Literature Review Report
on
Impact of static sea surface topography variations on ocean surface waves

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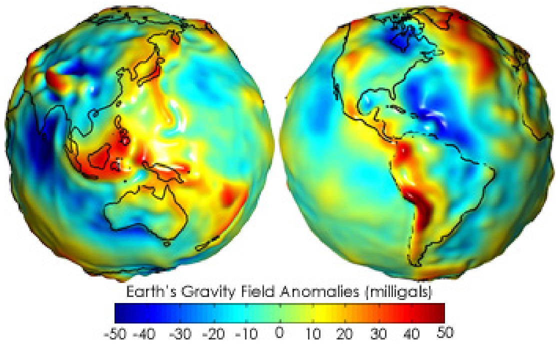
Contents

Chapter 1

Introduction

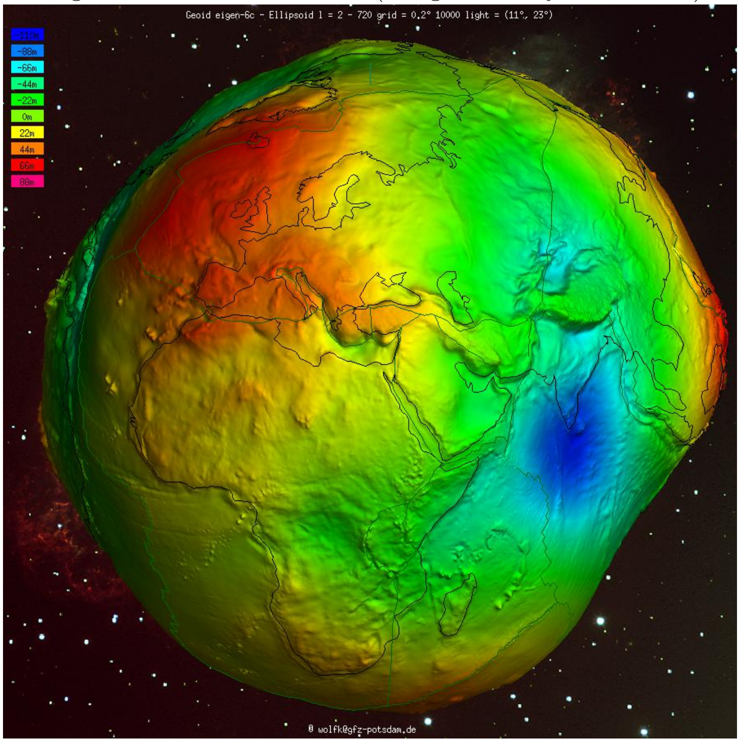
While it is generally assumed in various scientific and engineering applications that the gravity on Earth surface is uniform, which is usually characterised by with a standard gravity value $g_0 \approx 9.8067 m/s^2$, this is not the reality. Due to the uneven distribution of masses above and below Earth's crust, the gravitational attraction at every point on Earth is distinct. The quantity *gravity anomaly* is used to measure the discrepancy between the actual magnitude of gravity and the reference gravity g_0 on earth surface. Thanks to advanced observing satellites, the variation of gravity on Earth surface can nowadays be measured precisely. Figure 1.1 shows the measurement data of gravity anomaly by satellites. It is seen that over the Indian Ocean the gravity is significantly weaker than the reference gravity, while above the Ring of Fire in the Pacific Ocean the gravity is significantly stronger. The unit to measure gravity anomaly is usually *mgal* or *milligal*. 1 mgal is equivalent to $0.00001 m/s^2$. On the Earth surface the gravity anomaly typically ranges from -50 mgal to $+50$ mgal depending on the location on Earth. In this sense, the gravity field on Earth is *weakly non-uniform*.

Figure 1.1: Observed Gravity Anomaly (Image courtesy to NASA)



A collection of fluid is said to be in hydrostatic equilibrium when the pressure gradient is balanced entirely by external force, such as the gravity, so that the fluid flows with zero or steady velocity. The highly turbulent oceanic motions practically never satisfy the hydrostatic condition. However, the time-average of oceanic motions can be regarded as an approximation to the steady state of motion. In this sense, by measuring the time-average surface elevation of the ocean, an approximation to the hydrostatic ocean elevation, known as *mean sea level* can be determined. Due to the gravity variation, the mean sea level is also not a perfect sphere on Earth surface. The field measurement is shown in figure 1.2. The mean sea level is known as the *static sea surface topography.* Due to a weaker gravity, the mean sea level in the Indian Ocean is naturally also lower than the average and vice verse in the Ring of Fire.

Figure 1.2: Mean Sea Level (Image courtesy to ICGEM)



Most of the analytical and numerical models in geophysical fluid dynamics are based on the assumption that gravity is uniform. Hence it will be interesting to examine the effects of imposing a non-uniform gravitation field on fluid. In particular, in this project the behaviour of ocean surface wave in the weakly non-uniform gravitation field will be examined.

While the main focus of the project is the ocean surface wave, ranging from swells to tsunami waves, it should be highlighted that the analytical techniques developed in this project are more general and are very likely also be applicable for surface wave in fluids in other fields of scientific studies and industrial applications.

Chapter 2

Basis Terminology

2.1 Governing Equations

2.1.1 Equations of Motion in a Non-rotating Coordinate System

The governing equations for fluid are given by the continuum equations of motions.

Denote $\mathbf{x}$ to be the position vector. $\rho = \rho(\mathbf{x}, t)$ is the density of fluid, $\mathbf{u} = \frac{dx}{dt}(\mathbf{x}, t)$ is the velocity field, $p = p(\mathbf{x}, t)$ is the pressure field, $\Phi = \Phi(\mathbf{x}, t)$ is the potential for conservative force and $\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t)$ is the non-conservative forces.

Continuity Equation:

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

If the fluid is incompressible, so that the density remains constant over time, then $\frac{d\rho}{dt} = 0$ and gives the incompressible continuity equation:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

Momentum Equation:

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p - \rho\nabla\Phi + \mathbf{F} \tag{2.3}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{2.4}$$

is the total derivative following individual fluid elements.

2.1.2 Equations of Motion in a Rotating Coordinate System

The continuity equation remains invariant in both frames of reference. The momentum equation, however, is transformed into the following form:

$$\rho[\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega} \times \mathbf{u}] = -\nabla p - \rho\nabla(\Phi + \Phi_c) + \mathbf{F} \tag{2.5}$$

where $\boldsymbol{\Omega}$ is the uniform angular velocity of the rotating frame, Φ_c is the conservative force field associated to the centrifugal force due to the rotating frame.

2.2 Transformation of Coordinates

2.2.1 Notations

Commonly in practice, Cartesian coordinates (x, y, z) are used to study small to medium scale of geophysical fluid dynamics problem, where x, y, z are independent spatial coordinates.

For specific problems, it may happen that the use of an alternative coordinate system can simplify the analysis. In the following text, a specific transformation on the vertical-coordinates will be performed. A general description is given below.

Suppose r is a general vertical coordinate which is monotonic with z. Transformation from (x, y, z, t) to (x, y, r, t) makes $z = z(x, y, r, t)$ a dependent variable. Any function $F = F(x, y, z, t)$ can be rewritten in the new coordinates $F = F(x, y, z(x, y, r, t), t) = \tilde{F}(x, y, r, t)$. Applying the chain rule yields

$$\left.\frac{\partial \tilde{F}}{\partial x}\right|_r = \left.\frac{\partial F}{\partial x}\right|_z + \left.\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right|_r \tag{2.6}$$

The same principle also applies for $\frac{\partial \tilde{F}}{\partial y}|_r$ and $\frac{\partial \tilde{F}}{\partial t}|_r$, where the vertical bar means partial differentiation holding the suffix constant. Meanwhile

$$\frac{\partial \tilde{F}}{\partial r} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial r} \tag{2.7}$$

In the following text, if a variable, a scalar function, or a vector function is marked with a tilde, i.e. $\tilde{(\cdot)}$, it implicitly means the coordinates (x, y, r, t) is used to express it. Denote $\nabla_H F = (\frac{\partial F}{\partial x}|_z, \frac{\partial F}{\partial y}|_z)$ and $\tilde{\nabla}_H \tilde{F} = (\frac{\partial \tilde{F}}{\partial x}|_r, \frac{\partial \tilde{F}}{\partial y}|_r)$. Hence, based on the chain rule, the gradient operator in the basis of Cartesian unit vector with respect to (x, y, r, t) is obtained:

$$\begin{aligned}\nabla F &= (\nabla_H F, \frac{\partial F}{\partial z}) \\ &= (\tilde{\nabla}_H \tilde{F} - \frac{\partial \tilde{F}}{\partial r}\frac{\partial r}{\partial z}\tilde{\nabla}_H z, \frac{\partial \tilde{F}}{\partial r}\frac{\partial r}{\partial z})\end{aligned}$$

Note that $\frac{\partial \tilde{F}}{\partial r} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial r}$ (chain rule) and $\frac{\partial z}{\partial r} = \frac{1}{\frac{\partial r}{\partial z}}$ (reciprocals of partial derivatives). Similarly for any vector field $\mathbf{V} = (V_x, V_y, V_z)$ in Cartesian coordinates, $\mathbf{V}_H = (V_x, V_y)$ and $\nabla_H \cdot \mathbf{V}_H = \frac{\partial V_x}{\partial x}|_z + \frac{\partial V_y}{\partial y}|_z$ can be defined such that

$$\nabla \cdot \mathbf{V} = \nabla_H \cdot \mathbf{V}_H + \frac{\partial V_z}{\partial z}$$

2.2.2 Transformation of equations

First $\nabla_H F$ is related to $\tilde{\nabla}_H \tilde{F}$, based on the chain rule:

$$\nabla_H F = \tilde{\nabla}_H \tilde{F} - \frac{\partial F}{\partial z}\tilde{\nabla}_H(z) \tag{2.8}$$

Second $\nabla_H \cdot \mathbf{V}_H$ is related to $\tilde{\nabla}_H \cdot \tilde{\mathbf{V}}_H$, also based on chain rule:

$$\nabla_H \cdot \mathbf{V}_H = \tilde{\nabla}_H \cdot \tilde{\mathbf{V}}_H - \frac{\partial \mathbf{V}_H}{\partial z} \cdot \tilde{\nabla}_H(z) \tag{2.9}$$

In the new coordinate system, the vertical velocity $\dot{r} = \frac{dr}{dt}$ has to be handled with care. To represent $w = w(x, y, z, t)$ by $\tilde{w}(x, y, r, t)$, consider the total differential $d\tilde{w}$ and divide it by dt on $\tilde{w}(x, y, r, t)$, it follows that:

$$\begin{aligned} w(x, y, z, t) &= \tilde{w}(x, y, r, t) \\ &= \frac{\partial z}{\partial t}\bigg|_r + u\frac{\partial z}{\partial x}\bigg|_r + v\frac{\partial z}{\partial y}\bigg|_r + \dot{r}\frac{\partial z}{\partial r} \\ &= \frac{\partial z}{\partial t}\bigg|_r + \tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(z) + \frac{dr}{dt}\frac{\partial z}{\partial r} \end{aligned}$$

Therefore, the total derivative of $\tilde{F}$ with respect to the coordinates (x, y, r, t) is given by:

$$\begin{aligned} \frac{d\tilde{F}}{dt} &= \frac{dF}{dt} \\ &= \frac{\partial F}{\partial t}\bigg|_z + u\frac{\partial F}{\partial x}\bigg|_z + v\frac{\partial F}{\partial y}\bigg|_z + w\frac{\partial F}{\partial z} \\ &= (\frac{\partial \tilde{F}}{\partial t}\bigg|_r - \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}\bigg|_r) + u(\frac{\partial \tilde{F}}{\partial x}\bigg|_r - \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\bigg|_r) + v(\frac{\partial \tilde{F}}{\partial y}\bigg|_r - \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\bigg|_r) + w(\frac{\partial F}{\partial z}) \\ &= (\frac{\partial \tilde{F}}{\partial t}\bigg|_r + u\frac{\partial \tilde{F}}{\partial x}\bigg|_r + v\frac{\partial \tilde{F}}{\partial x}\bigg|_r) + (\tilde{w} - \frac{\partial z}{\partial t}\bigg|_r - u\frac{\partial z}{\partial x}\bigg|_r - v\frac{\partial z}{\partial y}\bigg|_r)\frac{\partial F}{\partial z} \\ &= (\frac{\partial \tilde{F}}{\partial t}\bigg|_r + \tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(\tilde{F})) + \frac{dr}{dt}\frac{\partial z}{\partial r}\frac{\partial F}{\partial z} \\ &= \frac{\partial \tilde{F}}{\partial t}\bigg|_r + \tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(\tilde{F}) + \dot{r}\frac{\partial \tilde{F}}{\partial r} \end{aligned}$$

Note that $\frac{\partial \tilde{F}}{\partial r} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial r}$ (chain rule). The partial derivative $\frac{\partial \tilde{w}}{\partial r}$ is given by:

$$\begin{aligned} \frac{\partial \tilde{w}}{\partial r} &= \frac{\partial}{\partial r}(\frac{dz}{dt}) \\ &= \frac{\partial}{\partial r}(\frac{\partial z}{\partial t}\bigg|_r) + \frac{\partial}{\partial r}(\tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(z)) + \frac{\partial}{\partial r}(\dot{r}\frac{\partial z}{\partial r}) \\ &= \frac{\partial}{\partial t}\bigg|_r(\frac{\partial z}{\partial r}) + \frac{\partial \tilde{\mathbf{u}_H}}{\partial r} \cdot \tilde{\nabla}_H(z) + \tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(\frac{\partial z}{\partial r}) + \frac{\partial \dot{r}}{\partial r}\frac{\partial z}{\partial r} + \dot{r}\frac{\partial}{\partial r}(\frac{\partial z}{\partial r}) \\ &= \frac{d}{dt}(\frac{\partial z}{\partial r}) + \frac{\partial \tilde{\mathbf{u}_H}}{\partial r} \cdot \tilde{\nabla}_H(z) + \frac{\partial \dot{r}}{\partial r}\frac{\partial z}{\partial r} \end{aligned}$$

Then the divergence of velocity field in (x, y, r, t) coordinates become:

$$\begin{aligned}
\nabla \cdot \mathbf{u} &= \nabla_H \cdot \mathbf{u}_H + \frac{\partial w}{\partial z} \\
&= \tilde{\nabla}_H \cdot \tilde{\mathbf{u}_H} - \frac{\partial \mathbf{u}_H}{\partial z} \cdot \tilde{\nabla}_H z + \frac{\partial \tilde{w}}{\partial r}\frac{\partial r}{\partial z} \\
&= \tilde{\nabla}_H \cdot \tilde{\mathbf{u}_H} - \frac{\partial \mathbf{u}_H}{\partial z} \cdot \tilde{\nabla}_H z + \big[\frac{d}{dt}(\frac{\partial z}{\partial r}) + \frac{\partial \tilde{\mathbf{u}_H}}{\partial r} \cdot \tilde{\nabla}_H(z) + \frac{\partial \dot{r}}{\partial r}\frac{\partial z}{\partial r}\big]\frac{\partial r}{\partial z} \\
&= \tilde{\nabla}_H \cdot \tilde{\mathbf{u}_H} + \frac{\partial r}{\partial z}\big[\frac{d}{dt}(\frac{\partial z}{\partial r})\big] + \frac{\partial \dot{r}}{\partial r} \\
&= \tilde{\nabla}_H \cdot \tilde{\mathbf{u}_H} + \frac{\partial \dot{r}}{\partial r} + \frac{d}{dt}\big[\ln(\frac{\partial z}{\partial r})\big]
\end{aligned}$$

Note that $\frac{\partial \tilde{\mathbf{u}_H}}{\partial r} = \frac{\partial \mathbf{u}_H}{\partial z}\frac{\partial z}{\partial r}$ (chain rule) and $\frac{\partial z}{\partial r} = \frac{1}{\frac{\partial r}{\partial z}}$ (reciprocals of partial derivatives) are used in the derivation.

