Modelling and Simulating Large Scale Multi-Carrier Energy Networks

Literature Review

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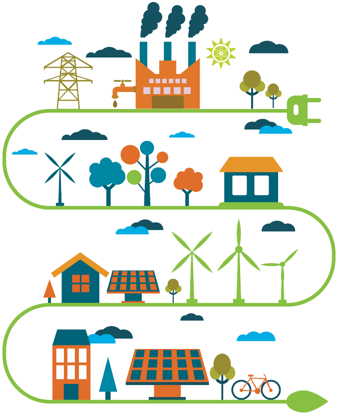
Presentation Outline

1. Problem Statement
2. Energy Networks
   - Electricity Networks
   - Gas Networks
   - Multi-Carrier Energy Networks
3. Solution Methods
4. Literature
5. Open Problems and Research Scope



01

Problem Statement

Problem Statement

Modelling of multi-carrier energy networks

- Optimization and planning
- Incorporation of renewables

Aim of the literature review:

- Physical models
- Solvers and preconditioners
- Recent literature and research
- Identify open problems
- Preliminary research scope and questions

02

Energy Networks

Electricity Networks

- Steady-state AC power system
  - Link: Transmission line
  - Node: Bus, generator, load
- System variables:
  - Voltage angle δ_i
  - Voltage magnitude $|V_i|$
  - Real power P_i
  - Reactive power Q_i

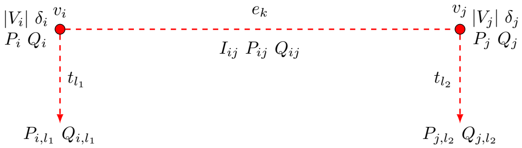


Table: Electricity network node types

Node type	Specified	Unknown
PQ/load bus	P, Q	$\|V\|$, δ
PV/generator bus	P, $\|V\|$	Q, δ
Slack bus	$\|V\|$, δ	P, Q

Electricity Networks: Load Flow Equations

Principles:

- Complex power $S_i = V_i {I_i}^* = P_i + \boldsymbol{i} Q_i$
- Complex nodal power equals injected complex power, $\sum_l S_{i,l} = S_i$

Load-flow equations: $\boldsymbol{F}^e(\boldsymbol{x}^e) = 0$, where

$$\boldsymbol{F}^e(\boldsymbol{x}^e) = \begin{bmatrix} \boldsymbol{F}^P(\boldsymbol{x}^e) \\ \boldsymbol{F}^Q(\boldsymbol{x}^e) \end{bmatrix} \quad \boldsymbol{x}^e = \begin{bmatrix} \boldsymbol{\delta} \\ |\boldsymbol{V}| \end{bmatrix}$$

$$\begin{cases} \boldsymbol{F}_i^P(\boldsymbol{x}^e) = P_i + |V_i|^2 G_{ii} + \sum_{j, j \neq i} |V_i||V_j| \left(G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij} \right) & \text{Real power} \\ \boldsymbol{F}_i^Q(\boldsymbol{x}^e) = Q_i - |V_i|^2 B_{ii} - \sum_{j, j \neq i} |V_i||V_j| \left(B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right) & \text{Reactive power} \end{cases}$$

Gas Networks

- Gas system
  - Link: Pipe, compressor, valve
  - Node: Sink, source, junction
- System variables:
  - Nodal pressure p_i
  - Link gas flow q_k

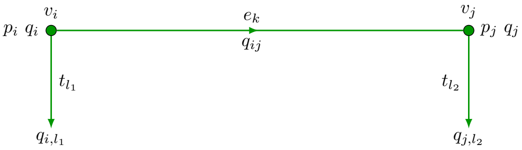


Table: Gas network node types

Node type	Specified	Unknown
Load node	q	p
Slack node	p	q

Gas Networks: Load-Flow Equations

Principles:

- Conservation of mass: Flows into the node equals flow out of the node
- Unknown link flow: Relation between nodal pressures and gas mass flow

