Additive Schwarz Preconditioning in Modelling Multi-Carrier Energy Networks

Modelling and Simulating Large-Scale Multi-Carrier Energy Networks

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TU Delft

Energy Transition

NU.nl 21-04-2026

"Stroomnet vol in deel van provincie Utrecht: ook woningen komen op de wachtlijst"

NOS Nieuws 15-01-2026

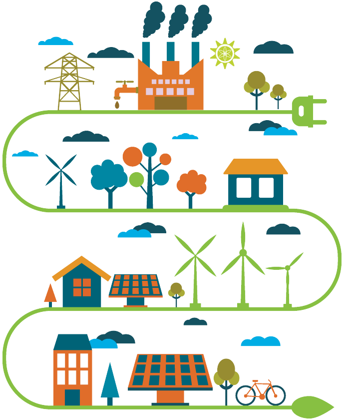
"Tekort aan duurzame energie kan Nederland miljarden euro's kosten"

TenneT 10-06-2026

"Stroomvoorziening in Nederland dreigt razendsnel in de knel te komen"

Presentation Outline

1. Background
2. Research Setup and Methodology
3. Results
4. Conclusion and Discussion



01

Background

Research Question

How does an additive Schwarz preconditioner perform when applied to a Krylov solver in the solution process of the steady-state load flow equations of large scale coupled gas-electricity energy networks?

Background:

1. Gas, electricity and coupled networks
2. Steady-state load flow analysis
3. Iterative Krylov solvers
4. Additive Schwarz Preconditioning

Energy Systems

Carriers of energy:

- Gas
- Electricity
- Multi-carrier

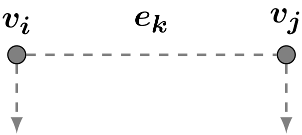


Figure: Energy network

Steady-state load flow equations

Flow of energy in a network

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$$

- $\boldsymbol{x}$: Variables
- $\boldsymbol{F}(\boldsymbol{x})$: Load flow equations

} Carrier-dependent

Conservation of energy:

$$\sum \text{energy in} = \sum \text{energy out}$$

Electricity Networks: Load Flow Equations

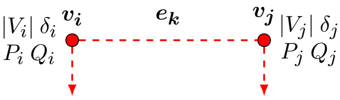


Figure: Electricity network

System variables:

- Voltage angle δ_i
- Voltage magnitude $|V_i|$
- Real power P_i
- Reactive power Q_i

Table: Electrical node types

Node type	Specified	Unknown
Load bus	P_i, Q_i	$\lvert V_i \rvert, \delta_i$
Generator bus	$P_i, \lvert V_i \rvert$	Q_i, δ_i
Slack bus	$\lvert V_i \rvert, \delta_i$	P_i, Q_i

$$\boldsymbol{F}^e(\boldsymbol{x}^e) = \begin{bmatrix} \boldsymbol{F}^P(\boldsymbol{x}^e) \\ \boldsymbol{F}^Q(\boldsymbol{x}^e) \end{bmatrix} \qquad \boldsymbol{x}^e = \begin{bmatrix} \boldsymbol{\delta} \\ |\boldsymbol{V}| \end{bmatrix}$$

$$\begin{cases} \boldsymbol{F}_i^P(\boldsymbol{x}^e) = P_i + |V_i|^2 G_{ii} + \sum_{j, j \neq i} |V_i||V_j| \left(G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij} \right) & \text{Real power} \\ \boldsymbol{F}_i^Q(\boldsymbol{x}^e) = Q_i - |V_i|^2 B_{ii} - \sum_{j, j \neq i} |V_i||V_j| \left(B_{ij} \cos \delta_{ij} - G_{ij} \sin \delta_{ij} \right) & \text{Reactive power} \end{cases}$$

Gas Networks: Load Flow Equations

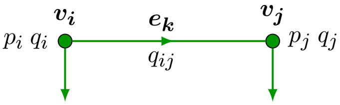


Figure: Gas network

System variables:

- Nodal pressure p_i
- Injected flow q_i
- Link flow q_{ij}

Table: Electrical node types

Node type	Specified	Unknown
Load node	q_i	p_i
Slack node	p_i	q_i

$$\boldsymbol{F}^g(\boldsymbol{x}^g) = \begin{bmatrix} \boldsymbol{F}^q(\boldsymbol{x}^g) \\ \boldsymbol{F}^L(\boldsymbol{x}^g) \end{bmatrix} \qquad \boldsymbol{x}^g = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}$$

$$\begin{cases} \boldsymbol{F}_i^q(\boldsymbol{x}^g) = \sum_{j,j\neq i} q_{ji} - \sum_{j,j\neq i} q_{ij} - q_i \\ \boldsymbol{F}_k^L(\boldsymbol{x}^g) = \Delta p_{ij} - (C^g)^{-2} f |q_{ij}| q_{ij} \end{cases}$$

Conservation of mass

Pipe flow

Coupled Networks: Load Flow Equations

Conversion units:

- Power-to-gas: Electricity $\rightarrow$ Gas
- Gas-fired generator: Gas $\rightarrow$ Electricity

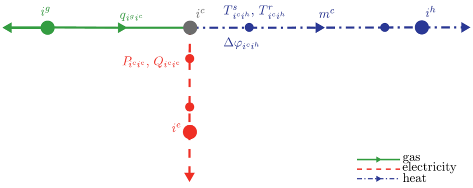


Figure: Multi-carrier energy network

- Coupling node: Coupling equations $\boldsymbol{F}^c(\boldsymbol{x}^c)$
- Dummy links: Coupling variables $\boldsymbol{x}^c$

Coupling variables:

- Real power: P_c
- Link flow: q_c

More coupling variables than equations

$\downarrow$

Additional node types (BCs) required

Table: Additional node types

Carrier	Node type	Specified	Unknown
Gas	Reference Load node	q_i, p_i	-
Electricity	PQV bus	$P_i, Q_i, \|V_i\|$	δ_i

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{F}^g(\boldsymbol{x}^g, \boldsymbol{x}^c) \\ \boldsymbol{F}^e(\boldsymbol{x}^e, \boldsymbol{x}^c) \\ \boldsymbol{F}^c(\boldsymbol{x}^c) \end{bmatrix} = \mathbf{0}, \ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^g \\ \boldsymbol{x}^e \\ \boldsymbol{x}^c \end{bmatrix}$$

Non-Linear Solution Method: Newton-Raphson

Non-linear load flow equations

$$\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$$

Iterative solution method:

1. Initial guess x_0
2. Compute next iterate x_1
3. Continue until $\boldsymbol{F}(\boldsymbol{x}_k) < \epsilon$

1^{st}-order multi-variate Taylor expansion around x_k

$$J_F(\boldsymbol{x}_k)\Delta\boldsymbol{x}_{k+1} = -\boldsymbol{F}(\boldsymbol{x}_k)$$
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta\boldsymbol{x}_{k+1}$$

Jacobian matrix:

$$J_F = \nabla\boldsymbol{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial x_1} & \cdots & \frac{\partial F_N}{\partial x_N} \end{bmatrix}$$

Linear Solution Methods

General linear system $A\boldsymbol{x} = \boldsymbol{f}$

Direct solution methods:

- LU factorisation
- Small systems
- Not scalable

Iterative Krylov subspace methods:

$$\boldsymbol{x}_k \in \boldsymbol{x}_0 + \mathcal{K}_k(A, \boldsymbol{r}_0)$$

$$\mathcal{K}_k(A, \boldsymbol{r}_0) = \text{span}\left\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{k-1}\boldsymbol{r}_0\right\}$$

Generalised minimum residual (GMRES) method

- Larger systems
- Scalable

Additive Schwarz Preconditioning

Transform the linear system:
Easier to solve + same solution

Preconditioning matrix M^{-1}

$$M^{-1}A\boldsymbol{x} = M^{-1}\boldsymbol{f}$$

Additive Schwarz preconditioner

1. Divide into subdomains
2. Compute local solutions
3. Combine local solutions
4. Compute global solution

Overlap-level δ: Propagation of information

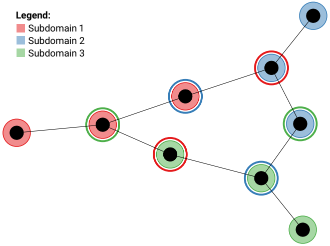


Figure: case9 network, 3 subdomains

Basic $$M_{\mathrm{AS}}^{-1} = \sum_i R_i^{\delta^T} A_i^{-1} R_i^{\delta}$$

Interpolated $$M_{\mathrm{IAS}}^{-1} = \sum_i R_i^{\delta^T} \boldsymbol{w}_i A_i^{-1} R_i^{\delta}$$

Restricted $$M_{\mathrm{RAS}}^{-1} = \sum_i R_i^{0^T} A_i^{-1} R_i^{\delta}$$

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Research Question

How does an additive Schwarz preconditioner perform when applied to a Krylov solver in the solution process of the steady-state load flow equations of large scale coupled gas-electricity energy networks?