Continuity Equation

It follows that the continuity equation in (x, y, r, t) coordinates becomes:

$$\frac{d\tilde{\rho}}{dt} + \tilde{\rho}\big[\tilde{\nabla}_H \cdot \tilde{\mathbf{u}_H} + \frac{\partial \dot{r}}{\partial r} + \frac{d}{dt}\big(\ln(\frac{\partial z}{\partial r})\big)\big] = 0 \tag{2.10}$$

Momentum Equation

The transformation of momentum equation, is unfortunately more complicated. Here the transformed momentum equation in a rotating coordinate frame will be derived. First consider the horizontal components of the momentum equations:

$$\rho[\frac{d\mathbf{u}_H}{dt} + \big(2\mathbf{\Omega} \times \mathbf{u}\big)_H] = -\nabla_H p - \rho\nabla_H(\Phi + \Phi_c) + \mathbf{F}$$

whereas

$$\nabla_H(p) = \tilde{\nabla}_H(\tilde{p}) - \frac{\partial \tilde{p}}{\partial r}\frac{\partial r}{\partial z}\tilde{\nabla}_H(z)$$

and

$$\nabla_H(\Phi + \Phi_c) = \tilde{\nabla}_H(\tilde{\Phi} + \tilde{\Phi_c}) - \frac{\partial(\tilde{\Phi} + \tilde{\Phi_c})}{\partial r}\frac{\partial r}{\partial z}\tilde{\nabla}_H(z)$$

Hence the horizontal components of the momentum equation is obtained:

$$\begin{aligned}
\tilde{\rho}[\frac{d\tilde{\mathbf{u}_H}}{dt} + \big(2\tilde{\mathbf{\Omega}} \times \tilde{\mathbf{u}}\big)_H] = -\tilde{\nabla}_H\tilde{p} - \tilde{\rho}\tilde{\nabla}_H(\tilde{\Phi} + \tilde{\Phi_c}) + \tilde{\mathbf{F}}|_H \\
+ \big[\frac{\partial \tilde{p}}{\partial r} + \tilde{\rho}\frac{\partial(\tilde{\Phi} + \tilde{\Phi_c})}{\partial r}\big]\frac{\partial r}{\partial z}\tilde{\nabla}_H(z)
\end{aligned} \tag{2.11}$$

For the vertical component, observe first that:

$$\frac{dr}{dt} = \frac{\partial r}{\partial z}\big[\tilde{w} - \frac{\partial z}{\partial t}\Big|_r - \tilde{\mathbf{u}_H} \cdot \tilde{\nabla}_H(z)\big] \tag{2.12}$$

In order to derive the momentum equation containing $\frac{d^2r}{dt^2}$, the total derivative operator should be applied once more to the above equation. The term $\frac{dw}{dt}$ will then be replaced with the aid of momentum equation. This is cumbersome and it will be shown that this equation is not needed in the shallow water approximation. Hence the exact form of $\frac{d^2r}{dt^2}$ will not be presented here.

Chapter 3

Gravity and Geopotential

3.1 Basic Definitions

3.1.1 Gravitational Potential and Force

The presence of mass always induces conservative gravitational force and hence potential. The attractive potential Φ_a due to mass density $\rho(x, y, z)$, is given by the Poisson's equation:

$$\nabla^2 \Phi_a = -4\pi G \rho \tag{3.1}$$

where G is the universal gravitational constant. In integral form, over an arbitrary volume ν containing mass, according to Newton's law of gravitational force, the potential can also rewritten as following:

$$\Phi_a(x, y, z) = G \int_\nu \frac{\rho(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} dx' dy' dz' \tag{3.2}$$

$d\nu = dx' dy' dz'$ is the element of volume.

In free space without mass, the gravitational potential is naturally given by the Laplace equation:

$$\nabla^2 \Phi_a = 0 \tag{3.3}$$

subject to suitable boundary conditions. Mathematically, a function satisfying the Laplace equation is called a harmonic function.

The negative gradient of Φ_a, denoted as $\vec{g_a} = -\nabla \Phi_a$, is the attractive gravitational force. It is noted that the negative sign indicates the gravity points from high potential to low potential.

3.1.2 Geopotential on a rotating body

In this section, a rotating body with fixed rotational axis and angular speed ω is considered.

In a rotating frame of reference, whose z-axis is aligned with the rotational axis, denote $d(x, y, z)$ to be the the shortest distance between the point $\vec{r} = (x, y, z)$ and the fixed rotational axis. Hence d is given by $d(x, y, z) = \sqrt{x^2 + y^2}$. Then define a 'centrifugal' potential Φ_c as follow:

$$\Phi_c = \frac{1}{2} \omega^2 d^2 \tag{3.4}$$

Φ_c does not satisfies the Laplace equation (3.3), which means it is not harmonic.

Taking the gradient of $-\Phi_c$ gives $-\nabla\Phi_c = -\omega^2 d\nabla(d)$. It is noted that $\nabla d = -\frac{(x,y,0)}{\sqrt{x^2+y^2}}$ is a unit vector in the centripetal direction of rotation. Therefore $\vec{g_c}(x, y, z) = -\nabla\Phi_c$ is the non-fictitious centrifugal force acting on a unit mass located at position (x, y, z) in the rotating frame of reference.

Assume that the Earth rotates with a fixed angular velocity. Take the origin of rotating frame as the approximate of the Earth centre, with z axis being aligned with the rotational axis. The Geopotential $\Phi(x, y, z)$ due to the rotating Earth is given by the sum of attractive potential and centrifugal potential:

$$\Phi = \Phi_a + \Phi_c \tag{3.5}$$

It is worthwhile to point out that the Earth needs not be a perfect sphere or ellipsoid in the above expression. Uneven distribution of mass, either due the non-uniform interior structures or topographical features on the Earth surface, is already taken into account in the determination of Φ_a. Although Φ_c is not harmonic, the centrifugal force is conservative since $\nabla \times g_c = -\nabla \times \nabla\Phi_c = \vec{0}$. Hence the Geopotential Φ is a well defined scalar potential. The negative gradient of the Geopotential is known as the *effective gravity* $\vec{g}$:

$$\vec{g} = -\nabla\Phi = \vec{g_a} + \vec{g_c} \tag{3.6}$$

and is what resting mass in rotating frame exactly 'feels'.

3.1.3 Mean Sea Level: A Specific Equipotential Surface

Since Geopotenial Φ is a well-defined scalar potential, equipotential surfaces can be formed by collecting points (x, y, z) such that $\Phi(x, y, z)$ equals a real number. On an equipotential surface, the negative gradient of the Geopotential, or the effective gravity $\vec{g}$, is always perpendicular to the surface.

The Mean Sea Level Potential is nothing but simply a specific scalar value Φ_0 to form an equipotential surface. The set of points (x, y, z) lying on this equipotential surface such that $\Phi(x, y, z) = \Phi_0$ coincides with the undisturbed sea surface, and thus is known as the *Mean Sea Level* or *Geoid*.

3.2 Gravity and Gravity Anomaly

In this section, several concepts related to gravity and gravity anomaly will be presented. It is worthwhile to point out that accurate determination of gravity is a broad scientific discipline in Geophysics. There is no unified definition to the term 'gravity anomaly'. Various treatments are used to measure the geometric shape of Earth and its gravitational field. In this report only the relatively simple and idealised definitions for 'gravity anomaly' will be presented. In the end, a definition that can be better justified in mathematical sense will be used in later sections.

The term 'anomaly' literally means quantities or qualities that deviate from the standard expectation. Hence it is necessary to define the 'normal' gravity first.

3.2.1 Reference Gravity

Consider the Geopotential Φ defined in equation (3.5). Φ is approximately decomposed into two parts: normal potential $\bar{\Phi}$ and disturbing potential Φ'.

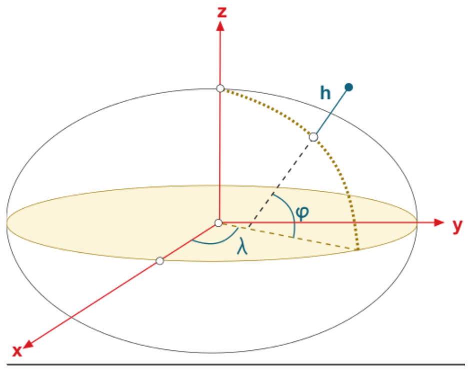
$$\Phi = \bar{\Phi} + \Phi' \tag{3.7}$$

The normal potential $\bar{\Phi}$ is defined as follow: The equipotential surface for $\bar{\Phi} = constant$ forms a ellipsoid. In particular, the equipotential surface for

$$\bar{\Phi} = \Phi_0 \tag{3.8}$$

is the best approximate for the Geoid in the sense of a least squares fit. Ellipsoidal coordinates (h, λ, φ) with reference to this ellipsoid are defined as in figure 3.1.

Figure 3.1: Ellipsoidal coordinates (Image courtesy of European Space Agency)



Over the ellipsoidal coordinates, the normal potential $\bar{\Phi}$ is independent of λ and thus:

$$\Phi(h, \lambda, \varphi) = \bar{\Phi}(h, \varphi) + \Phi'(h, \lambda, \varphi) \tag{3.9}$$

Denote, with the aid of the ellipsoidal coordinates, a function $h = N(\lambda, \varphi)$ such that the point $(N(\lambda, \varphi), \lambda, \varphi)$ lies on the Geoid, it then gives

$$\Phi(N(\lambda, \varphi), \lambda, \varphi) = \bar{\Phi}(0, \varphi) = \Phi_0 \tag{3.10}$$

It is noted that $N(\lambda, \varphi)$ is used in practice in the *Geodetic Reference System* in Global Positioning System(GPS).

With the aid of the ellipsoid, the normal gravity γ is defined as the negative gradient of $\bar{\Phi}$, that is,

$$\gamma = -\nabla\bar{\Phi} \tag{3.11}$$

$\gamma = \gamma(h, \varphi)$ is only dependent on the height h from the reference ellipsoid surface and latitude φ. Over the open ocean, the reference gravity γ_0 is given by simply choosing $h = 0$ in γ. Hence the reference gravity is simply a function of latitude φ by:

$$\gamma_0(\varphi) = -\nabla\bar{\Phi}(0, \varphi). \tag{3.12}$$

Finally, it is noted that over continents with solid mass above Geoid, the definition of reference gravity γ_0 has to be corrected due to topographical features.

However, since in this project the effects of gravity on ocean waves are studied, the land-based adjustments are not discussed. In addition, the normal potential $\bar{\Phi}$ already takes into account the centrifugal potential discussed in the previous section. The disturbing potential Φ' is entirely due to Earth's non-uniform distribution of mass in or on Earth.

3.2.2 Gravity Disturbance

As its name suggests, the gravity disturbance vector $\vec{\delta g}$ is given by the negative gradient of the disturbance potential Φ', that is:

$$\begin{aligned}\vec{\delta g} &= -\nabla\Phi' = -\nabla(\Phi - \bar{\Phi}) \\ &= \vec{g} - \vec{\gamma}\end{aligned}$$

However, it should be noted that $\vec{\delta g}(h, \lambda, \varphi) = \vec{g}(h, \lambda, \varphi) - \vec{\gamma}(h, \varphi)$ is dependent on all (h, λ, φ). Typically in Geophysics, the quantity 'gravity disturbance' δg is counter-intuitively NOT given by the magnitude of $\vec{\delta g}$, but the difference of magnitude between $-\nabla\Phi$ and $-\nabla\Phi'$, that is,

$$\delta g = |\vec{g}| - |\vec{\gamma}| \tag{3.13}$$

This is because the orientation of $\vec{g}$ in space could not be practically measured by gravimeters traditionally. However, such definition does not favour mathematical analysis since δg does not possess a nice mathematical property. On the other hands, $\vec{\delta g}$ is trivially curl-free. The disturbing potential Φ' associated with $\vec{\delta g}$ can also be shown to be harmonic. Therefore, in the following text, only the gravity disturbance vector $\vec{\delta g}$ will be considered.

3.2.3 Gravity Anomaly

Qualitatively speaking, Gravity Anomaly refers to the difference between effective gravity $\vec{g}$ at a certain location and the reference gravity $\vec{\gamma}$ defined in section 3.2.1.

There are multiple definitions to the term 'Gravity Anomaly' in Geodesy. It turns out that most definitions used by Geophysicists are not suitable for mathematical analysis. The common definitions include the Classical (cf. Hofmann-Wellenhof & Moritz, 2005), the Modern (cf. Molodensky et al., 1962) and the Topography-Reduced Gravity Anomaly (cf. Hofmann-Wellenhof & Moritz, 2005). Yet none of these consider the direction of gravities. Merely the magnitude of gravities were considered for practical reasons. In this section only the definitions of Classical and Modern Gravity Anomaly will be presented to illustrate the concepts.

Classical Gravity Anomaly

The classical gravity anomaly Δg_{cl} is simply given by the magnitude of effective gravity $\vec{g}$ on the Geoid minus the magnitude of reference gravity $\vec{\gamma}$ on the reference ellipsoid (3.8) at the same longitude λ and latitude φ in the reference ellipsoidal coordinates, that is:

$$\Delta g_{cl} = |\vec{g}(N(\lambda, \varphi), \lambda, \varphi)| - |\vec{\gamma}(h = 0, \varphi)| = |\vec{g}(N(\lambda, \varphi), \lambda, \varphi)| - |\vec{\gamma_0}(\varphi)| \tag{3.14}$$

It is highlighted again that classical gravity anomaly depends only on longitude λ and latitude φ but not on the height h from the surface of the ellipsoid. When the Earth surface is projected onto a Cartesian coordinates, the classical gravity anomaly g_{cl} is essentially a 2-dimensional scalar field and is determined by two spatial variables (x, y) on the projected horizontal Earth surface.

Modern Gravity Anomaly

The actual and general definition of Modern Gravity Anomaly Δg_m involves the consideration of actual topography and other concepts. However over open ocean the definition can be simplified into:

$$\Delta g_m = |\vec{g}(h, \lambda, \varphi)| - |\vec{\gamma}(h - N(\lambda, \varphi), \varphi)| \tag{3.15}$$

which is the magnitude of the effective gravity $\vec{g}$ at (h, λ, φ) minus the magnitude of normal gravity $\vec{\gamma}$ at a relative height $h - N(\lambda, \varphi)$ to mean sea-level at the same longitude λ and latitude φ. Hence the Modern Gravity Anomaly can be interpreted as the difference in gravity at (h, λ, φ) and mean sea-level. Modern Gravity Anomaly is essentially a scalar field that is determined by three spatial variables.

3.3 Approaches to Evaluate Geopotential

3.3.1 Traditional Approaches to Evaluate Gravity Anomaly

Over the past centuries, Geophysicists have developed several approximation schemes to obtain the correct gravity at particular points with respect to local topography. These correction methods include Latitude correction, Free-air correction, Bouguer correction and Terrain correction. These schemes were proposed to correct the magnitude of measured gravity and reduce the effect of above-surface geology. After these corrections, gravity anomaly due to inhomogeneous density material in crust or mantle can be obtained and can be applied for other purposes, such as surveying, positioning and so on.

In this report, the details of these correction methods will not be presented. Interested readers can refer to Turcotte and Schubert (2002) for detailed review and discussion of these methods.

A modern approach can be used to determined the Geopotential above Earth numerically, based on spherical harmonics. This will be presented in the next section.

3.3.2 Modern Approach to Compute Geopotential and Geoid

In section 3.1, the quantity Geopotential Φ is defined. It is also shown that Geopotential on Earth can be decomposed into two parts, the attractive potential Φ_a and the centrifugal potential Φ_c in equation (3.5). Recall that Φ_a actually satisfies the Poisson equation (3.1). In particular, in free space the right-hand side of the Poisson equation (3.1) vanishes and the equation reduces into the Laplace equation (3.3).