Load-flow equations: $\boldsymbol{F}^g(\boldsymbol{x}^g) = 0$, where

$$\boldsymbol{F}^g(\boldsymbol{x}^g) = \begin{bmatrix} \boldsymbol{F}^q(\boldsymbol{x}^g) \\ \boldsymbol{F}^L(\boldsymbol{x}^g) \end{bmatrix} \quad \boldsymbol{x}^g = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}$$

$$\boldsymbol{F}^g(\boldsymbol{x}^g) = \begin{bmatrix} \boldsymbol{F}^q(\boldsymbol{x}^g) \\ \boldsymbol{F}^L(\boldsymbol{x}^g) \end{bmatrix} \begin{cases} \boldsymbol{F}_i^q(\boldsymbol{x}^g) = \sum_{j,j\neq i} q_{ji} - \sum_{j,j\neq i} q_{ij} - q_i & \text{Conservation of mass} \\ \boldsymbol{F}_k^L(\boldsymbol{x}^g) = \Delta p_{ij} - (C^g)^{-2} f|q_{ij}|q_{ij} & \text{Pipe flow} \end{cases}$$

Multi-Carrier Energy Networks

- Coupling node: No associated variables
- Dummy links: Only represent coupling, same link parameters as the coupled SC network

Coupling equations depend on the specific coupling units

$$F_i^c(x^c) = 0, \quad x^c = \begin{bmatrix} q_c \\ P_c \\ Q_c \end{bmatrix}$$

Table: Conversion units in MCN

Input	Output	Abbr.	Example
Gas	Electricity	G2P	Gas-fired generator
Electricity	Gas	P2G	Electrolyser

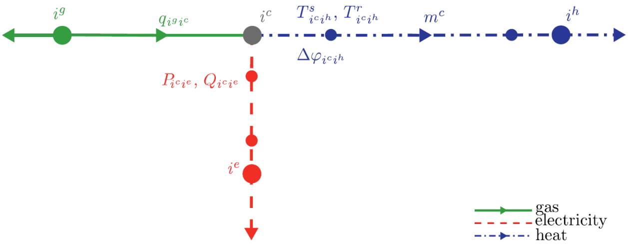


Figure: Coupling node

Multi-Carrier Energy Networks

Additional node types due to extra BCs required

↓

SC networks can become over/under determined

Load-flow equations coupled network:

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{F}^g(\boldsymbol{x}^g, \boldsymbol{x}^c) \\ \boldsymbol{F}^e(\boldsymbol{x}^e, \boldsymbol{x}^c) \\ \boldsymbol{F}^c(\boldsymbol{x}^c) \end{bmatrix} = \mathbf{0}, \ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^g \\ \boldsymbol{x}^e \\ \boldsymbol{x}^c \end{bmatrix}$$

Table: Node types in a multi-carrier (gas–electricity) network

Carrier	Node type	Specified	Unknown
Gas	Load node	q	p
	Slack node	p	q
	Ref. load node	p, q	–
	Slack node	–	p, q
Electricity	PQ/load bus	P, Q	$\|V\|, \delta$
	PV/generator bus	$P, \|V\|$	Q, δ
	Slack bus	$\|V\|, \delta$	P, Q
	PQδ bus	P, Q, δ	$\|V\|$
	QVδ bus	$Q, \|V\|, \delta$	P
	PQVδ bus	$P, Q, \|V\|, \delta$	–

03

Solution Methods

Newton-Raphson

Iteratively solve non-linear systems, using the **Jacobian** matrix and update $\boldsymbol{s}_k = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k$

$$J_F(\boldsymbol{x}_k)\boldsymbol{s}_k = -\boldsymbol{F}(\boldsymbol{x}_k) \quad J_F = \nabla \boldsymbol{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial x_1} & \cdots & \frac{\partial F_N}{\partial x_N} \end{bmatrix}$$

Jacobian of a coupled gas-electricity network

$$J_{F^c} = \begin{pmatrix} J_{F^g} & \mathbf{0} & \frac{\partial \boldsymbol{F}^g(\boldsymbol{x}^g, \boldsymbol{x}^c)}{\partial \boldsymbol{x}^c} \\ \mathbf{0} & J_{F^e} & \frac{\partial \boldsymbol{F}^e(\boldsymbol{x}^e, \boldsymbol{x}^c)}{\partial \boldsymbol{x}^c} \\ \mathbf{0} & \mathbf{0} & \frac{\partial \boldsymbol{F}^c(\boldsymbol{x}^c)}{\partial \boldsymbol{x}^c} \end{pmatrix}$$

Blocks are generally not square due to new node types

Linear Systems

General linear system of the form

$$A\boldsymbol{x} = \boldsymbol{f}$$

Direct computation of the inverse is expensive and inefficient, especially for larger systems

Solution methods:

- Direct: Factorize A in a product of matrices such that computation of $\boldsymbol{x}$ is simplified and less expensive
- Iterative: Improve an approximation of $\boldsymbol{x}$ iteratively

Direct Solution Methods

Commonly used direct solution method: **LU factorization**

$$A = LU,$$

with L lower-triangular and U upper-triangular. Determine solution vector $\boldsymbol{x}$ by solving

$$L\boldsymbol{y} = \boldsymbol{f}, \quad U\boldsymbol{x} = \boldsymbol{y}.$$

LU decomposition is used often in solving power flow systems:

- Straightforward to work with and implement, standard in the power systems community
- Used for solving small SC and MC networks

Iterative Solvers for Linear Systems

Initial guess $\boldsymbol{x}_0$ and iteratively improve until iterate $\boldsymbol{x}_k$ is sufficiently close to exact solution $\boldsymbol{x}$.

