Numerical Modeling of a Turbine

Modeling of Non-Smooth Dynamics of lamellas

Efficient solutions for ODEs with periodic boundary conditions

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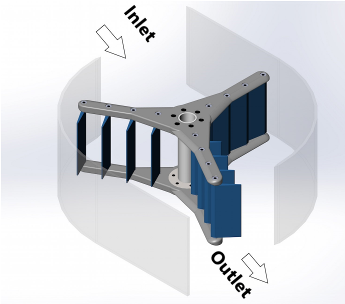
Background

- The turbine is called Oryon Watermill (OWM).
- Developed by Deep Water Energy BV, Netherlands.

Key features

- Modular build
- Operates under low pressure head conditions.
- 'Special' design of the rotor arm.



Numerical Model

- What are the expectations from the Numerical Model?

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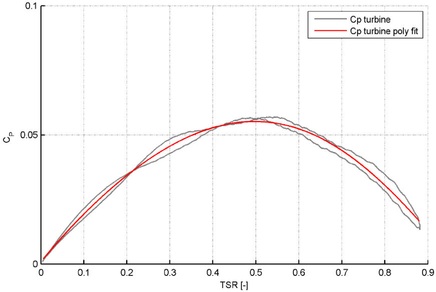
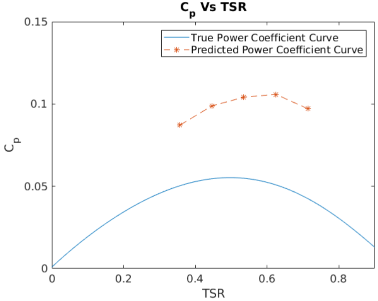
Performance Characteristic Curve

Co-efficient of Power v/s Tip Speed Ratio

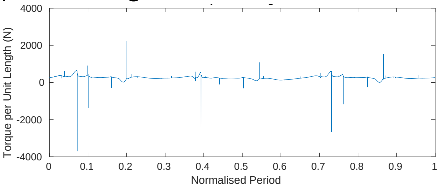


Current Numerical Model

- Computationally intensive
- No agreement with experimental result.

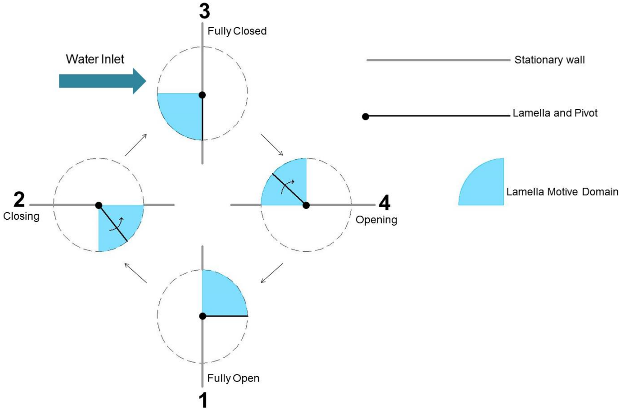


- Spikes in torque time signal.



Solution to the spikes in torques

- Torque transfer mechanism- Water → Lamella → Shaft
- Fluid-Structure Interaction problem.



Governing equations for lamella motion

Newton's Second Law

Mathematically expressed as an ODE-

$$\mathcal{M}\ddot{\mathbf{x}}(t) = \mathbf{h}(t)$$

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Algebraic Constraints

$$\|$$

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Differential **A**lgebraic **E**quation

Governing equations for lamella motion

Newton's Second Law

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$$\parallel$$

Differential **A**lgebraic **E**quation

Non-smooth Dynamics

Position and velocity vectors are not smooth functions of time.

Numerical Solution

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- Event Driven
  - Separate the non-smooth motion.
  - Integrate the smooth part until collision.
  - Solve the impact problem at the discontinuity.
  - Reset the ODE.

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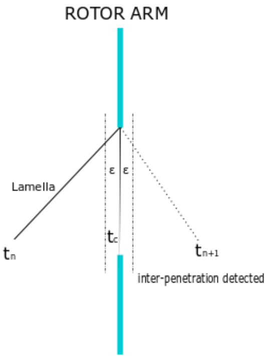
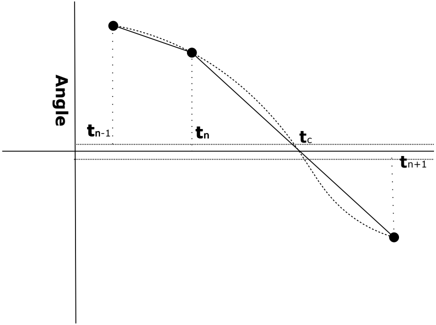
  - Solve the impact problem at the discontinuity.
  - Reset the ODE.
- Time Stepping
  - Discretize the entire DAE with the inequalities.
  - Less administrative effort
  - **Problems**
    - Small time step size.
    - Poor accuracy as compared to the event-driven approach.
    - Inability to model the partial elastic behavior correctly.