If spherical coordinates (r, λ, φ) are used, any solution satisfying the Laplace equation, particularly the attractive potential Φ_a, can be expanded and exactly represented by the spherical harmonics:

$$\Phi_a = \frac{GM}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R}{r}\right)^l P_{lm}(\sin\varphi)(C_{lm}^W \cos(m\lambda) + S_{lm}^W \sin(m\lambda)) \qquad (3.16)$$

The notations are given by

R - reference radius, usually an estimate of Earth's radius

GM - product of gravitational constant and mass of the Earth

l, m - degree, order of spherical harmonics

P_{lm} - normalised Lengendre functions

C_{lm}^W, S_{lm}^W - coefficients known as Stokes' coefficients

The coefficents C_{lm}^W and S_{lm}^W are determined numerically subject to boundary conditions, given by the on surface topography and Earth's density distribution. Hence, essentially determination of Φ_a through the Laplace equation (3.3) is a boundary-value problem.

In numerical models, it is impossible to consider infinitely many degrees l of spherical harmonics. Only the terms up to a certain integer l_{max} are considered. Hence the numerical solution of attractive potential Φ_a to the Laplace equation (3.3) is given by:

$$\Phi_a = \frac{GM}{r} \sum_{l=0}^{l_{max}} \sum_{m=0}^{l} \left(\frac{R}{r}\right)^l P_{lm}(\sin\varphi)(C_{lm}^W \cos(m\lambda) + S_{lm}^W \sin(m\lambda)) \qquad (3.17)$$

and the coefficients C_{lm}^W and S_{lm}^W are determined subject to boundary conditions in the same manner.

In this project, only the effect of gravity on ocean waves are studied, thus the exact procedures to set up the boundary conditions subject to topographical data, as well as the numerical schemes to carry out the computation will be omitted in this report. Interested readers may refer to Martinec (1998).

Suppose all coefficients C_{lm}^W and S_{lm}^W are known, equation (3.17) can be exactly differentiated. The attractive gravity $\vec{g_a} = -\nabla\Phi_a$ can be written out explicitly with a closed form mathematical expression. On the other hand, the centrifugal potential Φ_c can be represented exactly in spherical coordinates (r, λ, φ) by:

$$\Phi_c = \frac{1}{2}\omega^2 r^2 \cos^2\varphi \qquad (3.18)$$

Therefore, the Geopotential $\Phi = \Phi_a + \Phi_c$ (cf. equation (3.5)) can be expressed and differentiated exactly in closed form up to spherical harmonic order $l = l_{max}$. Every quantity defined in section 3.2 thus has a closed form mathematical expression in free space.

In this project, both the idealised and actual Geopotential will be considered. In the former case, the explicit expression of the Geopotential will be derived and directly applicable. Examples will be given in the section 'Adapted Models'.

To deal with the realistic Geopotential Φ on Earth, output from operational numerical models will be extracted. In both cases, the effective gravity $\vec{g} = -\nabla\Phi$ defined in section 3.1.2 and he gravity disturbance vector $\vec{\delta g}$ defined in section 3.2.2 will be used directly in later sections.

Chapter 4

Standard Shallow Water Model

Shallow Water Model is a basic yet powerful tool to study geophysical fluid dynamics, especially when it comes to large-scale motion or waves in fluid on a rotating planet. The 'shallowness' refers to the case that the horizontal length scale of motion is large relative to the vertical length scale, which then leads to systematic approximations and simplifications of the equations of motion. In this chapter the standard one-layer shallow water model will be introduced. Note that throughout the chapter, the gravity g is uniform and always points downwards in the z direction of Cartesian coordinates, when Cartesian coordinates systems are referred to.

4.1 Derivation of Shallow Water Equations

4.1.1 Characteristic Scale Analysis

Denote the D and L to be the length scale of vertical and horizontal motions of certain inviscid and incompressible fluid. The aspect ratio δ of the motion is given by:

$$\delta = \frac{D}{L} \tag{4.1}$$

If $\delta \ll 1$, the fluid motion is said to be *shallow*.

Continuity Equation

Denote U and W to be the characteristic scales of the horizontal and vertical velocity. The incompressible continuity equation (2.2), in Cartesian coordinates given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.2}$$

thus poses a constraint to the vertical motion:

$$W = \mathcal{O}(\delta U) \tag{4.3}$$

Hence, if the aspect ratio δ of the length scales is small, vertical motion within fluid is limited. In this case, the fluid motion can be approximated to be a two-dimensional flow which is independent of z coordinates.

Momentum Equation

Consider the general momentum equation in a rotating frame, given by (2.5).

It is first assumed that the pressure p of the fluid can be decomposed into a sum of hydrostatic pressure p_s and variation pressure p'. The hydrostatic pressure p_s refers to the pressure when the fluid remains at rest, so that the hydrostatic pressure balances with all external and velocity independent forces. In other words, hydrostatic pressure p_s does not alter the velocity of fluid flow. It is the variation pressure p' that regulates the velocity of fluid.

In Cartesian coordinates (x, y, z), denote u, v, w to be the velocity components in (x, y, z) direction and f to be the Coriolis frequency. It is additional assumed the *f-plane approximation* is used, so that the Coriolis frequency f is constant over the Cartesian plane. The momentum equation (2.5) is thus given by a system of partial differential equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = \frac{-1}{\rho}\frac{\partial p'}{\partial x} + \frac{-1}{\rho}\frac{\partial p_s}{\partial x} \tag{4.4a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = \frac{-1}{\rho}\frac{\partial p'}{\partial y} + \frac{-1}{\rho}\frac{\partial p_s}{\partial y} \tag{4.4b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p'}{\partial z} + \frac{-1}{\rho}\frac{\partial p_s}{\partial z} - g \tag{4.4c}$$

In order to obtain $\frac{\partial p_s}{\partial x}$, $\frac{\partial p_s}{\partial y}$ and $\frac{\partial p_s}{\partial x}$, put $u, v, w = 0$ in (4.4) and take $p' = 0$. The hydrostatic pressures under *hydrostatic balance* is thus given by:

$$\frac{\partial p_s}{\partial x} = 0 \tag{4.5a}$$

$$\frac{\partial p_s}{\partial y} = 0 \tag{4.5b}$$

$$\frac{\partial p_s}{\partial z} = -\rho g \tag{4.5c}$$

Note that there are no horizontal gradients of the hydrostatic pressure. In the vertical direction the gravity force $-\rho g$ is balanced by the pressure gradient $\frac{\partial p_s}{\partial z}$.

For motions on Earth, the Coriolis frequency f is given by the angular speed of Earth Ω and latitude ϕ via $f = 2\Omega\sin(\phi)$. Hence, the higher latitude the fluid flow, the stronger effects of Earth's rotation can it experience. The *Rossby Number* Ro, given by

$$\mathrm{Ro} = \frac{U}{fL} \tag{4.6}$$

is a dimensionless number. It is reminded here that U is the scale of the horizontal velocity and L is the length scale of the motion. The Rossby number measures the relative influence of rotation frame to the motion. The higher the Rossby number is, the weaker the Earth's rotation affects the fluid dynamics.

Under careful examinations of the characteristic length and velocity scales in (4.4), it can shown that, regardless of the Rossby number, $\frac{\partial p'}{\partial z}$ scales with at most $\mathcal{O}(\delta^2)$. Hence in the 'shallow' fluid with $\delta \ll 1$, $\frac{\partial p'}{\partial z}$ can be approximated to be 0. This also justifies the validity of decomposing the pressure into hydrostatic p_s and variation p' components. The detailed analysis can be found in section 3.3 of Pedlosky 1979.

4.1.2 Hydrostatic Assumption

The results of characteristic scale analysis discussed in the previous section revealed that the variation pressure p' is independent of vertical z coordinate. Hence it naturally follows that the vertical gradient of the pressure $\frac{\partial p}{\partial z}$ is given by the hydrostatic p_s component only:

$$\begin{aligned}\frac{\partial p}{\partial z} &= \frac{\partial p_s}{\partial z} + \frac{\partial p'}{\partial z} \\ &= -\rho g + 0\end{aligned}$$

Hence a handy formula for the vertical gradient of pressure $\frac{\partial p}{\partial z}$, which is known as the *hydrostatic assumption*, is obtained:

$$\frac{\partial p}{\partial z} = -\rho g \tag{4.7}$$

In the ocean, it is usually assumed the pressure p on the ocean surface is always the atmospheric pressure p_a, regardless of the surface elevation. Suppose the surface elevation of oceanic water at (x, y) at time t is given by $z = h(x, y, t)$, imposing the boundary condition $p(x, y, h) = p_a$ to (4.7) and solving it give the pressure p at (x, y, z):

$$p(x, y, z) = \rho g(h(x, y) - z) + p_a \tag{4.8}$$

The horizontal gradient of p naturally follows from (4.8) that:

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} \tag{4.9a}$$

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial h}{\partial y} \tag{4.9b}$$

These equations together with (4.5) will be plugged into the momentum equation (4.4) to derive the shallow water equation in section 4.1.5.

4.1.3 Depth-Averaged Velocity

In section 4.1.1, it is suggested that when the 'shallowness' condition $\delta \ll 1$ holds, the vertical velocity w of the fluid is almost negligible and thus the fluid motion can be viewed as atwo-dimensional flow.

This can be interpreted in two ways. The first is mathematically more straight-forward. The fluid flow is assumed to be purely two-dimensional, so that the horizontal velocities $u = u(x, y)$ and $v = v(x, y)$ are both independent of z coordinates. Then the vertical gradients of u and v, given by $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ respectively, therefore vanish.

Another interpretation is closer to the physical reality. In practice the velocities u and v in fluid are not completely independent of z due to presence of viscosity. However, by vertically averaging over the depth of fluid $H = H(x, y)$, averaged values of u and v along the vertical direction can be yielded and are denoted as $\bar{u}$ and $\bar{v}$:

$$\bar{u} = \frac{1}{H}\int_H u dz \tag{4.10a}$$

$$\bar{v} = \frac{1}{H}\int_H v dz \tag{4.10b}$$

The quantities $\bar{u}$ and $\bar{v}$ are known as *depth-averaged* velocities. The depth averaged quantities are, of course, independent of z. These can be plugged into the continuity equation when depth-averaging is performed, which will be presented in the next section. It turns out that the depth-averaged quantities will lead to the same set of equation of motions for the shallow fluid, known as shallow water equations. For mathematical simplicity, the first interpretation will be adopted in the coming sections.

4.1.4 Depth-Averaged Continuity Equation

Consider a single layer of fluid flowing over a rigid time-independent bottom floor given by $z = h_B(x, y)$. Suppose also that at time t, the surface elevation of the fluid at (x, y) is given by $z = h(x, y, t)$. Therefore the instantaneous depth of fluid $H = H(x, y, t)$ is given by $H = h - h_B$.

Perform integration over depth H to the incompressible equation (4.2). Note that u, v are independent of z, and so as $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$. It is also noticed that $\int_{h_B}^{h} \frac{\partial w}{\partial z} dz = w|_{(z=h)} - w|_{(z=h_B)}$:

$$\int_{h_B}^{h} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) dz = 0 \tag{4.11}$$

$$(h_B - h)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (w|_{(z=h)} - w|_{(z=h_B)}) = 0 \tag{4.12}$$

Impose the kinematic boundary condition on the fluid surface $z = h$ and no-normal flow condition at the rigid boundary $z = h_B$:

$$w|_{(z=h)} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \tag{4.13}$$

$$w|_{(z=h_B)} = u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} \tag{4.14}$$

After simplifications the *depth-averaged continuity equation* is obtained:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0 \tag{4.15}$$

Equation (4.15) is known as the equation corresponding to the conservation of mass in the Shallow Water Model.

4.1.5 Shallow Water Equations

First collect different assumptions discussed in previous sections to simplify the momentum equation. Recall that u and v are independent of z. Substituting equations (4.5) and (4.9) into the original momentum equation (4.4) gives:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} \tag{4.16a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} \tag{4.16b}$$

These two equations are known as the the momentum equation of the *Shallow Water Model*.

Denote the velocity vector $\vec{v_H} = (u, v, 0)$ and the Coriolis factor $\vec{f} = (0, 0, f)$ in Cartesian coordinates, and $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ is used for the notation of the total derivative operator. Together with the conservation of mass (4.15), the *Shallow Water Model* can be summarised into:

$$\frac{dH}{dt} + H\nabla \cdot (\vec{v_H}) = 0 \tag{4.17a}$$

$$\frac{d\vec{v_H}}{dt} + \vec{f} \times \vec{v_H} = -g\nabla(h) \tag{4.17b}$$

4.1.6 Potential Vorticity in the Shallow Water

Consider the vorticity $\vec{\omega} = \nabla \times \vec{v_H}$ in the Shallow Water Model. Since $\vec{v_H}$ lies on the horizontal plane, the vorticity vector $\vec{\omega}$ only contains z-component. Denote the z component to be ω.

By taking the curl operator $\nabla\times$ to the momentum equation (4.17b), followed by a series of vector identities, it can be shown that the following equation is equivalent to momentum equation (4.17b).

$$\frac{d\omega}{dt} + (\omega + f)\nabla \cdot (\vec{v_H}) = 0 \tag{4.18}$$

The detailed derivation of above equation can be found in any introductory geophysical fluid dynamics textbook. Interested readers may, for example, refer to Marshall 2008.

On the other hand, manipulating the mass conservation equation (4.17a) gives

$$-(\omega + f)\nabla \cdot (\vec{v_H}) = \frac{\omega + f}{H}\frac{dH}{dt} \tag{4.19}$$

Merging equation (4.18) and (4.19), and noticing that the Coriolis factor f is independent of time t so that $\frac{d\omega}{dt} = \frac{d(\omega+f)}{dt}$, it turns out that:

$$\begin{aligned} \frac{d(\omega + f)}{dt} &= \frac{\omega + f}{H}\frac{dH}{dt} \\ \frac{1}{H}\frac{d(\omega + f)}{dt} - \frac{(\omega + f)}{H^2}\frac{dH}{dt} &= 0 \\ \frac{d}{dt}(\frac{\omega + f}{H}) &= 0 \end{aligned}$$

Define $q = \frac{\omega+f}{H}$ to be the *potential vorticity*. Then it follows that the potential vorticity q remains invariant along a fluid element in the shallow water model:

$$\frac{d}{dt}(q) = 0 \tag{4.20}$$

Hence the potential vorticity q is a conservative quantity in the shallow water model and equation (4.20) is known as the conservation of potential vorticity.

The potential vorticity q plays a significant role in understanding large-scale geophysical fluid flow. Interested readers may refer to Pedlosky 1979 for a detailed discussion.

4.2 Derivation of Shallow Water Waves

Waves refer to the small perturbation from certain reference states of motion. In the case of shallow water wave, the 'perturbation' mainly deals with the surface elevation relative to the reference level of surface elevation $h = h(x, y, t)$.

Decompose the surface elevation $h(x, y, t)$ into a sum of mean elevation $h_0(x, y)$ and perturbed elevation $\eta(x, y, t)$. The mean elevation $h_0(x, y)$ refers to the surface elevation when the fluid is under hydrostatic balance, so that the reference velocity $\vec{v_H}_0 = \vec{0}$. Hence the velocity $\vec{v_H}$ consists purely of the perturbed velocity $\vec{v_H}' = (u', v', 0)$.