The exact solution is unknown, so the residual vector is often used

$$\boldsymbol{r}_k = \boldsymbol{f} - A\boldsymbol{x}_k.$$

Basic iterative methods (BIMs):

- Based on relaxation steps
- Not often used on their own in practice

Krylov subspace methods

- Based on the Krylov subspace
- Commonly used in practice

Krylov Subspace Methods

Form solution approximations using the **Krylov subspace**

$$\boldsymbol{x}_k \in \boldsymbol{x}_0 + \mathcal{K}_k(A, \boldsymbol{r}_0)$$

$$\mathcal{K}_k(A, \boldsymbol{r}_0) = \text{span}\left\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{k-1}\boldsymbol{r}_0\right\}$$

Computing an orthogonal basis:

- Arnoldi's orthogonalization method
- Lanczos' bi-orthogonalization method

Steps general Krylov method:

1. Expand the Krylov subspace
2. Update the basis vectors
3. Form the projected matrix
4. Solve the reduced system
5. Update approximate solution
6. Compute residual
7. Stop if method converged

Generalized Minimum Residual (GMRES)

Find an orthogonal basis using Arnoldi's method

$$AV_k = V_k H_k + \boldsymbol{w}_k e_k^T$$

Basis matrix V_k: span $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} =$ span $\left\{\boldsymbol{v}_0, A\boldsymbol{v}_0, \ldots, A^{k-1}\boldsymbol{v}_0\right\}$
Hessenberg matrix H_k: How A acts inside $\mathcal{K}_k$

$$\text{Find } \boldsymbol{x}_k \text{ that minimizes } ||\boldsymbol{f} - A\boldsymbol{x}||$$

$$\downarrow$$

$$\text{Find } \boldsymbol{y}_k \text{ that minimizes } ||\beta\boldsymbol{e}_1 - \bar{H}_k\boldsymbol{y}_k||, \text{ and update } \boldsymbol{x}_k = \boldsymbol{x}_0 + V_k\boldsymbol{y}_k$$

Bi-Conjugate Gradient Stabilized (Bi-CGSTAB)

Find an orthogonal basis using Lanczos' bi-orthogonalization method

$$AV_k = V_k T_k + \delta_{k+1} \boldsymbol{v}_k e_k^T, \quad A^T W_k = W_k T_k^T + \beta_{k+1} \boldsymbol{w}_k e_k^T$$

Basis matrices V_k, W_k, Tri-diagonal matrix T_k

Find $\boldsymbol{x}_k$ that minimizes $||\boldsymbol{f} - A\boldsymbol{x}||$

$\downarrow$

Find $\boldsymbol{y}_k$ that minimizes $||\beta \boldsymbol{e}_1 - \bar{T}_k \boldsymbol{y}_k||$, using bi-orthogonality, and update

$$\boldsymbol{x}_k = \boldsymbol{x}_0 + V_k \boldsymbol{y}_k$$

Compute residuals of the form $\boldsymbol{r}_k = \Upsilon_k(A)\phi(A)\boldsymbol{r}_0$, stabilization polynomial $\Upsilon_k(t) = (1 - \omega_k t)\Upsilon_{k-1}(t)$.

Preconditioning

Transform the linear system into one that is easier to solve

- Non-singular matrix M
- Easy to solve $M\boldsymbol{x} = \boldsymbol{f}$

Left preconditioning:

$$M^{-1}A\boldsymbol{x} = M^{-1}\boldsymbol{f}$$

Right preconditioning:

$$AM^{-1}\boldsymbol{u} = \boldsymbol{f}, \ \boldsymbol{x} = M^{-1}\boldsymbol{u}$$

Preconditioned methods

$$\mathcal{K}_k(M^{-1}A, M^{-1}\boldsymbol{r}_0)$$
$$\mathcal{K}_k(AM^{-1}, \boldsymbol{r}_0)$$