Event Driven Approach





Event Driven Approach

Constraints are formulated as contacts

- **Colliding Contact**
  Change in velocity on collision and bodies move apart with different velocities.
- **Resting Contact**
  Two bodies after collision are resting on each other.

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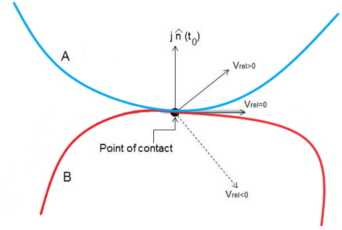

- $v_{rel} > 0 \rightarrow$ the bodies are not contacting after t_c.
- $v_{rel} < 0 \rightarrow$ The bodies are in colliding contact after t_c.
- $v_{rel} = 0 \rightarrow$ The bodies are in resting contact after t_c.

Colliding and Resting Contact

- **Colliding Contact**
  At the instant of collision-
  - Calculate relative velocities.
  - Add impulse $j = -M_a(1 + \epsilon)v_{rel}^{-}$
  - Reset ODE.

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Colliding and Resting Contact

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  At the instant of collision-
  - Calculate relative velocities.
  - Add impulse $j = -M_a(1 + \epsilon)v_{rel}^-$
  - Reset ODE.
- **Resting contact**
  Contact force equals the force acting on the lamella exerted by the fluid.

Summary of non-smooth dynamics for lamella motion

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- ODE Formulation

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- Detection of time of collision

Summary of non-smooth dynamics for lamella motion

- ODE Formulation
- Detection of time of collision
- Detection of the type of contact
  - Colliding Contact
  - Resting Contact

Coupled system of nonlinear ODEs

Governing equation for lamella motion

$$\mathcal{M}\ddot{\mathbf{x}}(t) = \mathbf{h}(t) + \mathbf{w}(t); \tag{1}$$

Governing equation for fluid flow

$$\frac{d\mathbf{u}}{dt} + \mathbf{R}(\mathbf{u}) = 0; \tag{2}$$

- RHS of ODE 1 from forces due to fluids.
- RHS of ODE 2 from lamella motion.
- Resulting system of nonlinear ODEs has a periodic solution.

Requirements for the numerical method

- The method should be faster than the direct time integration.
- The method should have low memory requirements.
- The method should be easy to implement with minimum modifications to the solver.

Methods for solving the coupled system

Standard Methods for nonlinear ODEs with periodic solutions-

- Shooting Method
- Finite Difference Method
- Collocation Method

Specific Methods for fast analysis of periodic flows-

- Multitime multigrid method
- Time Linearization method
- Time spectral (harmonic balance) method

Harmonic Balance Method

$$\frac{du}{dt} + R = 0; \tag{3}$$

Fourier series expansion of u(t) with n harmonics reads:

$$u(t) = \sum_{j=0}^{n} \mathfrak{u}_j e^{ij\omega t}, \tag{4}$$

and the expansion for R(t) reads:

$$R(t) = \sum_{j=0}^{n} \mathfrak{R}_j e^{ij\omega t}, \tag{5}$$

where $\mathfrak{u}_j$ and $\mathfrak{R}_j$ are the Fourier co-efficients.

Harmonic Balance Method

Inserting equations (4) and (5) in equation (3) we obtain,

$$\omega \sum_{j=0}^{n} ij\mathfrak{u}_j e^{ij\omega t} + \sum_{j=0}^{n} \mathfrak{R}_j e^{ij\omega t} = 0.$$

$$\text{n equations for sine} \begin{cases} -1\omega\mathfrak{u}_{c_1} + \mathfrak{R}_{s_1} = 0; \\ -2\omega\mathfrak{u}_{c_2} + \mathfrak{R}_{s_2} = 0; \\ \vdots \\ -n\omega\mathfrak{u}_{c_n} + \mathfrak{R}_{s_n} = 0; \end{cases} \tag{6}$$

$$\text{center-} \qquad \mathfrak{R}_0 = 0 \tag{7}$$

$$\text{n equations for cosine} \begin{cases} 1\omega\mathfrak{u}_{s_1} + \mathfrak{R}_{c_1} = 0; \\ 2\omega\mathfrak{u}_{s_2} + \mathfrak{R}_{c_2} = 0; \\ \vdots \\ n\omega\mathfrak{u}_{s_n} + \mathfrak{R}_{c_n} = 0; \end{cases} \tag{8}$$

$$\omega\mathcal{A}\boldsymbol{\mathfrak{u}} + \boldsymbol{\mathfrak{R}} = 0. \tag{9}$$