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1. Gas, electricity and coupled networks
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4. Additive Schwarz Preconditioning

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Research Setup and Methodology

Methodology

Test network data:

- Gas
- Electricity
- Coupled: Gas + Electricity

Experiment setup:

- Non-linear solver: Newton-Raphson (NR)
- Linear solver: GMRES(m)
- Initialisation

Parameters:

- Overlap level
- Number of subdomains
- AS type
- Partitioning algorithm
- Network configuration
- Coupling

Implementation

1. **Partition the energy network graph**
2. **Include the links (gas only)**
3. Translate the graph partition to the Jacobian
4. Compute local updates
5. Construct the AS preconditioner

Link inclusion method:

- ~~Only outgoing links~~
- All adjacent links

Requirement: $\#\text{links} \geq \#\text{nodes}$

$\downarrow$

Non-singular Jacobian

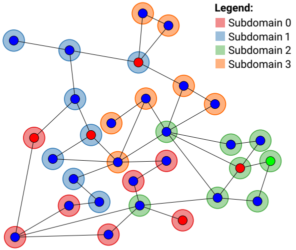

Figure: case30 network, ParMETIS

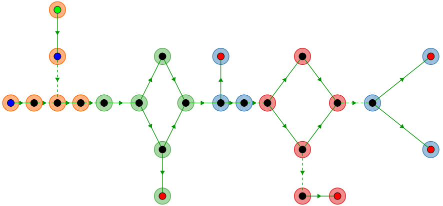

Figure: GasLib-24 network, ParMETIS

Implementation

1. Partition the energy network graph
2. Include the links (gas only)
3. **Translate the graph partition to the Jacobian**
4. Compute local updates
5. Construct the AS preconditioner

Carrier	Node/link type	Equation $\boldsymbol{F}$	Variable $\boldsymbol{x}$
Elec.	PQ node	$\boldsymbol{F}_i^P(\boldsymbol{x}^e),\ \boldsymbol{F}_i^Q(\boldsymbol{x}^e)$	$\lvert V_i \rvert,\ \delta_i$
	PV node	$\boldsymbol{F}_i^P(\boldsymbol{x}^e)$	δ_i
	Slack node	-	-
Gas	Load node	$\boldsymbol{F}_i^q(\boldsymbol{x}^g)$	p_i
	Slack node	-	-
	Link	$\boldsymbol{F}_k^L(\boldsymbol{x}^g)$	q_k

Include Jacobian element $J(\boldsymbol{x})_{ij} = \partial \boldsymbol{F}_i / \partial \boldsymbol{x}_j$, only when $\boldsymbol{F}_i$ and $\boldsymbol{x}_j$ are both in the subdomain

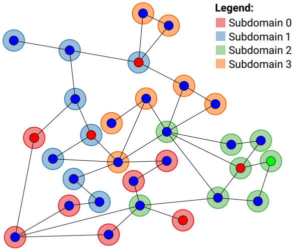

Figure: case30 network, ParMETIS

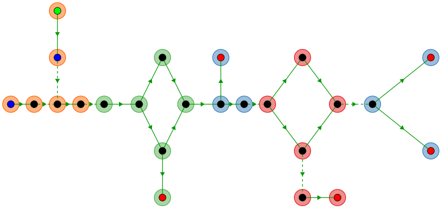

Figure: GasLib-24 network, ParMETIS

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Implementation

1. Partition the energy network graph
2. Include the links (gas only)
3. Translate the graph partition to the Jacobian
4. **Compute local updates**
5. **Construct the AS preconditioner**

Local update of subdomain i:

$$A_i \boldsymbol{x}_i = \boldsymbol{b}_i$$

Method:

- Direct solve, LU decomposition
- Alternatively: Incomplete LU, solve iteratively