When the gravity is uniform, the free surface of fluid lies on a flat plane and thus $h_0(x, y) = h_0$ is constant over the Cartesian spatial coordinates (x, y).

$$h(x, y, t) = h_0 + \eta(x, y, t) \tag{4.21}$$

$$\vec{v_H}(x, y, t) = \vec{0} + \vec{v_H}'(x, y, t) \tag{4.22}$$

Since $\vec{v_H}$ consists of only the perturbed terms, the advection term $\vec{v_H} \cdot \nabla \vec{v_H}$ is a second-order perturbation term and is thus negligible compared with $\frac{\partial \vec{v_H}}{\partial t}$. Hence the shallow water equations (4.17) can be linearised, by keeping only the first-order perturbed term, which gives:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[\vec{v_H}'(h_0 - h_B(x, y))\right] = 0 \tag{4.23a}$$

$$\frac{\partial \vec{v_H}'}{\partial t} + \vec{f} \times \vec{v_H'} = -g\nabla(\eta) \tag{4.23b}$$

Note that the term $H_0(x, y) = h_0 - h_B(x, y)$ actually represents the mean depth of fluid column at (x, y). Define the mass flux $\vec{V}$ as $\vec{V} = \vec{v_H}' H_0$. Then after rearranging terms the above two equations are expressed concisely by:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \vec{V} = 0 \tag{4.24a}$$

$$\frac{\partial \vec{V}}{\partial t} + \vec{f} \times \vec{V} = -gH_0\nabla(\eta) \tag{4.24b}$$

Partially differentiating the first equation with respect to t and taking the gradient of the second, after some manipulations, the two equations can be reduced into a partial differential equation only for the surface elevation η. The equation is known as the *shallow water wave equation* in a rotating frame:

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta - \nabla \cdot (gH_0\nabla\eta)\right] - gfJ(H_0, \eta) = 0 \tag{4.25}$$

where $J(H_0, \eta) = \frac{\partial H_0}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial H_0}{\partial y}\frac{\partial \eta}{\partial x}$.

The detailed manipulations of terms in the derivations can be found from Pedlosky 1979.

4.2.1 Surface Gravity Wave

In the remaining part of this chapter, the mean depth of water H_0 will be assumed to be constant to illustrate the key concepts. Hence the term $gfJ(H_0, \eta)$ in equation (4.25) vanishes.

Consider a simplified case when the Coriolis factor $\vec{f} = \vec{0}$. This can be interpreted as the motions in a non-rotating frame or motions near the equator of the Earth, where $f = 2\Omega \sin(\phi)$ is approximately 0. It is reminded here that Ω is the angular velocity of rotating Earth and ϕ is the latitude.

The shallow water wave equation thus reduces into the standard second-order gravity wave equation:

$$\frac{\partial^2 \eta}{\partial t^2} - gH_0 \nabla^2 \eta = 0 \tag{4.26}$$

The dispersion relation in one-dimensional flow is thus given by

$$\omega^2 = gH_0 k^2 \tag{4.27}$$

Hence the group velocity is given by $c_g = c = \sqrt{gH_0}$. The interpretation of the dispersion relation and group velocity can be found in Chapter 5.

4.2.2 Poincare Wave

The Poincare Wave is also known the *Inertio-gravity* waves because of both the effects from Earth's rotation f and gravity g.

The assumption that the mean depth of water H_0 is constant over spatial coordinates (x, y) remains. Hence equation (4.25) reduces into

$$\frac{\partial}{\partial t}\left[(\frac{\partial^2}{\partial t^2} + f^2)\eta - (gH_0 \nabla^2 \eta) \right] = 0 \tag{4.28}$$

The dispersion relation can be shown as:

$$\omega^2 = c^2 (\frac{f^2}{c^2} + k^2) \tag{4.29}$$

so that the group velocity $c_g = \frac{kc^2}{\sqrt{f^2 + c^2 k^2}}$, where $c = \sqrt{gH_0}$ is the group velocity of a standard gravity wave without any rotation.

In the *short wave limit*, in which $\frac{f^2}{c^2} \ll k^2$, the dispersion relation of Poincare wave reduces into the dispersion of a standard surface gravity wave (4.27). This is the scenario where the wavelength of wave is too small to experience the Earth's rotational effects.

In the *long wave limit*, in which $k^2 \ll \frac{f^2}{c^2}$, the dispersion relation of Poincare wave is given by

$$\omega^2 = f^2 \tag{4.30}$$

Hence the Poincare waves are generated with fixed frequency regardless of the initial mechanisms of wave generation. Such a wave exhibits a special form which is known as *inertial oscillations*. Detailed explanation can be found in Pedlosky 1979.

Chapter 5

Wave Phenomenon in Single Layered Fluid

In this chapter, basic wave kinematics will be presented. Due to variation in water depth, waves in fluid exhibit several universal wave characteristics. Only linear waves will be considered in this chapter. The assumption of 'shallowness' of the fluid layer in shallow water model is usually considered.

5.1 Wave Kinematics: Ray Theory

Wave is defined as a propagating disturbance of physical quantities relative to an equilibrium state. Under the presence of wave, energy can be transferred without actual transport of mass.

The mathematically simplest wave is known as plane wave $\eta(x,t)$, which in one spatial dimension is given by:

$$\eta(x,t) = \mathrm{Re}(a\exp[i(kx-\omega t)]) \tag{5.1}$$

where a, k, ω are constants and are known as wave amplitude, wave number and angular frequency the wave respectively. $\mathrm{Re}(z)$ is the real part of a complex number z. Since the real part is always considered, the notation is omitted from now on. The phase of the plane wave is denoted as $S(x,t) = kx - \omega t$. While keeping the phase $S(x,t)$ constant, if k is positive, it can seen that plane wave η propagates in the direction of positive x-direction with phase speed $c_p = \frac{k}{\omega}$.

The plane wave (5.1) can be generalised into

$$\eta(x,t) = a(x,t)\exp(iS(x,t)) \tag{5.2}$$

such that the wave amplitude $a(x,t)$ is dependent on both spatial and temporal coordinates, and the phase $S(x,t)$ is a function of (x,t). If the amplitude $a(x,t)$ varies slowly relative to the phase function $S(x,t)$, such wave is called nearly-plane wave. In this case, the wave number $k(x,t)$ and angular frequency ω is defined as:

$$k(x,t) = \frac{dS}{dx} \tag{5.3}$$

$$\omega(x,t) = -\frac{dS}{dt} \tag{5.4}$$

The conservation of crest equation immediately follows from the definitions:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{5.5}$$

Consider the propagation of constant phase S, by setting the total differential $dS = 0$, the velocity at which a constant phase propagates, called the phase velocity c_p, is given by

$$c_p = \frac{\omega}{k} \tag{5.6}$$

Furthermore, denote the properties of the medium on which the wave travels via a parameter $\lambda(x,t)$. Such properties in fluid waves can be the density of fluid, the depth of water, gravity, etc. A *dispersion relation* between the angular frequency $\omega(x,t)$, wave vector $k(x,t)$ and medium properties $\lambda(x,t)$ can be obtained as the following form via some differentiable function σ:

$$\omega(x,t) = \sigma(k(x,t); \lambda(x,t)) \tag{5.7}$$

Denote the group velocity c_g by $c_g = \frac{\partial \sigma}{\partial k}$. Then the equation for conservation of crest (5.5) can be rewritten in the form of an advection equation of k:

$$\frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = -\frac{\partial \sigma}{\partial \lambda}\frac{\partial \lambda}{\partial x} \tag{5.8}$$

By differentiating the dispersion relation (5.7) with respect to t and applying the conservation of crest (5.5), it can also be shown that the advection of ω satisfies

$$\frac{\partial \omega}{\partial t} + c_g \frac{\partial k}{\partial x} = \frac{\partial \sigma}{\partial \lambda}\frac{\partial \lambda}{\partial t} \tag{5.9}$$

Hence both k and ω advect via the characteristic curve $x = r(t)$ given by:

$$\frac{dr}{dt} = c_g = \frac{\partial \sigma}{\partial k} \tag{5.10}$$

These characteristic curves $r(t)$ are known as the wave rays. Along wave rays, the quantities wave number k and angular frequency ω varies based on the right-hand sides of equation (5.8) and (5.9). Hence, equipped with initial and boundary conditions as well as additional information on the variation of the wave amplitude a along the wave rays, the full wave can be constructed over the full space-time domain.

The higher dimension generalisation to the Ray theory can be found in any introductory wave dynamics texts. Yet the principles involved are entirely the same.

5.2 Airy Linear Wave Theory

Airy Linear Wave Theory is based on the two-dimensional potential flow theory, namely the horizontal velocity $u = \frac{dx}{dx}$ and vertical velocity $w = \frac{dz}{dt}$, can be given by a scalar velocity potential ϕ by $u = \frac{\partial \phi}{\partial x}$ and $w = \frac{\partial \phi}{\partial z}$.

Under the assumptions of incompressible and irrotational fluid, the momentum equations in the Navier-Stokes equations can be replaced by the Bernoulli

Equation. Together with the original continuity equation, the new set of equations can be linearised into the following form:

Continuity Equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{5.11}$$

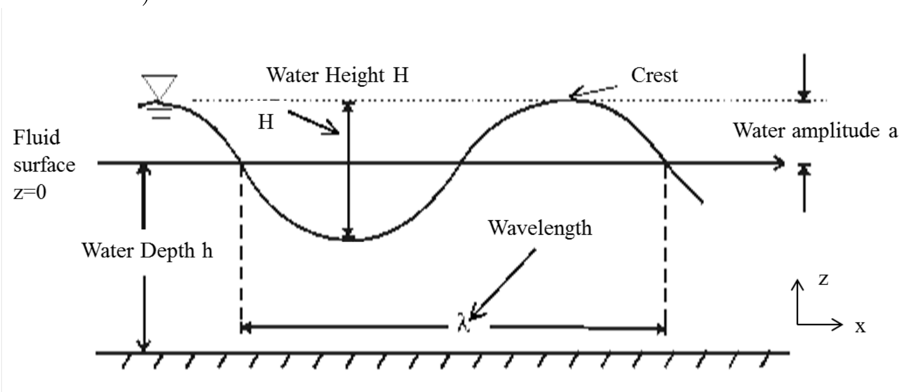
Momentum Equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\|\nabla \phi\|^2 + \frac{P}{\rho} + gz = 0 \tag{5.12}$$

where $\|\cdot\|$ is the l^2 norm, P is the pressure, ρ is the density and g is the gravity.

Consider a single layer with uniform density flowing on a gently sloped impermeable bottom $z = -h(x)$ as shown in figure 5.1.

Figure 5.1: Physical setting for Airy Wave Theory (Image courtesy of web.mit.edu)



Imposing kinematic and dynamic boundary conditions on the free surface $z = \eta(x,t)$, and kinematic boundary condition at bottom $z = -h(x)$ yields three extra equations:

$$\frac{d\eta}{dt} = \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta(x,t)} \tag{5.13}$$

$$P|_{z=\eta(x,t)} = 0 \tag{5.14}$$

and

$$\frac{dh}{dt} = \left.\frac{\partial \phi}{\partial z}\right|_{z=-h(x)} \tag{5.15}$$

The momentum equations can be further linearised:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + gz = 0 \tag{5.16}$$

These equations can be used to study the growth and propagation of small amplitude perturbations on the surface of a one-layered fluid. If the perturbation is in a form of a plane wave, with its amplitude a much smaller relative to the wavelength $\lambda = \frac{2\pi}{k}$, it can shown that the solutions to the linearised momentum equation (5.16) satisfying the boundary conditions are:

$$\phi(x,z,t) = A\cosh(k(z+h))\exp[i(kx-\omega t)] \tag{5.17}$$

and correspondingly

$$\eta(x,t) = (\frac{i\omega}{g})A\cosh(kh)\exp[i(kx-\omega t)] \tag{5.18}$$

where A is some constant determined by initial conditions. It can be shown over a gentle and time independent bottom slope $z = -h(x)$, the dispersion relation is given by:

$$\omega^2 = gk\tanh(kh) \tag{5.19}$$

The detailed derivations can be found in any introductory fluid mechanics texts.

Given the water depth h, the dispersion relation restricts the angular frequency and wave number.

Two other important quantities, phase velocity c_p and group velocity c_g for this monochromatic wave are given by

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)} \tag{5.20}$$

$$c_g = \frac{d\omega}{dk} = \frac{1}{2}c_p\sqrt{1 + kh\frac{1-\tanh^2(kh)}{\tanh(kh)}} \tag{5.21}$$

The phase velocity c_p measures the speed at which a phase in the wave traveled in space and the group velocity c_g measures the propagation speed of the overall wave form and energy of the wave.

It is worthwhile to note that in shallow water, that is in the limit of $kh \ll 1$,

$$\omega^2 = ghk^2 \tag{5.22}$$

so that

$$c_p = c_g = \sqrt{gh} \tag{5.23}$$

Hence the wave propagates with speed proportional to the the square root of water depth h in shallow water. In this sense, shallow water wave 'feels' the bottom.

Whereas in the limit of $kh \gg 1$, that is in the so-called 'deep water',

$$c_p = 2c_g = \sqrt{\frac{g}{k}} \tag{5.24}$$

and are both independent of water depth h. The deep water wave cannot 'feel' the bottom and propagates with speed only dependent on its frequency ω, which is determined at the instance when the wave was generated.

Furthermore, it can also be shown that the mean energy per unit horizontal area transported by linear wave $\langle E \rangle$ can be given by:

$$\langle E \rangle = \langle \int_{-h}^{0} \rho(\frac{1}{2}u^2 + gz)dz \rangle = \frac{1}{2}\rho g a^2 \tag{5.25}$$

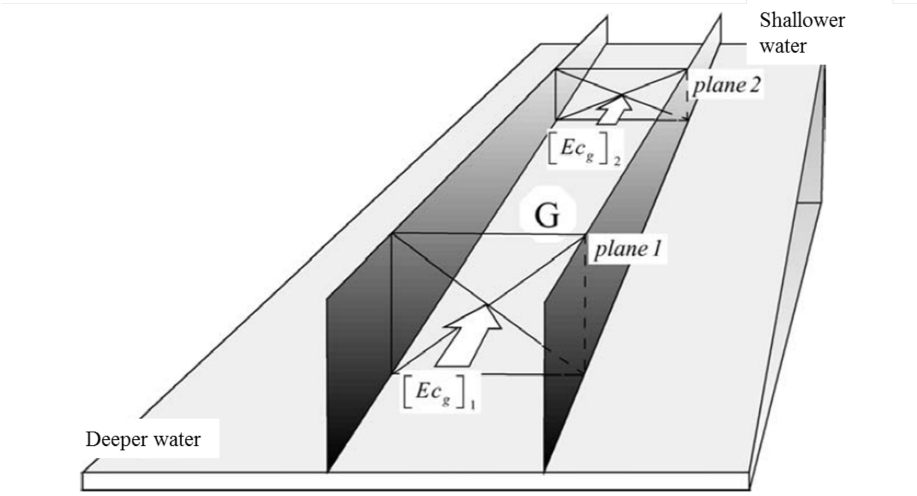
where $\langle \cdot \rangle$ is the average over time. In other words, waves energy, including both the kinetic and potential energy, is proportional to the square of wave amplitude a^2. The conservation of energy plays a crucial role in the shoaling process, which will be introduced in the next section.

5.3 Shoaling

In the previous section, the dispersion relation for a plane wave in (single-layered) fluid is presented. In this section, the dispersion relation and the energy equation will be applied to understand the phenomenon called 'shoaling'.

Shoaling is the process of growing wave amplitude when shallow wave travels from a deep zone to shallow zone. To illustrate the principle, consider a simplified case when a shallow water wave travels from open ocean to the coast in figure 5.2.

Figure 5.2: Control volume G (Image courtesy of Holthuijsen 2007)



Over a control volume G of fluid aligned in the direction of the wave (shown by the arrow in the above figure). Assume there is no dissipation of energy when a wave front travels from plane 1 to plane 2, the energy flux passing though plane 1 has to balance that passing from plane 2.