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Harmonic Balance Method

Frequency to time domain transformation-

$$\mathfrak{u} = \mathcal{E}\hat{\mathbf{u}}(t).$$

The operator $\mathcal{E}$ is given by:

$$\mathcal{E} = \frac{2}{2n+1}\begin{bmatrix} \sin(\omega t_1) & \sin(\omega t_2) & \sin(\omega t_3) & \dots & \sin(\omega t_{2n+1}) \\ \sin(2\omega t_1) & \sin(2\omega t_2) & \sin(2\omega t_3) & \dots & \sin(2\omega t_{2n+1}) \\ \vdots & \vdots & \vdots & & \vdots \\ \sin(n\omega t_1) & \sin(n\omega t_2) & \sin(n\omega t_3) & \dots & \sin(n\omega t_{2n+1}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ \cos(\omega t_1) & \cos(\omega t_2) & \cos(\omega t_3) & \dots & \cos(\omega t_{2n+1}) \\ \cos(2\omega t_1) & \cos(2\omega t_2) & \cos(2\omega t_3) & \dots & \cos(2\omega t_{2n+1}) \\ \vdots & \vdots & \vdots & & \vdots \\ \cos(n\omega t_1) & \cos(n\omega t_2) & \cos(n\omega t_3) & \dots & \cos(n\omega t_{2n+1}) \end{bmatrix}; \qquad (10)$$

$$\omega\mathcal{A}\mathcal{E}\hat{\mathbf{u}} + \mathcal{E}\hat{\mathbf{R}} = 0.$$

Multiplying from left, the inverse transform operator $(\mathcal{E}^{-1})$, we have:

$$\omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E})\hat{\mathbf{u}} + \hat{\mathbf{R}}(\hat{\mathbf{u}}) = 0. \qquad (11)$$

Harmonic Balance Method

- The derivative term is converted to a source term with the left multiplication of the operator $\mathcal{E}^{-1}\mathcal{A}\mathcal{E}$
- $\omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E})\hat{\mathbf{u}} + \hat{\mathbf{R}}(\hat{\mathbf{u}}) = 0$ can be solved as coupled stationary problems.

$$\mathcal{E}^{-1}\mathcal{A}\mathcal{E} = \frac{2}{2n+1}\begin{bmatrix} 0 & B_1 & B_2 & B_3 & \dots & \dots & B_{2n} \\ -B_1 & 0 & B_1 & B_2 & B_3 & & B_{2n-1} \\ -B_2 & -B_1 & 0 & B_1 & B_2 & & \vdots \\ -B_3 & -B_2 & -B_1 & 0 & B_1 & & \vdots \\ \vdots & & & & \ddots & & B_2 \\ \vdots & & & & & \ddots & B_1 \\ -B_{2n} & \dots & \dots & -B_3 & -B_2 & B_1 & 0 \end{bmatrix};$$

$B_i = \sum_{k=1}^{n} k\sin(k\omega j t_1); \quad j = \{1, ..., 2n\}.$

Harmonic Balance Method

Pseudo time marching method:

$$\frac{d\hat{\mathbf{u}}}{d\tau} + \omega\mathcal{B}\hat{\mathbf{u}} + \hat{\mathbf{R}}(\hat{\mathbf{u}}) = 0;$$
$$\text{where} \quad \mathcal{B} = \mathcal{E}^{-1}\mathcal{A}\mathcal{E}. \tag{12}$$

$$\frac{\hat{\mathbf{u}}^{k+1} - \hat{\mathbf{u}}^{k}}{\Delta\tau} = -[\omega\mathcal{B}\hat{\mathbf{u}} + \hat{\mathbf{R}}(\hat{\mathbf{u}}^{k+1})]. \tag{13}$$

$\hat{\mathbf{R}}(\hat{\mathbf{u}}^{k+1})$ is linearized using a Taylor Series expansion:

$$\hat{\mathbf{R}}(\hat{\mathbf{u}}^{k+1}) = \hat{\mathbf{R}}(\hat{\mathbf{u}}^{k}) + \mathcal{J}_R\Delta\hat{\mathbf{u}} + \mathcal{O}(\Delta\hat{\mathbf{u}}^2), \tag{14}$$

where $\mathcal{J}_R$ is the Jacobian matrix of the residual vector in block diagonal form.

