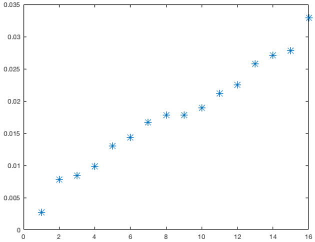
$$\mathbf{PA}\hat{\mathbf{x}} = \mathbf{Pb} \tag{8}$$

P: Deflation matrix

Z: Deflation-subspace matrix



(a) Eigenvalues of **PA**



(b) Eigenvalues of **A**

Proper Orthogonal Decomposition (POD)

Known solutions:

$$\mathbf{A}\mathbf{x}_i = \mathbf{b}_i \tag{9}$$

$$\mathbf{X} = [\mathbf{x}_1 \;\; \ldots \;\; \mathbf{x}_m] \tag{10}$$

Correlation matrix:

$$\mathbf{R} = \frac{1}{m}\mathbf{X}\mathbf{X}^\top \tag{11}$$

Eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \lambda_m \tag{12}$$

Proper Orthogonal Decomposition (POD)

Correlation matrix:

$$\mathbf{R} = \frac{1}{m}\mathbf{X}\mathbf{X}^\top \tag{13}$$

Eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \lambda_m \tag{14}$$

$$\frac{\max\limits_{1 \le l \le m} \sum_{i=1}^{l} \lambda_i(\mathbf{R})}{\sum_{i=1}^{m} \lambda_i(\mathbf{R})} \le \alpha \tag{15}$$

where $0 < \alpha \le 1$

Basis matrix:

$$\mathbf{\Psi} := \begin{bmatrix} \psi_1 & \ldots & \psi_l \end{bmatrix} \tag{16}$$

Basis matrix as deflation-subspace matrix $\mathbf{Z}$

Deflated Preconditioner Conjugate Gradient

$$\mathbf{M}^{-1}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{M}^{-1}\mathbf{P}\mathbf{b} \tag{17}$$

P: Deflation matrix
Z: Deflation-subspace matrix
M: Preconditioner

Deflated Preconditioner Conjugate Gradient

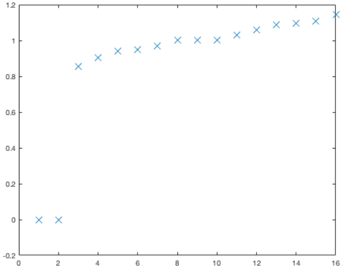


Figure: Eigenvalues of $\mathbf{M}^{-1}\mathbf{PA}$

$$\kappa_2(\mathbf{M}^{-1}\mathbf{PA}) \leq \kappa_2(\mathbf{M}^{-1}\mathbf{A}) \tag{18}$$

ROM-based Preconditioner

$$\mathbf{M}_{rom}^{-1} = \mathbf{M} + \mathbf{Q}(1 - \mathbf{AM}) \tag{19}$$

Q: Correction matrix

Z: Deflation-subspace matrix

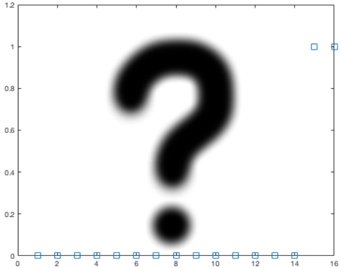
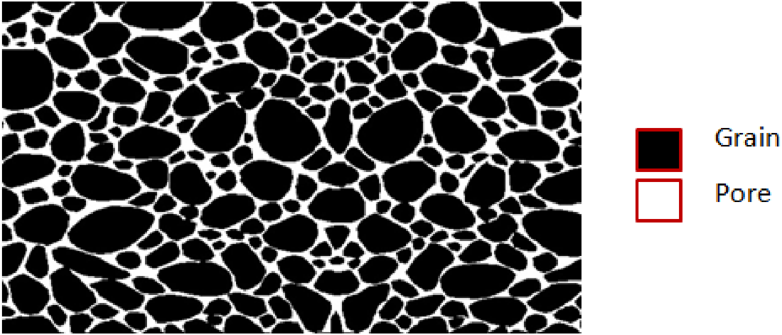
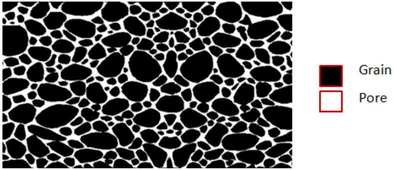


Figure: Eigenvalues of $\mathbf{M}_{rom}^{-1}\mathbf{A}$

Porous Media



Reservoir Simulation



Mass conversation law + Darcy's velocity =

$$c_t \rho \varphi \frac{\partial p}{\partial t} - \nabla \left(\rho \frac{\mathbf{K}}{\mu} (\nabla p - \rho g \nabla d) \right) = \rho q \qquad (20)$$

ρ	fluid density	φ	rock porosity
$\mathbf{K}$	rock permeability	μ	fluid viscosity
g	gravity constant	d	reservoir depth
q	source term	c_t	total compressibility

p pressure (unknown)

Incompressible Model

Fluid density and rock porosity does not depend on pressure
+ more assumptions

$$-\frac{1}{\mu}\nabla(\mathbf{K}\nabla p) = q. \tag{21}$$

Discretization:

$$\mathbf{T}\mathbf{p} = \mathbf{q} \tag{22}$$

Constant Compressible Model

Fluid density depends on pressure
$+$ more assumptions

$$\varphi \frac{\partial \rho(p)}{\partial t} - \frac{\rho_0}{\mu} \nabla(\mathbf{K} \nabla p) = \rho(p) q. \tag{23}$$

Discretization:

$$\mathbf{V} \frac{\boldsymbol{\rho}(\mathbf{p}^{k+1}) - \boldsymbol{\rho}(\mathbf{p}^{k})}{\Delta t^k} + \mathbf{T}\mathbf{p}^{k+1} = \mathbf{q}(\mathbf{p}^{k+1}). \tag{24}$$

Newton-Raphson

1. Initial: $\mathbf{p}^0, \varepsilon$
2. **While** $\left|\mathbf{p}^{k+1} - \mathbf{p}^k\right| > \varepsilon$
3. **Solve** $\mathbf{J_f}(\mathbf{p}^k)\delta\mathbf{p} = -\mathbf{f}(\mathbf{p}^k)$
4. **Update** $\mathbf{p}^{k+1} = \mathbf{p}^k + \delta\mathbf{p}$
5. $k = k + 1$

Research Question

		DPCG	ROM-based prec
1.	Complexity	**?**	**?**
2.	Memory	**?**	**?**
3.	Convergence	**?**	**?**
4.	Iterations	**?**	**?**
5.	Error	**?**	**?**

Summary

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