Setting up the balance equation over a infinitesimal control volume it can be derived that (c.f. Holthuijsen 2007), with a being the wave amplitude:

$$\frac{d}{dx}\big(c_g E\big) = \frac{d}{dx}\big[c_g(\frac{1}{2}\rho g a^2)\big] = 0 \tag{5.26}$$

In particular, over a small control volume G it can be approximated that

$$a_2 = \sqrt{\frac{c_{g_1}}{c_{g_2}}}a_1 \tag{5.27}$$

where a_1 and a_2 are the wave amplitudes at position 1 and 2 in the figure 5.2 respectively. Due to the 'shallowness', the group speed of the wave is given by (5.23). Since the water depth at position 1 is larger than that in position 2, c_{g_1} is also larger than c_{g_2}. Therefore, equation (5.27) tells a_2 is greater than a_1.

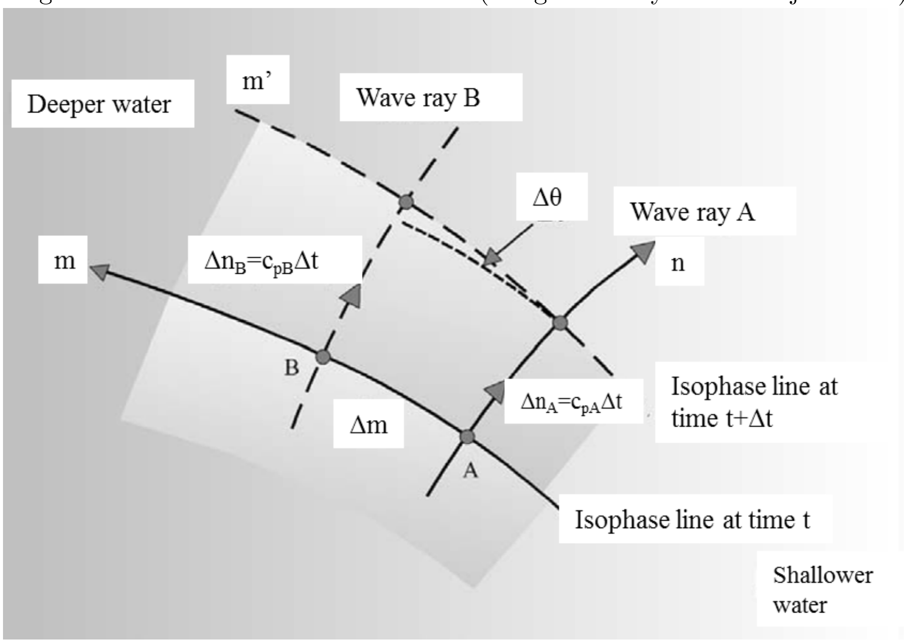
Practically, the wave amplitude does not grow infinitely when oceanic wave approaches the coast. Another process, known as wave breaking, starts to dissipate energy when the wave amplitude is sufficiently large. The detailed mechanisms of wave breaking is currently still poorly-understood and will not be studied in this project.

5.4 Refraction

The refraction phenomenon is not a unique feature for a wave in a fluid. Refraction occurs in all kinds of mechanical and electromagnetic waves, which is the characteristics that wave always deflected towards the region where it can travel with lower propagation velocity.

To illustrate the principle, consider the a shallow water wave front m (=a line collecting points with same phase) travelling in a direction that is not parallel to the water depth contour. A sketch is given in figure 5.3. Note that the greyness in the figure refers to the relative depth of water.

Figure 5.3: Refraction in shallow water (Image courtesy of Holthuijsen 2007)



The solid line m is the wave front at time t. Since the water depth at point A is lower than that in point B, the wave front at point A travels at a lower phase speed than that at point B. After a short time interval Δt, the wave front m at A and B respectively travels along the direction of wave ray via n_A and n_B. The wave front at time $t + \Delta t$ hence becomes m'. The new wave front is therefore deflected with $\Delta\theta$ towards the shallow region. Under the assumption that there is no underlying current, consider an infinitesimal time interval Δt, it can be derived that the rate of deflection is characterised by:

$$\frac{d\theta}{dn} = -\frac{1}{c_p}\frac{\partial c_p}{\partial m} \tag{5.28}$$

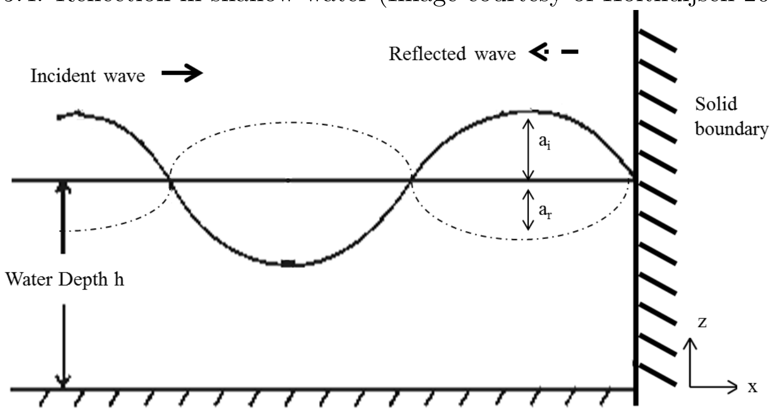
Refraction of a surface gravity wave due to the variation in water depth does not happen in deep water. The phase speed c_p in deep water depends only on the angular frequency ω but not water depth h. However, the wave does not have to be 'shallow', as long as the deep water approximation does not hold so that the full dispersion relation (5.19) is valid, refraction takes place.

5.5 Reflection

Same as refraction, reflection is also a universal characteristic of waves. However, unlike the refraction, dispersion relation does not play important role in the reflection phenomenon.

When wave propagates and reaches some solid boundaries, without sufficiently large wave energy to break the solid, the wave cannot penetrate the solid. The wave energy originally propagating in the direction orthogonal to the boundary surface reflects fully or partially in the opposite direction. Such phenomenon is known as reflection. To illustrate the principle, consider the 1-dimensional water surface wave over uniform bottom as in figure 5.4.

Figure 5.4: Reflection in shallow water (Image courtesy of Holthuijsen 2007)



Denote a_i and a_r to be the wave amplitude of the incident and the reflected wave. Mathematically, by imposing the boundary condition that the fluid remains at rest at the solid boundary, it can be shown when an incident monochromatic wave $\eta_i(x,t) = a_i cos(\omega t - kx)$ hits a boundary, the reflected monochromatic wave $\eta_r(x,t) = a_r cos(\omega t + kx)$ is induced to satisfy the boundary condition. Hence the surface elevation $\eta(x,t)$ is given by the sum of η_i and η_r and can be simplified into:

$$\eta = \eta_i + \eta_r = (a_i - a_r) sin(\omega t - kx) + 2a_r \cos(kx)\sin(\omega t) \tag{5.29}$$

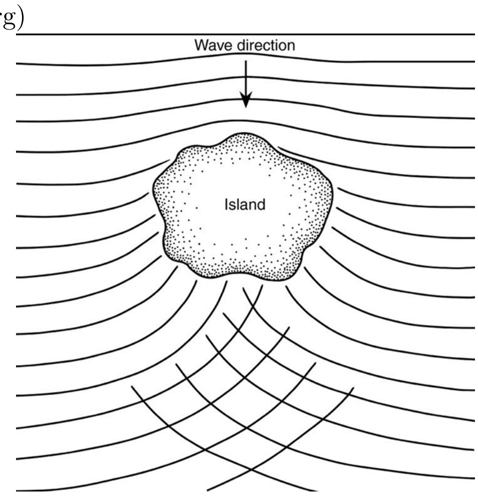
In particular, if the wave energy is perfectly reflected so that $a_r = a_i$, the surface elevation $\eta(x,t) = 2a_i \cos(kx)\sin(\omega t)$ is a standing wave. In practice, the ratio of a_r to a_i depends on the material at the boundary and has to be empirically measured.

When the water depth h is non-uniform, in shallow water the wave number $k = k(x)$ becomes dependent on the spatial coordinate x. Analysis of the mathematical behaviour of reflection in such a case becomes nontrivial, yet numerical simulation is still possible by imposing proper boundary conditions at the solid boundary inside the computational domain. The details of boundary conditions can be found in Leveque 2002.

5.6 Diffraction

Similar to reflection, diffraction is a universal wave characteristic that occurs when waves reach an obstacle in the fluid medium. Diffraction refers to the phenomenon that wave rays deflect into the region without wave rays at diminishing wave amplitudes. Diffraction is particularly crucial when wave passes slits or above-water obstacles. An illustration of diffraction is given in figure 5.5.

Figure 5.5: Diffraction in open ocean due to above water island (Image courtesy of jeb.biologists.org)



The phenomenon can be explained physically based on Huygens-Fresnel principle, which says every points on a wave front is the source of a spherical wave.

Mathematical analysis of diffraction is nontrivial. Unlike refraction and reflection, diffraction is typically not analyzed via the dispersion relation nor the linearised momentum equation. Instead, diffraction is better described in the frequency-space based equation, known as Helmholtz equation, under suitable boundary conditions. In this project the Helmholtz equation will likely not be studied, hence the detailed treatment of the Helmholtz equation and diffraction phenomenon will not be presented here. Interested readers may refer to the textbook Leblond and Mysak 1981 for detailed discussion.

Chapter 6

Adapted Model

In this chapter, the adapted model will be presented. A adapted wave equation for ocean surface wave defined on geopotential height will be presented. The inertial frame without non-conservative force will be investigated first.

6.1 Basic Definitions

Define Mean-Sea Level Set Msl: It contains a the points (x, y, z) such that

$$\Phi(x, y, z) = \Phi_0 \tag{6.1}$$

where Φ_0 is an empirically found value. In other words, $Msl = \{(x, y, z) : \Phi(x, y, z) = \Phi_0\}$. Given (x, y), there is unique $z_0 = z_0(x, y)$ such that (x, y, z_0) belongs to Msl. The work done required to transport a unit mass from any point to MSl to any arbitrary point (x, y, z) is known as the potential difference between (x, y, z) and MSl, and is given by:

$$\Psi(x, y, z) = -\int_{(\bar{x},\bar{y},\bar{z})}^{(x,y,z)} -\nabla\Phi \cdot d\vec{l} \tag{6.2}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is any point on Msl. Since Φ is a conservative field, the above integral is path independent. Hence an equivalent definition can be given by:

$$\Psi(x, y, z) = \int_{z_0(x,y)}^{z} \frac{\partial\Phi}{\partial z}(x, y, s)ds \tag{6.3}$$

which takes a straightly vertical path integral, or,

$$\Psi(x, y, z) = \Phi(x, y, z) - \Phi_0 \tag{6.4}$$

which is essentially the potential difference.

Define the Geopotential height Z as following:

$$Z = Z(x, y, z) = \frac{\Psi(x, y, z)}{g_0} \tag{6.5}$$

where g_0 is a constant reference gravity. Note that Ψ takes into account only the geopotential induced by masses which are time-independent. The geopotential induced by ocean water is neglected. Z has the same physical dimension with physical height z. Given any (x, y), the mapping from Z to z is monotonic and one-to-one.

The geopotential height Z will serve as r in the chapter 2.2 to transform the equations.

6.2 Properties of Geopotential Height

A very nice property of Geopotential Height Z comes from the fact that $\Phi = g_0 Z + \Phi_0$ (c.f. equation (6.4)), so that a very handy vertical gradient is given by:

$$\frac{\partial \Phi}{\partial Z} = g_0 \tag{6.6}$$

which is the reference gravity independent of any spatial or temporal coordinates.

Furthermore, assume that the gravity $\vec{g}$ is *weakly non-uniform*: $\vec{g}$ consists of a uniform component $-g_0\hat{k}$ and a perturbed non-linear conservative component $\vec{g'}(x, y, z) = -\nabla\Phi'(x, y, z)$, such that $\|\vec{g'}\| \ll g_0$. that is:

$$\vec{g}(x, y, z) = -g_0\hat{k} + \vec{g'}(x, y, z) \tag{6.7}$$

It follows that, fixing (x, y), the potential difference Ψ at (x, y, z) and mean sea level is given by

$$\begin{aligned}\Psi(x, y, z) &= -\int_{z_0(x,y)}^{z} [-g_0 + \vec{g'}(x, y, z)]ds \\ &= g_0(z - z_0) + \int_{z_0(x,y)}^{z} \nabla\Phi'(x, y, s)ds\end{aligned}$$

$$\Psi(x, y, z) = g_0(z - z_0(x, y)) + [\Phi'(x, y, z) - \Phi'(x, y, z_0(x, y))] \tag{6.8}$$

where $z = z_0(x, y)$ is the mean sea level at horizontal location (x, y). The term $g_0(z - z_0)$ can be attributed to the geopotential Φ induced by a the reference ellipsoid discussed in section 3.2.1, while $\Phi'(x, y, z)$ can be interpreted as the disturbing geopotential Φ' discussed in section 3.2.2 due to local topographical features.

It follows that the geopotential height Z becomes:

$$\begin{aligned}Z &= \frac{\Psi}{g_0} \\ &= z + \frac{\Phi'(x, y, z)}{g_0} - (z_0(x, y) + \frac{\Phi'(x, y, z_0)}{g_0})\end{aligned}$$

Denote $Z_0(x, y) = z_0(x, y) + \frac{\Phi'(x,y,z_0)}{g_0}$, which is independent of z, the transformation from (x, y, z) to (x, y, Z) can be given by

$$Z = z + \frac{\Phi'(x, y, z)}{g_0} - Z_0(x, y) \tag{6.9}$$

Now the partial derivative of Z with respect to z (keeping x, y constant) is considered, note that $Z_0(x, y)$ is independent of z. Denote $\frac{\partial \Phi'}{\partial z}$ by g'_z.

ATTENTION: Instead of defining $g'_z = -\frac{\partial \Phi'}{\partial z}$, which is the typical definition for conservative potential and force, here the definition is given by $-g'_z = -\frac{\partial \Phi'}{\partial z}$. The motivation is that by defining g'_z in this way, the effective gravity $\vec{g}$ in z direction is given by $-(g_0 + g'_z)$, so that when g'_z is positive, the magnitude of gravity in z-direction is $g_0 + g'_z > g_0$.

$$\begin{aligned}\frac{\partial Z}{\partial z} &= 1 + \frac{1}{g_0}\frac{\partial \Phi'}{\partial z} \\ &= 1 + \frac{g'_z}{g_0}\end{aligned}$$

The partial derivative of Z with respect to z

$$\frac{\partial Z}{\partial z} = 1 + \frac{g'_z}{g_0} \tag{6.10}$$

will be the most crucial ingredient in the later derivation of equations and analysis. It is noted that, since $Z = Z(x, y, z)$, since Z is a smooth function, according to the reciprocal rule of partial derivative,

$$\frac{\partial z}{\partial Z} = \frac{1}{\frac{\partial Z}{\partial z}} = \frac{1}{1 + \frac{g'_z}{g_0}} \tag{6.11}$$

is also valid.

6.3 Definition of Water Depth

6.3.1 Classical case

Recall that mean sea level is known as a specific equipotential surface for the geopotential. When there is no gravity anomaly, the hydrostatic balance will give rise to a horizontal ocean surface topography. The bathymetry (water depth) $d(x, y)$ at a point (x, y) on a projected horizontal plane, is defined by the vertical distance between the hydrostatic surface elevation of the ocean and the bottom sea floor.

Now first consider a horizontal sea floor. The bathymetry $d(x, y)$ is trivially constant everywhere. See figure 6.1 for graphical illustration.