Harmonic Balance Method

Explicit $\omega\mathcal{B}\hat{\mathbf{u}}$

$$\left[\frac{\mathsf{V}\mathcal{I}}{\Delta\tau} + \mathcal{J}_{\mathsf{R}}\right]\Delta\hat{\mathbf{u}} = -\hat{\mathbf{R}}^{\mathsf{k}} - \omega\mathcal{B}\hat{\mathbf{u}}^{\mathsf{k}}, \tag{15}$$

$$\begin{bmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & E_{2n+1} \end{bmatrix}; \quad E_i = \frac{V}{\Delta\tau_i} + \mathcal{J}_{ts_{i,i}}$$

- Solve independently for each of the $2n+1$ stationary solutions
- Only the k^{th} snapshot of the Jacobian has to be stored. No extra memory.
- Can be easily parallelized.
- Restricts the size of the Courant-Friedrich-Lewy (CFL) number and thus, time step size.

Harmonic Balance Method

Implicit $\omega\mathcal{B}\hat{\mathbf{u}}$

$$\left[\frac{V\mathcal{I}}{\Delta\tau} + \mathcal{J}_R + \omega\mathcal{B}\right]\Delta\hat{\mathbf{u}} = -\hat{\mathbf{R}}^k - \omega\mathcal{B}\hat{\mathbf{u}}^k; \tag{16}$$

$$\begin{bmatrix} E_1 & H_{1,2} & H_{1,3} & \dots & H_{1,2n+1} \\ H_{2,1} & E_2 & & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & E_{2n} & H_{2n,2n+1} \\ H_{2n+1,1} & \dots & \dots & & E_{2n+1} \end{bmatrix}; \quad E_i = \frac{V}{\Delta\tau_i} + \mathcal{J}_{ts_{i,i}}; \quad H_{i,j} = V\omega\mathcal{B}_{i,j}$$

- Memory requirements and CPU time are more than the explicit approach.
- No restriction on CFL number and thus, on time step size.
- Significant modification in the solver.

Summary

- To solve the problem of unphysical spikes equations from the field of non-smooth dynamics to be used.
- The event-driven approach with the consideration of colliding and resting contacts is most appropriate.
- Complex problem of modeling lamella motion was reduced to simple ODE integration but with appropriate conditions.
- Coupled system of nonlinear ODEs with periodic solutions.
- The harmonic balance method is most suited for the current problem.

Research Questions

- How to apply the harmonic balance method to the coupled fluid-structure problem?
- Which of the two treatments-explicit or implicit, is the most appropriate?

Appendix

Methods Applied to a system of linear ODEs

Consider the a system of linear ODEs

$$\dot{\mathbf{x}}(t) = \mathcal{Q}\mathbf{x} + \mathbf{f}(t); \quad \mathbf{x}(0) = \mathbf{x}(T); \quad \mathbf{x} = [0 \quad 1]^T; \tag{17}$$

where $\mathcal{Q} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} \sin(\omega t) \\ \cos(\omega t) \end{bmatrix}$

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where $\mathcal{Q} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} \sin(\omega t) \\ \cos(\omega t) \end{bmatrix}$

The true solution of the above system is computed to be-

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\sin(\omega t) \\ \text{cost}(\omega t) + \frac{1}{\omega}(\sin(\omega t)) \end{bmatrix} \tag{18}$$

Harmonic balance solution

$$\dot{\mathbf{x}}(t) = \mathcal{Q}\mathbf{x} + \mathbf{f}(t); \tag{19}$$

Applying harmonic balance method, gives:

$$\omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E})\hat{\mathcal{X}} + \hat{\mathcal{R}} = 0.$$

where, for n harmonics we have

$$\mathcal{E}^{-1}\mathcal{A}\mathcal{E} \in \mathbb{R}^{2n+1\times 2n+1}; \quad \hat{\mathcal{X}} \in \mathbb{R}^{2n+1\times 2}; \quad \hat{\mathcal{R}}^{2n+1\times 2}$$

with

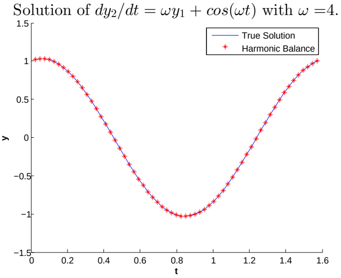
$$\hat{\mathcal{R}} = -\hat{\mathcal{X}}\mathcal{Q}^{T} - \mathcal{F}$$