Replace actions of A by actions of $M^{-1}A$ or AM^{-1}

Right preconditioning: Transformed variable is not directly computed

$$\boldsymbol{x}_k = \boldsymbol{x}_0 + M^{-1}V_k\boldsymbol{y}_k$$

Preconditioning Techniques

Incomplete LU
Partial LU decomposition

$$A = \tilde{L}\tilde{U} - R$$

LU factorization leads to fill-in

- ILU(0): Zero-pattern of A
- ILU(p): Allow p-level fill-ins
- Dropping tolerance: Drop fill-ins based on magnitude

Target matrix: Jacobians J_i, J_0 or FDLF matrix Φ

Additive and Multiplicative Schwarz
Jacobi and Gauss-Seidel

Divide into (overlapping) subdomains and compute the preconditioner application for each domain

$$M_{\mathrm{AS}}^{-1} = \sum R_i^T A_i^{-1} R_i$$

- Multiplicative: Use most recent information
- Additive: Update afterwards, parallel computations

04

Literature

State of the Art

- Ill-conditioned Jacobian matrices
- Choice of solution method
- Preconditioning
  - Incomplete LU
  - Additive Schwarz
- Domain decomposition in SC energy networks
- Solvers for multi-carrier energy networks
  - Decomposed methods
  - Integrated methods
- Partitioning methods

Ill-Conditioned Jacobian Matrices

Condition number of the Jacobian $\longrightarrow$ Convergence of Krylov solvers

Ill-conditioned Jacobian when:

General:

- Large networks

Electricity networks:

- High and extreme loading conditions
- Electrical properties: R/X ratio
- Topology: Radial

Choice of Solution Method

Direct methods ⟶ Scale poorly in problem size ⟶ Krylov solvers

GMRES

- Non-symmetric matrices
- Least number of matrix-vector calculations
- Free of problems that interrupt the iterative process
- Indicate whether other Krylov methods are more suitable

Bi-CGSTAB

- Variant of good-performing PCG

Preconditioning: ILU-GMRES

Roque et al. [8]:

- Electricity networks
- Effect of dropping tolerance
- Reordering

Nguyen, Romate, and Vuik [6]:

- Gas networks
- Conclusion: ILU not suitable

Idema et al. [2]:

- Electricity networks
- Reordering
- Target matrix J_0 preferable

Mori and Iizuka [5]:

- Electricity networks
- Less iterations required for ILU(2)

Preconditioning is problem-dependent

Various versions of ILU lead to customized well-performing preconditioners, but the search for the best parameter combination is complicated

Preconditioning: Additive Schwarz

Abhyankar, Smith, and Constantinescu [1]

- Very large power flow problems
- Overlapping restricted additive Schwarz preconditioner
- Speed-up compared to Jacobi preconditioner and direct method
- Speed-up saturation, slowdown for higher levels of overlap

$$M_{\mathrm{RAS}}^{-1} = \sum R_i^0 A_i^{-1} R_i^\delta$$

Domain Decomposition in SC Networks

Principle of divide and conquer

Kootte, Sereeter, and Vuik [3]: Coupling of transmission and distribution networks

- Unified vs splitting methods
- Splitting methods increased CPU time, especially for larger networks
- Recommendation: Use preconditioned Krylov methods for larger networks, parallelization

Zhang et al. [9]: Decoupling (looped) heat networks

- Radial systems lead to linear hydraulic equations
- Similar accuracy to unified methods
- Improving computational efficiency
- More robust against improper initialization

Solvers for Multi-Carrier Energy Networks

Focus on formulation and establishment of models, only a few studies focus on the solution methods [7].

Solving large integrated energy networks all at once remains an open challenge [6].

Solvers for MCN: Decoupled Approach

Markensteijn [4]:

- Solve single-carrier networks separately with direct inner solvers
- Solve coupled networks using interface conditions
- Decoupled formulation is equivalent to a permutation of the Jacobian of the coupled system

Zhang et al. [9]:

- Electro-thermal system, heat-load-following mode
- Solving the heating system, convert to electric power, solve the power system with direct inner solvers

Solvers for MCN: Integrated Approach

Small networks:

- Newton-Raphson, direct solver

Large networks, Qi, Li, and Bie [7]:

- Electro-thermal coupling system
- Inexact Newton, ILU preconditioned GMRES
- Significant acceleration effect on calculation speeds

Partitioning Methods

Markensteijn [4]:

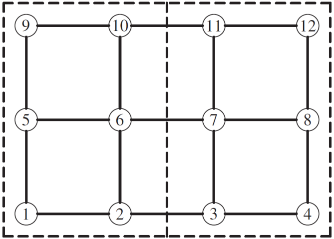
- Node-based partitioning
- Divide nodes based on carrier-type

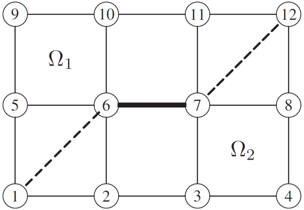
Zhang et al. [9]:

- Link-based partitioning
- Split links such that no loops occur

Abhyankar, Smith, and Constantinescu [1]:

- Multi-level graph partitioning algorithm





05

Open Problems and Research Scope

Open Problems and Research Scope

Research area: Fast and robust preconditioned Krylov methods for large MCN

- Investigate the effect of coupling on the spectral properties of the Jacobian matrix of a multi-carrier energy network.
- Partition based on carrier, using a non-unique permutation of the original Jacobian to create square blocks. Use a multiplicative/Gauss-Seidel method to solve the coupled system and apply ILU preconditioning to the blocks. What is the effect of different permutations?
- Apply an additive Schwarz preconditioner to a multi-carrier energy network. First investigate additive Schwarz preconditioning applied to gas (and electricity) networks.
- Investigate partitioning based on physics (carrier), graph structure and/or numerical effects and the effect on accuracy and efficiency.

Thank You for Your Attention

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References I

[1] Shrirang Abhyankar, Barry Smith, and Emil Constantinescu. "Evaluation of overlapping restricted additive Schwarz preconditioning for parallel solution of very large power flow problems". In: *Proceedings of the 3rd International Workshop on High Performance Computing, Networking and Analytics for the Power Grid*. Nov. 2013, pp. 1–8. ISBN: 9781450325103. DOI: 10.1145/2536780.2536784.

[2] Reijer Idema et al. "Scalable Newton-Krylov solver for very large power flow problems". In: *IEEE Transactions on Power Systems* 27.1 (Feb. 2012), pp. 390–396. ISSN: 08858950. DOI: 10.1109/TPWRS.2011.2165860.

[3] Marieke E. Kootte, B. Sereeter, and Cornelis Vuik. *Solving the Steady-State Power Flow Problem on Integrated Transmission-Distribution Networks: A Comparison of Numerical Methods*. IEEE, Oct. 2020. ISBN: 9781728171005. DOI: 10.1109/ISGT-Europe47291.2020.9248852.

[4] Anne S. Markensteijn. "Mathematical models for simulation and optimization of multi-carrier energy systems". PhD thesis. Delft University of Technology, 2021. DOI: 10.4233/uuid:044c083a-4eb1-4999-b789-db7150c4c7df.

[5] Hiroyuki Mori and Fumitaka Iizuka. "An ILU(p)-Preconditioner Bi-CGStab Methodfor Power Flow Calculation". In: *2007 IEEE Lausanne Power Tech*. IEEE, July 2007, p. 2235. ISBN: 9781424421909. DOI: 10.1109/PCT.2007.4538533.

References II

[6] Buu-Van Nguyen, Johan Romate, and Cornelis Vuik. "Performance of Krylov solvers for gas networks". In: *E-ENERGY 2025 - Proceedings of the 2025 16th ACM International Conference on Future and Sustainable Energy Systems*. Association for Computing Machinery, Inc, June 2025, pp. 489–494. ISBN: 9798400711251. DOI: 10.1145/3679240.3734600.

[7] Sen Qi, Gengfeng Li, and Zhaohong Bie. "Hybrid Energy Flow Calculation for Electric-Thermal Coupling System Based on Inexact Newton Method". In: *2019 IEEE Sustainable Power and Energy Conference (iSPEC)*. Beijing, China: IEEE, Nov. 2019. ISBN: 9781728149301. DOI: 10.1109/iSPEC48194.2019.8975152.

[8] Matheus M. Roque et al. "Numerical Investigation on the Solution of Ill-Conditioned Load Flow Linear Equations". In: *Journal of Control, Automation and Electrical Systems* 30.4 (Aug. 2019), pp. 580–588. ISSN: 21953899. DOI: 10.1007/s40313-019-00461-2.

[9] Suhan Zhang et al. "Partitional decoupling method for fast calculation of energy flow in a large-scale heat and electricity integrated energy system". In: *IEEE Transactions on Sustainable Energy* 12.1 (Jan. 2021), pp. 501–513. ISSN: 19493037. DOI: 10.1109/TSTE.2020.3008189.