AS construction: M_{AS}^{-1}

- Basic, interpolated and restricted types
- Compute the action on a vector $\boldsymbol{v}$

$$M_{\text{AS}}^{-1} = \sum_i R_i^{\delta^T} A_i^{-1} R_i^{\delta}$$

Algorithm: Action of M_{AS}^{-1} on $\boldsymbol{v}$

1: **Input:** $\boldsymbol{v}$
2: **for** $i = 1, \ldots, N$ **do**
3: $\quad \boldsymbol{z}_i = R_i^{\delta^T} A_i^{-1} R_i^{\delta} \boldsymbol{v}$
4: **end for**
5: $\boldsymbol{z} = \boldsymbol{z}_1 + \cdots + \boldsymbol{z}_N$
6: **Output:** $M_{AS}^{-1} \boldsymbol{v} = \boldsymbol{z}$

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Results

Gas Networks

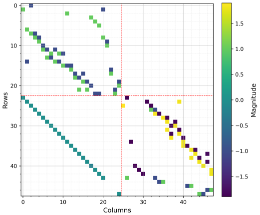

Figure: Structure, original

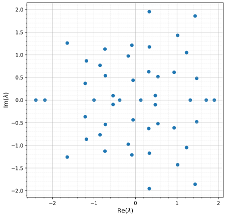

Figure: Eigenvalues, original

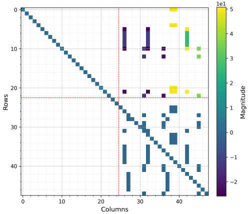

Figure: Structure, RAS

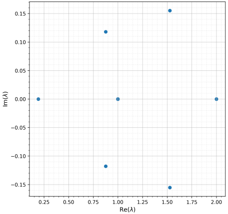

Figure: Eigenvalues, RAS

- Improved spread of eigenvalues and singular values
- Improved diagonal structure
- Less linear iterations than no preconditioner

Condition number:

- Original: $4.64 \cdot 10^2$
- RAS preconditioner: $6.46 \cdot 10^4$

Electricity Networks

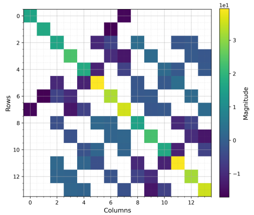

Figure: Structure, original

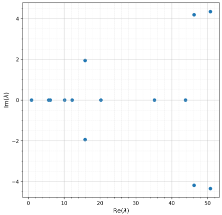

Figure: Eigenvalues, original

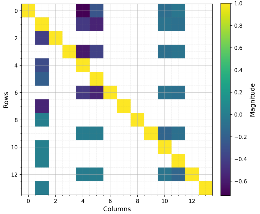

Figure: Structure, RAS

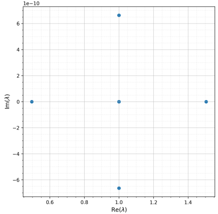

Figure: Eigenvalues, RAS

- Improved spread of eigenvalues and singular values
- Improved diagonal structure
- Less linear iterations than no preconditioner

Condition number:

- Original: $5.94 \cdot 10^1$
- RAS preconditioner: 6.10

Level of Overlap

Table: GasLib-40, RAS preconditioner, 4 subdomains

Overlap	Condition number	Linear iterations	Sub-system size
1-level	$1.16 \cdot 10^4$	54	$31.2\% - 40.5\%$
2-level	$6.97 \cdot 10^2$	55	$40.1\% - 53.6\%$
3-level	$9.88 \cdot 10^1$	42	$54.8\% - 69.0\%$
4-level	$5.77 \cdot 10^1$	43	$71.4\% - 82.1\%$

Table: case89pegase, RAS preconditioner, 5 subdomains

Overlap	Condition number	Linear iterations	Sub-system size
1-level	$3.13 \cdot 10^1$	93	30.9% - 49.1%
2-level	$1.84 \cdot 10^1$	62	52.7% - 79.4%
3-level	$1.05 \cdot 10^1$	52	76.4% - 97.6%

- Smaller condition number
- Less linear iterations
- Larger sub-system sizes
- Fewer restarts in GMRES(m)

Number of subdomains

Table: GasLib-40, RAS preconditioner, 2-level overlap

Subdomains	Condition number	Linear iterations	Sub-system size
2	$4.97 \cdot 10^2$	20	65.5%
4	$6.97 \cdot 10^2$	55	40.1% - 53.6%
6	$1.58 \cdot 10^3$	70	27.4% - 57.1%
14	$1.89 \cdot 10^3$	128	11.9% - 47.6%