The final system looks like:

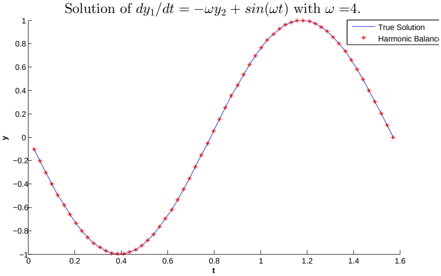
$$\omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E})\hat{\mathcal{X}} - \hat{\mathcal{X}}\mathcal{Q}^{T} = \mathcal{F}$$

The above system is a Sylvester equation.

Plots



Solution of $dy_2/dt = \omega y_1 + cos(\omega t)$ with ω =4.
True Solution
Harmonic Balance
y
t



Solution of $dy_1/dt = -\omega y_2 + sin(\omega t)$ with ω =4.
True Solution
y
t

Another formulation

Sylvester problem can be represented as a linear equation in this case.

$$\omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E})\hat{\mathcal{X}} - \hat{\mathcal{X}}\mathcal{Q}^{\mathsf{T}} = \mathcal{F} \Rightarrow \mathcal{Z}\hat{\mathbf{x}} = \mathbf{F} \tag{20}$$

where $\mathcal{Z} = \omega(\mathcal{E}^{-1}\mathcal{A}\mathcal{E}) - (\mathcal{I} \otimes \mathcal{Q})$

Observation

Comparing with $\mathcal{Z} = \mathcal{M} - \mathcal{N}$ splitting.
Properties of $\mathcal{Z}$ determine the convergence of the iterative process.

Van der Pol's equation

The Van der Pol's equation in reduced form:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ \mu(1 - y_1^2)y_2 - y_1 \end{bmatrix} =: -\mathbf{R} \tag{21}$$

Performing the harmonic balance transformation, we have:

$$\begin{aligned} & \omega\mathcal{B}\hat{\mathbf{Y}} + \hat{\mathbf{R}}(\hat{\mathbf{Y}}) = 0; \\ \text{where} \quad & \hat{\mathbf{Y}} \in \mathbb{R}^{2*(2n+1)\times 1}; \quad \hat{\mathbf{R}} \in \mathbb{R}^{2*(2n+1)\times 1}; \\ & \mathcal{B} \in \mathbb{R}^{2*((2n+1)\times(2n+1))} \end{aligned} \tag{22}$$

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Van der Pol's equation

Pseudo time stepping-

$$\frac{\mathrm{d}\hat{\mathbf{Y}}}{\mathrm{d}\tau} + \omega\mathcal{B}\hat{\mathbf{Y}} + \hat{\mathbf{R}} = 0. \tag{23}$$

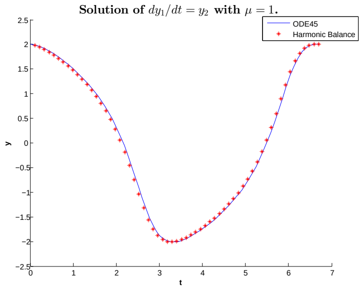
Explicit

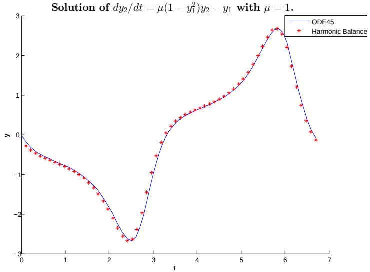
$$\left[\frac{\mathsf{V}\mathcal{I}}{\Delta\tau} + \mathcal{J}_{\mathsf{ts}}\right]\Delta\hat{\mathbf{Y}} = -\hat{\mathbf{R}}^{\mathsf{k}} - \omega\mathcal{B}\hat{\mathbf{Y}}^{\mathsf{k}}, \tag{24}$$

Implicit

$$\left[\frac{\mathsf{V}\mathcal{I}}{\Delta\tau} + \mathcal{J}_{\mathsf{ts}} + \mathcal{B}\right]\Delta\hat{\mathbf{Y}} = -\hat{\mathbf{R}}^{\mathsf{k}} - \omega\mathcal{B}\hat{\mathbf{Y}}^{\mathsf{k}}; \tag{25}$$

Plots





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Though the Roads been rocky, it sure feels good to me

-Bob Marley