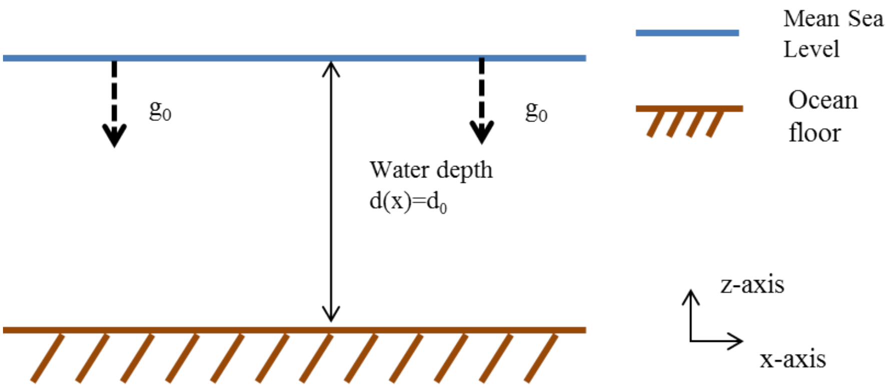
6.3.2 Adapted case

Bathymetry

When there is a gravity anomaly due to excess/deficit mass, the hydrostatic pressure gradient will still be balanced by the (conservative) gravity force. However, this will give rise to a non-flat equipotential surface, hence a non-uniform ocean surface topography.

In the same case of a horizontal sea floor, the bathymetry $d(x, y)$ is no longer constant but is spatially dependent. In figure 6.2 an idealised example is

Figure 6.1: Horizontal sea floor



presented. In this example, a spherical excess mass of radius R of density ρ_1, which is larger than the density of sea floor crust ρ_0 is placed underneath the ocean floor at depth R. For simplicity the origin of coordinate system is set at the intersection of the ocean floor and the excess mass.

Let $z = z_0(x, y)$ be the z coordinate of the mean sea level, that is, the set of points $\{(x, y, z_0(x, y))\}$ lies on the hydrostatic ocean surface. It naturally follows that the bathymetry $d(x, y)$ at (x, y) is given by $d(x, y) = z_0(x, y) - 0$.

Z transformation

Now consider that the potential induced by the excess mass. The gravitational potential Φ' at (x, y, z) due to the excess mass is defined by the work done needed to move a unit mass from infinity to (x, y, z). Note that in this case, Φ' is exactly the disturbing geopotential defined in section 3.2.2. Denote G to be the universal gravitational constant, $\Delta M = (\rho_1 - \rho_0)(\frac{4}{3}\pi R^3)$ to be the excess mass relative to the crust. The disturbing gravitational potential $\Phi'(x, y, z)$ induced by the excess mass is given by

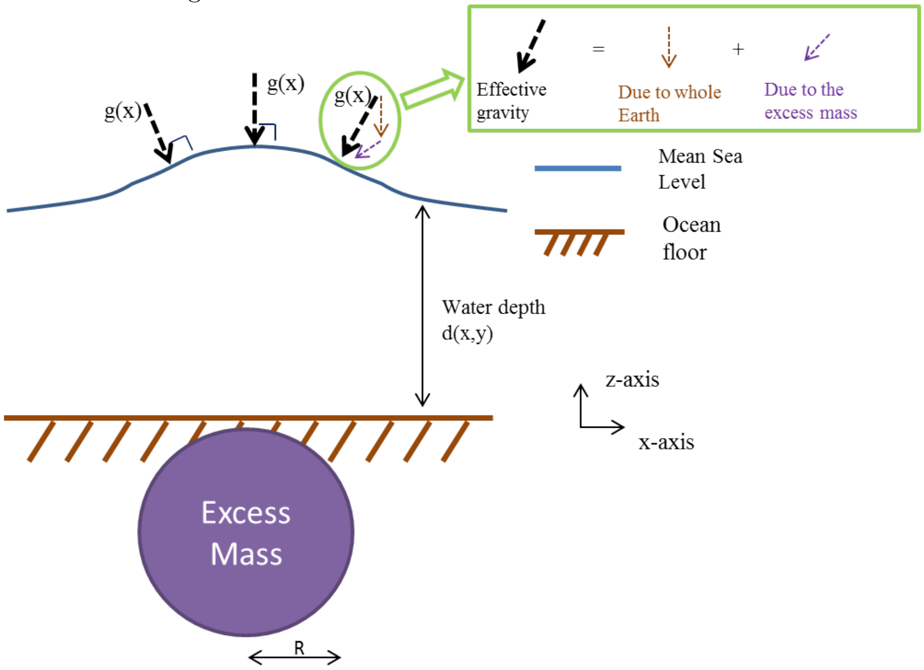
$$\Phi'(x, y, z) = -\frac{G(\Delta M)}{\sqrt{x^2 + y^2 + (z + R)^2}}$$

The geopotential difference relative to the mean sea level at (x, y, z) is then given by $\Psi = \Delta\Phi = g_0\big(z - z_0(x, y)\big) + \Phi'(x, y, z) - \Phi'(x, y, z_0(x, y))$. Therefore at (x, y), the coordinate transformation $Z = Z(x, y, z)$ is given by:

$$\begin{aligned} Z(x, y, z) &= \frac{\Psi}{g_0} \\ &= \frac{1}{g_0}\Big[\big(g_0 z + \Phi'(x, y, z)\big) - \big(g_0 z_0(x, y) + \Phi'(x, y, z_0(x, y))\big)\Big] \\ &= \frac{1}{g_0}\Big[\big(g_0 z - \frac{G(\Delta M)}{\sqrt{x^2 + y^2 + (z + R)^2}}\big) - \big(g_0 z_0(x, y) - \frac{G(\Delta M)}{\sqrt{x^2 + y^2 + (\bar{z}(x, y) + R)^2}}\big)\Big] \end{aligned}$$

In other words, denote, in the same manner as equation (6.9), $Z_0(x, y) = z_0(x, y) - \frac{G(\Delta M)}{g_0\sqrt{x^2+y^2+(\bar{z}(x,y)+R)^2}}$, which is independent of z, the transformation

Figure 6.2: Horizontal ocean floor with excess mass



from (x, y, z) to (x, y, Z) is given by

$$Z(x, y, z) = z - \frac{G(\Delta M)}{g_0\sqrt{x^2 + y^2 + (z + R)^2}} - Z_0(x, y) \tag{6.12}$$

Bathymetry in transformed coordinates

In equation (6.12), the exact transformation from (x, y, z) to (x, y, Z) is shown. Note that the ocean floor in (x, y, z) is given by the line $z = 0$, the point $(x, y, z = 0)$ lying on this line (0) is mapped to (x, y, Z) by:

$$\begin{aligned} Z(x, y, 0) &= 0 - \frac{G(\Delta M)}{g_0\sqrt{x^2 + y^2 + (0 + R)^2}} - Z_0(x, y) \\ &= -\frac{G(\Delta M)}{g_0\sqrt{x^2 + y^2 + R^2}} + (-z_0(x, y) + \frac{G(\Delta M)}{g_0\sqrt{x^2 + y^2 + (z_0(x, y) + R)^2}}) \\ &= -z_0(x, y) + \frac{G(\Delta M)}{g_0}(\frac{1}{\sqrt{x^2 + y^2 + (z_0(x, y) + R)^2}} - \frac{1}{\sqrt{x^2 + y^2 + R^2}}) \end{aligned}$$

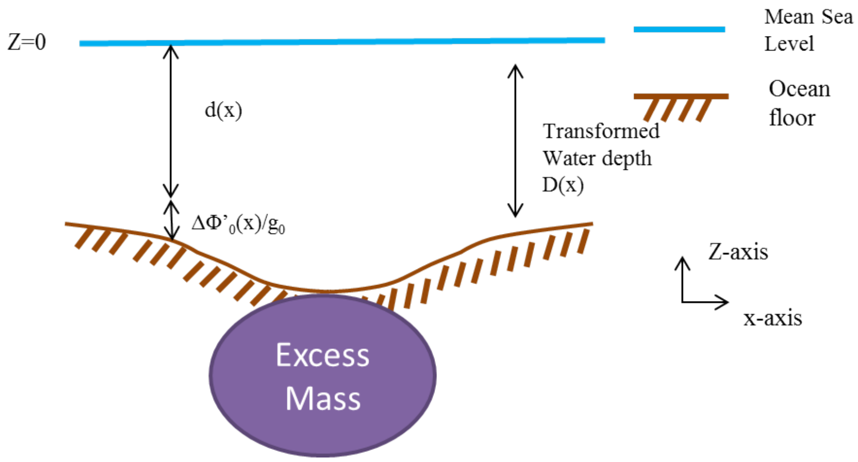
Recall that $Z = 0$ correspond to the mean sea level. Therefore, the 'vertical' distance between the ocean surface and bottom floor in Z coordinates $D(x, y)$

is given by

$$\begin{aligned} D(x,y) &= 0 - Z(x,y,0) \\ &= z_0(x,y) + \frac{G(\Delta M)}{g_0}\Big(\frac{-1}{\sqrt{x^2+y^2+(z_0(x,y)+R)^2}} - \frac{-1}{\sqrt{x^2+y^2+R^2}}\Big) \\ &= z_0(x,y) - \frac{G(\Delta M)}{g_0}\Big(\frac{-1}{\sqrt{x^2+y^2+R^2}} - \frac{-1}{\sqrt{x^2+y^2+(z_0(x,y)+R)^2}}\Big) \\ &= z_0(x,y) - \frac{1}{g_0}\left[\Phi'(x,y,0) - \Phi'(x,y,z_0(x,y))\right] \end{aligned}$$

The graph of $Z = D(x,y)$ is sketched in figure 6.3. It is obvious that a horizontal ocean floor in physical coordinates (x,y,z) is mapped into a non-flat floor in (x,y,Z) coordinate. Note that it is $D(x,y)$ rather than $d(x,y)$ that will be used in the derivation for the adapted wave equation (6.34) as the water depth D_0, this indicates that gravity anomalies somehow plays a role similar to topography variation in physical space, which controls the wave speed and hence reflection, refraction and shoaling of a shallow water wave.

Figure 6.3: Horizontal ocean floor with excess mass in (x,y,Z)



It is also interesting to realise that the second term in the above equation contains the term $\left[\Phi'(x,y,0) - \Phi'(x,y,z_0(x,y))\right]$, which is indeed the potential difference between seafloor $(x,y,0)$ and mean sea level $(x,y,z_0(x,y))$ due to the excess mass. Denote this potential difference by $\Delta\Phi_0{}'(x,y)$, it follows that

$$D(x,y) = z_0(x,y) - \frac{\Delta\Phi_0{}'(x,y)}{g_0} \tag{6.13}$$

Note that when the excess mass is beneath the crust, $\Delta\Phi_0{}'(x,y)$ is always negative, therefore $D(x,y)$ is always larger than $d(x,y) = \bar{z}(x,y)$. The example also indicates that gravity anomaly has effects in $\bar{z}$ implicitly and Φ' explicitly.

6.4 Transformation of equations

In the remaining chapter, all scalar fields and vector fields are understood to be expressed in (x, y, Z, t) unless otherwise specified. $\tilde{\nabla}_H$ is understood to be $(\frac{\partial}{\partial x}\big|_Z, \frac{\partial}{\partial y}\big|_Z)$, that is, keeping Z constant in the partial differentiation.

To transform the equations from (x, y, z, t) to (x, y, Z, t), take $r = Z$ in equation (2.10) and equation (2.11).

Continuity Equation:

$$\frac{d\rho}{dt} + \rho\big[\tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} + \frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big)\big] = 0 \tag{6.14}$$

Horizontal Momentum Equations:

$$\rho[\frac{d\mathbf{u}_H}{dt}] = -\tilde{\nabla}_H p - \rho\tilde{\nabla}_H(\Phi) + \big(\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z}\big)\frac{\partial Z}{\partial z}\tilde{\nabla}_H(z) \tag{6.15}$$

These equations are not easy to deal with. Hence it is necessary to further simplify them before analytically studying the properties of their solutions.

6.5 Additional simplifications

6.5.1 Incompressiblity

Density of water on the ocean surface does change due to temperature and salinity variation. However to simplify the analysis, such density variations are omitted. In other words, it is assumed that $\rho(x, y, Z) = \rho_0$, where ρ_0 is a constant. In the later text, the subscript in ρ_0 will be skipped so that $\rho(x, y, Z) = \rho$.

It follows that the continuity equation (6.14) becomes:

$$\tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} + \frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) = 0 \tag{6.16}$$

6.5.2 Horizontal Gradient of Geopotential

In the (horizontal) momentum equations (6.15) the horizontal gradient of geopotential with respect to Z, i.e. $\tilde{\nabla}_H(\Phi)$ is involved. However, recalling that $Z = \frac{\Phi - \Phi_0}{g_0}$, it is noticed that keeping Z constant is equivalent to keeping Φ constant. Hence the horizontal gradient of geopotential $\tilde{\nabla}_H(\Phi)$ simply vanishes.

$$\tilde{\nabla}_H(\Phi) = (\frac{\partial \Phi}{\partial x}\bigg|_Z, \frac{\partial \Phi}{\partial y}\bigg|_Z) = \vec{0} \tag{6.17}$$

6.5.3 Hydrostatic approximation

For large scale oceanic flows, the horizontal length scale (H) is usually much greater than the vertical length scale (D). This is known as the small aspect ratio $\delta = \frac{D}{H} \ll 1$ or *shallowness* of an oceanic flow. In section 2.8 of Pedlosky 1979, it is justified that for a *shallow* flow with $\delta \ll 1$, vertical pressure gradient $\frac{\partial p(x,y,z)}{\partial z}$ can be approximated by the hyrdostatic vertical pressure gradient $\rho\frac{\partial \Phi}{\partial z}$. An analogous result will be derived.

Following the derivation in Pedlosky 1979, the pressure gradient in hydrostatic condition is obtained by setting $\vec{u} = \vec{0}$ in the momentum equation. Hence equation(2.3) becomes:

$$\nabla p_s = -\rho \nabla \Phi$$

where p_s refers to the hydrostatic pressure. Since it is assumed that the density ρ is constant everywhere, ρ can be included in the gradient operator. Hence it follows that:

$$\nabla p_s = -\nabla(\rho\Phi) \tag{6.18}$$

$$\Rightarrow p_s = -\rho\Phi + p_0 \tag{6.19}$$

where reference pressure p_0 is some constant. Consider the gradient operator ∇ in Cartesian coordinate, along the z-axis, it is noted that:

$$\frac{\partial p_s}{\partial z} = -\rho \frac{\partial \Phi}{\partial z}$$

Multiplying both sides with $\frac{\partial z}{\partial Z}$ leads to:

$$\frac{\partial p_s}{\partial z}\frac{\partial z}{\partial Z} = -\rho \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial Z}$$

Recalling $p_s = p_s(x, y, z(x, y, Z, t), t)$ is a scalar field, by chain rule, the equalities $\frac{\partial p_s}{\partial Z} = \frac{\partial p_s}{\partial z}\frac{\partial z}{\partial Z}$ is justified. The same argument applies also for Φ. Thus the analogy of the hydrostatic pressure gradient in coordinates (x, y, Z, t) is established:

$$\frac{\partial p_s}{\partial Z} = -\rho \frac{\partial \Phi}{\partial Z} \tag{6.20}$$

It is worthwhile to point out that, unlike the classical case with uniformly downwards gravity $\vec{g} = -g_0\hat{k}$, $\frac{\partial p_s}{\partial x}|_z \neq 0$ and $\frac{\partial p_s}{\partial y}|_z \neq 0$. It is $\frac{\partial p_s}{\partial x}|_Z$ and $\frac{\partial p_s}{\partial y}|_Z$ that are equal to zero. The proof for $\frac{\partial p_s}{\partial x}|_Z = 0$ is shown:

$$\begin{aligned} \left.\frac{\partial p_s}{\partial x}\right|_Z &= \left.\frac{\partial(-\rho\Phi + p_0)}{\partial x}\right|_Z \\ &= -\rho \left.\frac{\partial(g_0 Z + \Phi_0)}{\partial x}\right|_Z + \left.\frac{\partial p_0}{\partial x}\right|_Z \\ &= -\rho g_0 \left.\frac{\partial Z}{\partial x}\right|_Z - \rho \left.\frac{\partial \Phi_0}{\partial x}\right|_Z + \left.\frac{\partial p_0}{\partial x}\right|_Z \end{aligned}$$

Recall that the reference pressure p_0 and reference geopotential Φ_0 are constants that are independent of any coordinates. The partial derivative $\left.\frac{\partial Z}{\partial x}\right|_Z$ vanishes because Z is kept invariant. Hence, every term on the right-hand side vanishes and yields:

$$\left.\frac{\partial p_s}{\partial x}\right|_Z = 0$$

The proof for $\frac{\partial p_s}{\partial y}|_Z = 0$ can be obtained exactly in the same way. Therefore, the horizontal gradient of hydrostatic pressure p_s becomes:

$$\frac{\partial p_s}{\partial x}\bigg|_Z = \frac{\partial p_s}{\partial y}\bigg|_Z = 0 \tag{6.21}$$

Following the characteristic scale analysis in section 3.3 of Pedlosky 1979, when gravity is uniform, in non-hydrostatic case $\frac{\partial p}{\partial z} = -\rho\frac{\partial \Phi}{\partial z} + O(\delta^2)$. In the next section it will be shown that an analogy:

$$\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z} = O(\delta^2) \tag{6.22}$$

is also valid. When the flow is shallow, i.e. $\delta \ll 1$, $O(\delta^2)$ is taken to be 0. A detailed argument will be given in the next chapter.