Table: case89pegase, RAS preconditioner, 2-level overlap

Subdomains	Condition number	Linear iterations	Sub-system size
2	$1.04 \cdot 10^1$	36	90.3% - 95.2%
4	$2.39 \cdot 10^1$	63	71.5% - 79.4%
6	$1.73 \cdot 10^1$	66	39.4% - 73.3%
16	$7.35 \cdot 10^1$	104	24.8% - 75.2%

- Larger condition number
- More linear iterations
- Smaller sub-system sizes
- Larger range of sub-system sizes

Additive Schwarz Types

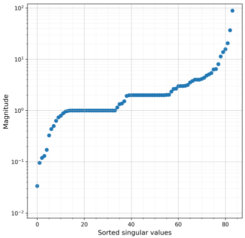

Figure: Basic AS

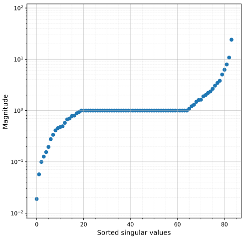

Figure: Interpolated AS

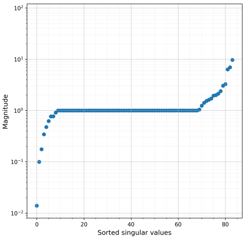

Figure: Restricted AS

Prefer Restricted AS:

- Singular value spread
- Condition number
- Linear iterations
- Error propagation

Table: GasLib-40 AS Types

AS Type	Condition number	Linear iterations
Basic	$2.63 \cdot 10^3$	92
Interpolated	$1.28 \cdot 10^3$	65
Restricted	$6.97 \cdot 10^2$	55

Table: case89pegase AS Types

AS Type	Condition number	Linear iterations
Basic	$4.55 \cdot 10^1$	95
Interpolated	$3.00 \cdot 10^1$	83
Restricted	$2.39 \cdot 10^1$	64

Reusing Previous RAS Preconditioner

RAS precond. at NR iteration k:

$$M_{\text{RAS},k}^{-1} = \sum_i R_i^{0\,T} J(\mathbf{x}_k)_i^{-1} R_i^\delta$$

Reuse RAS precond. of NR iteration j at NR iteration k, with $j < k$:

$$M_{\text{RAS},k}^{-1} = \sum_i R_i^{0\,T} J(\mathbf{x}_j)_i^{-1} R_i^\delta$$

- Cheaper
- Less accurate

Gas:

- Small j, much more linear iterations
- Large j, some more linear iterations
- No significant effect on final residual for $j > 0$

Electricity:

- Small effect linear iterations
- No significant effect on final residual

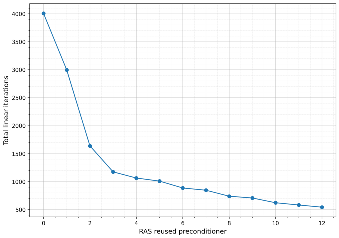

Figure: GasLib-4197

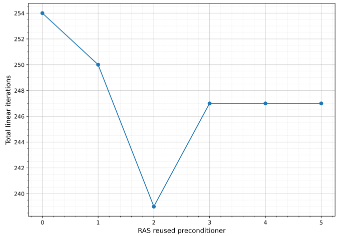

Figure: case9241pegase

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Coupled Energy Networks: Coupling

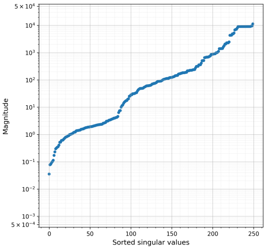


Figure: 1 P2G

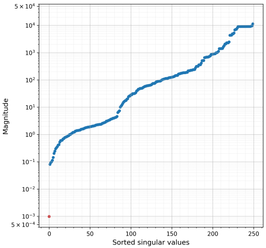


Figure: 1 GFG

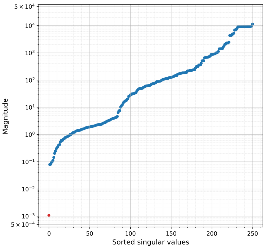


Figure: 1 P2G + 1 GFG

Power-to-gas: P2G

- PQV node

Gas-fired-generator: GFG

- Reference load node
- Stiff

Coupled Energy Networks: Coupling

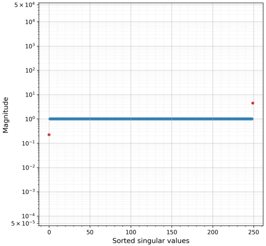


Figure: 1 P2G

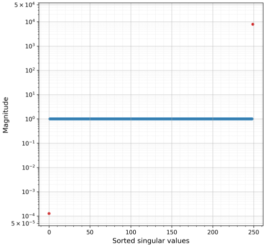


Figure: 1 GFG

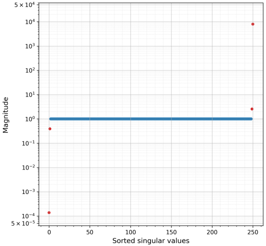


Figure: 1 P2G + 1 GFG

n coupling units

↓

$$\left.\begin{array}{l} n : \sigma_i > 1 \\ n : \sigma_i < 1 \end{array}\right\} \; 2n : \sigma_i \neq 1$$

$$\left.\begin{array}{l} \text{P2G: } \sigma_i \approx 10^{\pm\alpha} \\ \text{GFG: } \sigma_i \approx 10^{\pm\beta} \end{array}\right\} \; \alpha < \beta$$

Table: GasLib-4197 and case9241pegase, average linear iterations

P2G \ GFG	0	1	2	3	4
0	1.00	2.00	-	2.00	2.00
1	1.47	2.14	2.93	2.36	2.36
2	1.47	3.64	4.14	3.47	4.07
3	1.35	3.13	2.31	4.23	4.21

- One-sided interaction
- Two-sided interaction

Gas and Coupled Networks: Network Elements

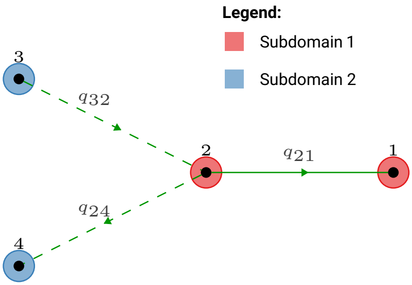


Figure: Example gas network

Boundary links:

- Compressors: $p_i = rp_j$
- Zero link flow: $\Delta p_{ij} = 0$
- Coupled: Compressor + reference load node

↓

Singular sub-system Jacobian

Propagation of boundary element errors: resistors

↓

Worsens conditioning

04

Conclusion and Discussion

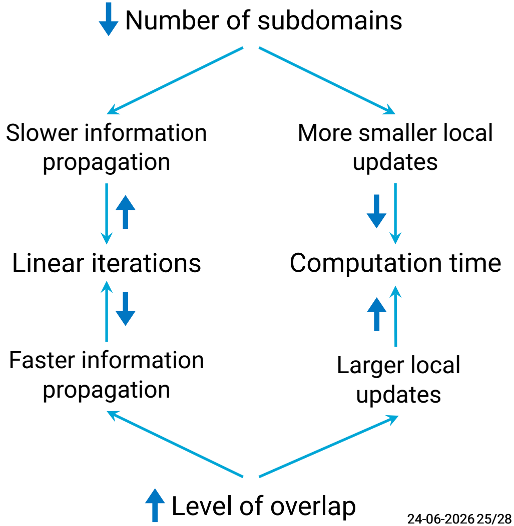
Conclusions: Network Elements & Partition

Gas network elements as boundary links:

- Compressors
- Zero link flow
- Ref. load node
- Resistors

Compressors, zero link flow, ref. load node: Singular sub-system Jacobian

Resistors: Worsens Jacobian conditioning



Conclusions: AS Types & Reusing RAS

AS Types

- Restricted AS
- Lowest number of linear iterations
- Prevent error propagation from boundary

Reusing a previous RAS preconditioner

- Gas:
  - Increase in linear iterations
  - Minimal effect on final residual with $M^{-1}_{\text{RAS},k}, k \geq 1$
  - Successful when construction of $M^{-1}_{\text{RAS},k}$ fails
- Electricity:
  - No increase in linear iterations
  - Minimal effect on final residual

Conclusions: Coupling

RAS preconditioned GMRES(m):

- Reference load node (GFG) $\rightarrow$ Worse conditioning
- n coupling units $\rightarrow 2n : \sigma_i \neq 1$, with $n : \sigma_i > 1$, and $n : \sigma_i < 1$.
- Increasing number of coupling units $n \rightarrow$ Increasing linear iterations.
- One-sided vs two-sided interaction

Limitations and Further Research

- Parallel implementation
- Test network data
- Partitioning
  - Single-carrier networks
  - Coupled networks
- Linear solver and stopping criteria
- Two-level AS preconditioning
- Modelling of energy networks

Thank You for Your Attention

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