It follows that the momentum equation in non-rotating shallow water becomes:

$$\frac{d\mathbf{u}_H}{dt} = \frac{-1}{\rho}\tilde{\nabla}_H(p) \tag{6.23}$$

6.5.4 Simplification of Continuity Equation

Suppose the vertical gravity perturbation g_z' is small compared with the reference gravity g_0, that is, $g_z' \ll g_0$, the continuity equation (6.16) can be simplified in the following ways.

In the incompressible continuity equation (6.16): $\tilde{\nabla}_H\cdot\mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} + \frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) = 0$, the last term can be considered and approximated in the following way:

From equation (6.11) and (6.10) it follows:

$$\begin{aligned}\frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) &= -\frac{d}{dt}\big(\ln(\frac{\partial Z}{\partial z})\big)\\ &= -\frac{d}{dt}\big(\ln(1+\frac{g_z'}{g_0})\big)\end{aligned}$$

Now note if $\frac{g_z'}{g_0} \ll 1$, which is the case being studied, the first order approximation $\ln(1+\frac{g_z'}{g_0}) \approx \frac{g_z'}{g_0}$ is justified. Expressing the total derivative $\frac{d}{dt}$ fully yields:

$$\begin{aligned}\frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) &\approx -\frac{d}{dt}(\frac{g_z'}{g_0})\\ &= -(\frac{\partial}{\partial t}\bigg|_Z + u\frac{\partial}{\partial x}\bigg|_Z + v\frac{\partial}{\partial y}\bigg|_Z + \dot{Z}\frac{\partial}{\partial Z})(\frac{g_z'}{g_0})\end{aligned}$$

Note that $\frac{g_z'}{g_0}$ is time-independent, the partial derivative with respect to t: $\frac{\partial}{\partial t}\big|_Z(\frac{g_z'}{g_0})$ vanishes.

$$\frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) \approx -\big(0 + u\frac{\partial}{\partial x}\bigg|_Z + v\frac{\partial}{\partial y}\bigg|_Z + \dot{Z}\frac{\partial}{\partial Z}\big)(\frac{g_z'}{g_0})$$

or equivalently,

$$\frac{d}{dt}\big(\ln(\frac{\partial z}{\partial Z})\big) \approx -\big(\mathbf{u}_H \cdot \tilde{\nabla}_H + \dot{Z}\frac{\partial}{\partial Z}\big)\big(\frac{g'_z}{g_0}\big) \tag{6.24}$$

Thus the continuity equation (6.16) can be approximated to first order by:

$$\tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} - \big(\mathbf{u}_H \cdot \tilde{\nabla}_H + \dot{Z}\frac{\partial}{\partial Z}\big)\big(\frac{g'_z}{g_0}\big) = 0 \tag{6.25}$$

6.6 Characteristic Scale Analysis

Denote, with respect to Z coordinate, H and U to be the horizontal length and velocity scale, D and W to be the vertical counterpart. In other words:

$$\begin{aligned} u, v &\sim \mathcal{O}(U) \\ \dot{Z} &\sim \mathcal{O}(W) \\ x, y &\sim \mathcal{O}(H) \\ Z &\sim \mathcal{O}(D) \end{aligned}$$

Also, define σ to be the scale of gravity perturbation g'_z relative to reference gravity g_0, that is

$$\sigma \sim \mathcal{O}(\frac{g'_z}{g_0})$$

6.6.1 Continuity Equation

Considering the dimension of each of the term in (6.25) yields:

$$\begin{aligned} \tilde{\nabla}_H \cdot \mathbf{u}_H &\sim \mathcal{O}(\frac{U}{H}) \\ \frac{\partial \dot{Z}}{\partial Z} &\sim \mathcal{O}(\frac{W}{D}) \\ \mathbf{u}_H \cdot \tilde{\nabla}_H\big(\frac{g'_z}{g_0}\big) &\sim \mathcal{O}(\frac{U\sigma}{H}) \\ \dot{Z}\frac{\partial}{\partial Z}\big(\frac{g'_z}{g_0}\big) &\sim \mathcal{O}(\frac{W\sigma}{D}) \end{aligned}$$

balancing these quantities via the continuity equation (6.25) gives:

$$\begin{aligned} \frac{U}{H} + \frac{W}{D} + \sigma\frac{U}{H} + \sigma\frac{W}{D} &= 0 \\ (\frac{U}{H} + \frac{W}{D})(1+\sigma) &= 0 \end{aligned}$$

If the gravity perturbation g'_z is weak, $\sigma \ll 1$, then $1+\sigma \approx 1$. The length scales H, D and velocity scales U, W are thus limited in the way same as classical case:

$$\frac{U}{H} + \frac{W}{D} = 0$$

It also follows that, in analogy to the classical case, the scale of vertical velocity W is constrained by the product of horizontal velocity U and the aspect ratio $\delta = \frac{D}{H}$ of length scales D and H.

$$W \sim \mathcal{O}(\frac{DU}{H}) = O(\delta U) \tag{6.26}$$

In the shallow water, i.e. $\delta \ll 1$, W is thus very small relative to U. Hence, the vertical momentum equation can be abandoned.

6.6.2 Momentum Equation

The characteristic scale analysis of the Momentum Equation presented in this section is tricky. The standard way to conduct the characteristic scale analysis is to derive the full momentum equations in all coordinates and consider the characteristic scales of each variable. However, since the momentum equation in the direction of the transformed vertical coordinate Z was not derived, the standard approach does not work. However, a closer look of the horizontal momentum equation (6.15) suggests an alternative to derive information of the characteristic scales. Although this alternative is not entirely rigorous, it provides a handy and sensible argument to the characteristic scales of the terms in the momentum equation.

Recall that the horizontal momentum equation in the transformed coordinate Z is given by equation (6.15):

$$\rho\frac{d\mathbf{u}_H}{dt} = -\tilde{\nabla}_H(p) - \rho\tilde{\nabla}_H(\Phi) + \big(\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z}\big)\frac{\partial Z}{\partial z}\tilde{\nabla}_H(z)$$

Note that the main difference between the adapted horizontal momentum equation and the standard one is the extra term $(\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z})\frac{\partial Z}{\partial z}\tilde{\nabla}_H(z)$. Hence, it suffices to give an estimate of the order of magnitude of this term.

Assume that the pressure p can be decomposed into a hydrostatic part p_s and variation part p', so that

$$p(x, y, Z, t) = p_s(x, y, Z) + p'(x, y, Z, t) \tag{6.27}$$

The hydrostatic pressure p_s can be determined by equation (6.19), which gives

$$\frac{\partial p_s}{\partial Z} = -\rho\frac{\partial \Phi}{\partial Z} \tag{6.28}$$

so that

$$\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z} = \frac{\partial p'}{\partial Z} \tag{6.29}$$

and only the vertical gradient of variation pressure p' has to be considered.

The second idea to realise is the relation between $\frac{\partial p'}{\partial z}$ and $\frac{\partial p'}{\partial Z}$, by the chain rule,

$$\frac{\partial p'}{\partial Z} = \frac{\partial p'}{\partial z}\frac{\partial z}{\partial Z} \tag{6.30}$$

$$\tag{6.31}$$

Note that by the recipical rule $\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial z} = 1$. Hence it yields:

$$\left(\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z}\right)\frac{\partial Z}{\partial z} = \frac{\partial p'}{\partial Z}\frac{\partial Z}{\partial z} \tag{6.32}$$

$$= \frac{\partial p'}{\partial z} \tag{6.33}$$

Therefore, it is sufficient to consider the characteristic scale of $\frac{\partial p'}{\partial z}$. This can be done via the vertical momentum equation in standard Cartesian coordinates (x, y, z):

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} - \rho(g_0 + g'_z) \tag{6.34}$$

Recall that the pressure p is decomposed into the hydrostatic and variation parts: $p = p_s + p'$. Considering the hydrostatic balance in the vertical momentum equation gives

$$\frac{\partial p}{\partial z} = -\rho(g_0 + g'_z) \tag{6.35}$$

Hence what remains in the momentum equation (6.34) is

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p'}{\partial z} \tag{6.36}$$

The characteristic scale analysis for this equation is well-discussed in many geophysical fluid dynamics texts, for instance Pedlosky 1979. In this report the result will be cited without proof

$$\frac{\partial p'}{\partial z} \sim \mathcal{O}(\delta^2) \tag{6.37}$$

and therefore

$$\left(\frac{\partial p}{\partial Z} + \rho\frac{\partial \Phi}{\partial Z}\right)\frac{\partial Z}{\partial z} \sim \mathcal{O}(\delta^2) \tag{6.38}$$

where δ is the aspect ratio of the vertical length scale to horizontal length scale. When $\delta \ll 1$, this term can be approximated to 0. This therefore justifies the validity of the claim (6.22) and the simplified horizontal momentum equation (6.23).

6.7 Derivation of Adapted Shallow Water Equations

In this section the derivation of wave equation in shallow water under a weakly non-uniform gravitational field will be presented.

6.7.1 Adapted Incompressible Continuity Equation

In this section, a conservative form of the continuity equation (6.25) will be derived. Denote, for scalar field $f = f(x, y, Z)$,

$$\tilde{\nabla} f = \tilde{\nabla}_H f + \frac{\partial}{\partial Z} f$$

and for vector field $\mathbf{u} = (\mathbf{u}_H, \frac{\partial \dot{Z}}{\partial Z})$,

$$\begin{aligned} \tilde{\nabla} \cdot \mathbf{u} &= \tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} \\ (\mathbf{u} \cdot \tilde{\nabla}) f &= (\mathbf{u}_H \cdot \tilde{\nabla}_H) f + \dot{Z} \frac{\partial f}{\partial Z} \end{aligned}$$

It must be noted that $\tilde{\nabla}$ and $\tilde{\nabla}\cdot$ are not gradient operator nor divergence operator, because (x, y, Z) is not an orthonormal coordinate system. The notation is imposed just to make the derivation concise. First notice that, by chain rule,

$$\begin{aligned} \tilde{\nabla} \cdot (\mathbf{u} \frac{-g_z'}{g_0}) &= \tilde{\nabla}_H \cdot \big(\mathbf{u}_H \frac{-g_z'}{g_0}\big) + \frac{\partial}{\partial Z} \big(\dot{Z} \frac{-g_z'}{g_0}\big) \\ &= \frac{-g_z'}{g_0} \big(\tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z}\big) + \big(\mathbf{u}_H \cdot \tilde{\nabla}_H + \dot{Z} \frac{\partial}{\partial Z}\big) \big(\frac{-g_z'}{g_0}\big) \\ &= \frac{-g_z'}{g_0} \big(\tilde{\nabla} \cdot \mathbf{u}\big) + \big(\mathbf{u} \cdot \tilde{\nabla}\big) \big(\frac{-g_z'}{g_0}\big) \end{aligned}$$

Rearranging the terms gives:

$$\big(\mathbf{u} \cdot \tilde{\nabla}\big) \big(\frac{-g_z'}{g_0}\big) = \tilde{\nabla} \cdot (\mathbf{u}_H \frac{-g_z'}{g_0}) - \frac{-g_z'}{g_0} \big(\tilde{\nabla} \cdot \mathbf{u}\big) \tag{6.39}$$

Second, recall that the first order approximation to continuity equation (6.25) under the new notation is given by:

$$\begin{aligned} \tilde{\nabla}_H \cdot \mathbf{u}_H + \frac{\partial \dot{Z}}{\partial Z} + \big(\mathbf{u}_H \cdot \tilde{\nabla}_H + \dot{Z} \frac{\partial}{\partial Z}\big) \big(\frac{-g_z'}{g_0}\big) &= 0 \\ \tilde{\nabla} \cdot \mathbf{u} + \big(\mathbf{u} \cdot \tilde{\nabla}\big) \big(\frac{-g_z'}{g_0}\big) &= 0 \end{aligned}$$

Therefore, by rearranging the terms in the above expression, the first-order continuity equation (6.25) can be expressed in the following form:

$$\tilde{\nabla} \cdot \mathbf{u} = -\big(\mathbf{u} \cdot \tilde{\nabla}\big) \big(\frac{-g_z'}{g_0}\big) \tag{6.40}$$

Plug in (6.39) to the right-hand side term in (6.40) gives:

$$\begin{aligned} \tilde{\nabla} \cdot \mathbf{u} + \tilde{\nabla} \cdot (\mathbf{u} \frac{-g_z'}{g_0}) &= \frac{-g_z'}{g_0} \big(\tilde{\nabla} \cdot \mathbf{u}\big) \\ \tilde{\nabla} \cdot \left[(1 - \frac{g_z'}{g_0}) \mathbf{u} \right] &= \frac{-g_z'}{g_0} \big(\tilde{\nabla} \cdot \mathbf{u}\big) \end{aligned}$$

Now note that the right-hand side of above equation contains the left-hand side of equation (6.40) again. Plug in equation (6.40) yields:

$$\begin{aligned}
\tilde{\nabla} \cdot \Big[(1 - \frac{g_z'}{g_0})\mathbf{u}\Big] &= \frac{-g_z'}{g_0}\big(- \big(\mathbf{u} \cdot \tilde{\nabla}\big)\big(\frac{-g_z'}{g_0}\big)\big) \\
&= -\frac{-g_z'}{g_0}\big((\mathbf{u}_H \cdot \tilde{\nabla}_H)\frac{-g_z'}{g_0} + \dot{Z}\frac{\partial}{\partial Z}\big(\frac{-g_z'}{g_0}\big)\big) \\
&= -\frac{1}{2}\big((\mathbf{u}_H \cdot \tilde{\nabla}_H)\big(\frac{-g_z'}{g_0}\big)^2 + \dot{Z}\frac{\partial}{\partial Z}\big(\frac{-g_z'}{g_0}\big)^2\big) \\
&= -\frac{1}{2}\big((\mathbf{u} \cdot \tilde{\nabla})\big(\frac{-g_z'}{g_0}\big)^2\big)
\end{aligned}$$

Now note that the term $\tilde{\nabla}(\frac{-g_z'}{g_0})^2$ contains the partial derivatives of quadratic term of perturbation $(\frac{-g_z'}{g_0})^2$. If the gravity perturbation g_z' is spatially smooth, that is $\frac{-g_z'}{g_0}$ does not oscillate in space, which is in general the case in practice, then the right-hand side of above equation is small and can be neglected. That is,

$$(\mathbf{u} \cdot \tilde{\nabla})\big(\frac{-g_z'}{g_0}\big)^2 \approx 0 \tag{6.41}$$

In other words, the first order approximation to the exact continuity equation (6.16) in (x, y, Z) coordinates, or the *adapted continuity equation* is given by:

$$\tilde{\nabla}_H \cdot \big((1 - \frac{g_z'}{g_0})\mathbf{u}_H\big) + \frac{\partial}{\partial Z}\big((1 - \frac{g_z'}{g_0})\dot{Z}\big) = 0 \tag{6.42}$$

By setting $g_z' = 0$ such that $Z = z$, this equation reduces to the standard incompressible continuity equation: $\nabla \cdot \mathbf{u} = 0$.

6.7.2 Adapted Depth-Averaged Continuity Equation

In this section, the adapted shallow water model which is able to deal with weakly non-uniform gravity, will derived. Since the shallow water model is a 'semi-3D' model in the sense that the horizontal velocities are depth-averaged and are independent of vertical coordinate z, it is justifiable also to a priori take a 'depth-averaged' vertical gravity perturbation g_z', that is, $g_z' = g_z'(x, y)$ is also independent of Z.

It follows from the adapted continuity equation (6.42) that:

$$\begin{aligned}
\frac{\partial}{\partial Z}\big[(1 - \frac{g_z'}{g_0})\dot{Z}\big] &= -\tilde{\nabla}_H \cdot \big[(1 - \frac{g_z'}{g_0})\mathbf{u}_H\big] \\
(1 - \frac{g_z'}{g_0})\frac{\partial}{\partial Z}(\dot{Z}) &= -\Big(\big[(1 - \frac{g_z'}{g_0})\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(1 - \frac{g_z'}{g_0})\big]\Big) \\
\frac{\partial}{\partial Z}(\dot{Z}) &= -\Big(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \frac{1}{1 - \frac{g_z'}{g_0}}\tilde{\nabla}_H(1 - \frac{g_z'}{g_0})\big]\Big) \\
\frac{\partial}{\partial Z}(\dot{Z}) &= -\Big(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H\big(\ln(1 - \frac{g_z'}{g_0})\big)\big]\Big)
\end{aligned}$$

Take the first order approximation $\ln(1-\frac{g_z'}{g_0}) \approx \frac{-g_z'}{g_0}$ on of above equation, then it follows

$$\frac{\partial \dot{Z}}{\partial Z} = -\bigg(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]\bigg) \tag{6.43}$$

Note that the same result without the aid of adapted continuity equation (6.42) can be obtained, if the assumption that $\frac{\partial Z}{\partial z} = (1+\frac{g_z'}{g_0})$ is independent of Z is made in equation (6.10). In that case, (6.11) gives $\frac{\partial z}{\partial Z} = \frac{1}{1+\frac{g_z'}{g_0}} \approx 1 - \frac{g_z'}{g_0}$ to first order. Plug this into the exact transformed incompressible continuity equation (6.16) will lead to the same result in (6.43). This shows the consistency with the approximation $\tilde{\nabla}(\frac{-g_z'}{g_0})^2 \approx 0$ in equation (6.41). Yet it is worthwhile to point out that (6.42) is more general, in the sense that it holds also when g_z' depends on Z (subject to that the perturbation of geopotential Φ' is conservative, which is taken for granted).

Now depth-averaging can finally be performed. Denote the surface elevation to be $Z = h(x,y,t)$ and depth of fluid to be $Z = h_B(x,y)$, such that the total depth of fluid $D(x,y,t)$ is given by $D(x,y,t) = h(x,y,t) - h_B(x,y)$. Integrating the adapted continuity equation (6.43) over depth from $Z = h_B$ to arbitrary $Z = Z_0 \leq h$ gives:

$$\int_{h_B}^{Z_0} \frac{\partial \dot{Z}}{\partial Z} dZ = \int_{h_B}^{Z_0} \bigg[-\bigg(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]\bigg)\bigg] dZ$$

$$\dot{Z}(x,y,Z_0,t) - \dot{Z}(x,y,h_b,t) = -(Z_0 - h_B)\bigg(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]\bigg)$$

Imposing the normal flow to be zero at the bottom boundary $Z = h_b$:

$$\dot{Z}(x,y,h_b,t) = u\frac{\partial h_B}{\partial x}\bigg|_Z + v\frac{\partial h_B}{\partial y}\bigg|_Z = \mathbf{u}_H \cdot \tilde{\nabla}_H(h_B) \tag{6.44}$$

Then an expression for $\dot{Z} = \dot{Z}(x,y,Z_0,t)$ can be yielded:

$$\dot{Z}(x,y,Z_0,t) = -(Z_0 - h_B)\bigg(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]\bigg) + \mathbf{u}_H \cdot \tilde{\nabla}_H(h_B) \tag{6.45}$$

Note that this is an indication that $\dot{Z}(x,y,Z,t) = \alpha(x,y,t)Z + \beta(x,y,t)$ follows a affine linear structure in Z coordinate, where $\alpha(x,y,t)$ and $\beta(x,y,t)$ are denoted to be some scalar function.

Now kinematic boundary condition on fluid surface $Z = h(x,y,t)$ is imposed:

$$\dot{Z}(x,y,h,t) = \frac{\partial h}{\partial t}\bigg|_Z + u\frac{\partial h}{\partial x}\bigg|_Z + v\frac{\partial h}{\partial y}\bigg|_Z = \frac{\partial h}{\partial t}\bigg|_Z + \mathbf{u}_H \cdot \tilde{\nabla}_H(h) \tag{6.46}$$

and take $Z_0 = h$ in (6.45), which gives:

$$\big[\frac{\partial h}{\partial t}\bigg|_Z + \mathbf{u}_H \cdot \tilde{\nabla}_H(h)\big] - \mathbf{u}_H \cdot \tilde{\nabla}_H(h_B) = -(h - h_B)\bigg(\big[\tilde{\nabla}_H \cdot \mathbf{u}_H\big] + \big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]\bigg)$$

$$\frac{\partial h}{\partial t}\bigg|_Z + \tilde{\nabla}_H \cdot \big((h - h_B)\mathbf{u}_H\big) = -(h - h_B)\big[\mathbf{u}_H \cdot \tilde{\nabla}_H(\frac{-g_z'}{g_0})\big]$$

Now recall that the water depth $D = h - h_B$ and that the bottom boundary $h_B = h_B(x, y)$ is time-independent, so that $\frac{\partial h}{\partial t}\big|_Z = \frac{\partial(h-h_B)}{\partial t}\big|_Z = \frac{\partial D}{\partial t}\big|_Z$. The above equation can thus be rewritten as

$$\frac{\partial D}{\partial t}\bigg|_Z + \tilde{\nabla}_H \cdot \big(D\mathbf{u}_H\big) = -D\big[\mathbf{u}_H \cdot \tilde{\nabla}_H\big(\frac{-g'_z}{g_0}\big)\big]$$
$$\frac{\partial D}{\partial t}\bigg|_Z + \frac{\partial(uD)}{\partial x}\bigg|_Z + \frac{\partial(vD)}{\partial y}\bigg|_Z = D\left[u\frac{\partial}{\partial x}\bigg|_Z + v\frac{\partial}{\partial y}\bigg|_Z\right]\big(\frac{g'_z}{g_0}\big)$$

or equivalently in a form resembling the mass conservation, by denoting $\frac{d}{dt}_H = \frac{\partial}{\partial t}\big|_Z + \mathbf{u}_H \cdot \tilde{\nabla}_H$ as a a 'horizontal' total derivative in coordinates (x, y, Z, t):

$$\frac{d}{dt}_H(D) + D\tilde{\nabla}_H \cdot \big(\mathbf{u}_H\big) = D\big[\mathbf{u}_H \cdot \tilde{\nabla}_H\big(\frac{g'_z}{g_0}\big)\big] \tag{6.47}$$

Equation (6.47) suggests that the quantity D is not conserved. A nonlinear sink/source $S = S(x, y, t, D, \mathbf{u}_H, g'_z) = D\big[\mathbf{u}_H \cdot \tilde{\nabla}_H\big(\frac{g'_z}{g_0}\big)\big]$ is associated along the path line of fluid. Note that when the perturbation is absent, that is $g'_z = 0$, this nonlinear sink/source term vanishes and equation (6.47) reduces to the classical conservative case.

6.7.3 Adapted Momentum Equation

Recall that the simplified momentum equation (6.23) has the following form in shallow water:

$$\frac{d\mathbf{u}_H}{dt} = \frac{-1}{\rho}\tilde{\nabla}_H p \tag{6.48}$$

Decompose the pressure p into the hydrostatic part p_s and dynamic part $\breve{p}$

$$p(x, y, Z, t) = p_s(x, y, Z) + \breve{p}(x, y, Z, t)$$

Taking the horizontal gradient of p, and noting by equation (6.21) that the horizontal gradient to hydrostatic part p_s vanishes, it follows that

$$\tilde{\nabla}_H p = \tilde{\nabla}_H \breve{p}$$

and integrating equation (6.22) over depth:

$$\int_h^Z \big(\frac{\partial p}{\partial Z}\big)dZ = -\rho\int_h^Z \big(\frac{\partial \Phi}{\partial Z}\big)dZ$$

Note that by equation (6.6), $\frac{\partial \Phi}{\partial Z} = g_0$, imposing the boundary condition on the fluid surface: $p(x, y, h(x, y), t) = p_0$ leads to:

$$p(x, y, Z, t) - p_0 = -\rho g_0(Z - h)$$

Hence it follows that

$$\frac{\partial p}{\partial x}\bigg|_Z = \rho g_0 \frac{\partial h}{\partial x}\bigg|_Z \tag{6.49a}$$
$$\frac{\partial p}{\partial y}\bigg|_Z = \rho g_0 \frac{\partial h}{\partial y}\bigg|_Z \tag{6.49b}$$

Therefore using equation (6.48) and (6.49),the adapted horizontal momentum equation in shallow water is given by:

$$\frac{\partial u}{\partial t}\bigg|_Z + u\frac{\partial u}{\partial x}\bigg|_Z + v\frac{\partial u}{\partial y}\bigg|_Z = -g_0\frac{\partial h}{\partial x}\bigg|_Z \tag{6.50a}$$

$$\frac{\partial v}{\partial t}\bigg|_Z + u\frac{\partial v}{\partial x}\bigg|_Z + v\frac{\partial v}{\partial y}\bigg|_Z = -g_0\frac{\partial h}{\partial y}\bigg|_Z \tag{6.50b}$$

or equivalently

$$\frac{d\mathbf{u}_H}{dt} = -g_0\tilde{\nabla}_H(h) \tag{6.51}$$

6.8 Derivation of Adapted Wave Equation in Shallow Water

In this section the small-amplitude motion on the surface of a fluid layer will be considered. These small amplitudes represent free oscillations or waves on a fluid surface.

Let the hydrostatic thickness of the fluid layer be $D_0(x, y)$. D_0 is known as the mean sea level. Then the hydrodynamics thickness of the fluid layer $Z = D(x, y, y) = D_0(x, y) + \eta(x, y, t)$. The assumption that $\eta \ll \bar{Z}$ implies only small amplitude motions on the surface are considered.

In addition it is assumed $\mathbf{u_H}$ is small enough, such that the advection term $\mathbf{u_H} \cdot \tilde{\nabla}_H(\mathbf{u_H})$ can be neglected compared to $\frac{\partial \mathbf{u_H}}{\partial t}\big|_Z$. Consider

$$u = 0 + u'$$
$$v = 0 + v'$$
$$D = D_0 + \eta$$

and plug these into the adapted momentum equation (6.51) and adapted depth-averaged continuity equation (6.47). By keeping only the first order perturbations it is obtained that:

$$\frac{\partial u'}{\partial t}\bigg|_Z = -g_0\frac{\partial \eta}{\partial x}\bigg|_Z \tag{6.52a}$$

$$\frac{\partial v'}{\partial t}\bigg|_Z = -g_0\frac{\partial \eta}{\partial y}\bigg|_Z \tag{6.52b}$$

$$\frac{\partial \eta}{\partial t}\bigg|_Z + \tilde{\nabla}_H \cdot \big(D_0\mathbf{u}_H{}'\big) = D_0\big[\mathbf{u}_H{}' \cdot \tilde{\nabla}_H\big(\frac{g_z'}{g_0}\big)\big] \tag{6.52c}$$

Define the mass flux $\mathbf{U} = (u'D_0, v'D_0) = (U, V)$, it then follows

$$\frac{\partial U}{\partial t}\bigg|_Z = -g_0 D_0\frac{\partial \eta}{\partial x}\bigg|_Z \tag{6.53}$$

$$\frac{\partial V}{\partial t}\bigg|_Z = -g_0 D_0\frac{\partial \eta}{\partial y}\bigg|_Z \tag{6.54}$$

$$\frac{\partial \eta}{\partial t}\bigg|_Z + \tilde{\nabla}_H \cdot \big(\mathbf{U}\big) = \big[\mathbf{U} \cdot \tilde{\nabla}_H\big(\frac{g_z'}{g_0}\big)\big] \tag{6.55}$$

Partially differentiate equation (6.55) with respect to t, while keeping Z constant gives:

$$\frac{\partial^2 \eta}{\partial t^2}\Big|_Z + \tilde{\nabla}_H \cdot \big(\frac{\partial \mathbf{U}}{\partial t}\big) = \big[\frac{\partial \mathbf{U}}{\partial t} \cdot \tilde{\nabla}_H \big(\frac{g'_z}{g_0}\big)\big] \tag{6.56}$$

Noting that from equation (6.53) and (6.54), $\frac{\partial \mathbf{U}}{\partial t} = -g_0 D_0 \tilde{\nabla}_H(\eta)$. Substituting this into equation (6.56), the adapted wave equation is finally obtained:

$$\frac{\partial^2 \eta}{\partial t^2}\Big|_Z + \tilde{\nabla}_H \cdot \big(-g_0 D_0 \tilde{\nabla}_H(\eta)\big) = \big[-g_0 D_0 \tilde{\nabla}_H(\eta) \cdot \tilde{\nabla}_H \big(\frac{g'_z}{g_0}\big)\big]$$

or, since reference gravity g_0 is spatially independent, equivalently:

$$\frac{\partial^2 \eta}{\partial t^2}\Big|_Z - \tilde{\nabla}_H \cdot \big(g_0 D_0 \tilde{\nabla}_H(\eta)\big) = -\big[D_0 \tilde{\nabla}_H \big(g'_z\big) \cdot \tilde{\nabla}_H(\eta)\big] \tag{6.57}$$

Note that when $g'_z = 0$, the right-hand side of equation (6.57) vanishes and equation (6.57) reduces to the classical shallow water gravity wave equation. Counter-intuitively, the gravity anomaly ${g_x}'$ and ${g_y}'$ in x and y direction play no role in the adapted model. This is because in the transformation of coordinates from (x, y, z, t) to (x, y, Z, t), consideration of ${g_x}'$ and ${g_y}'$ were omitted by choosing suitable path for the line integral in a conservative geopotential field. It is worthwhile to remind once more that the above equation is defined on coordinates (x, y, Z, t). The solution of equation (6.57) should be converted back to (x, y, z, t) in order to be comparable with the solutions from standard gravity wave equation and physical observation.

To end this chapter, the mathematical characteristics of (6.57) is highlighted. From now on $\frac{\partial}{\partial t}$ and ∇ automatically refer to $\frac{\partial}{\partial t}\big|_Z$ and $\tilde{\nabla}_H$. Since $g_0 D_0$ is strictly positive, denote:

$$c(x, y) = \sqrt{g_0 D_0}$$
$$\mathbf{A}(x, y) = D_0 \nabla\big(g'_z\big) = c^2 \nabla\big(\frac{g'_z}{g_0}\big)$$

then equation (6.57) is in the form of:

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla \cdot \big(c^2 \nabla(\eta)\big) = -\big[\mathbf{A} \cdot \nabla(\eta)\big] \tag{6.58}$$

The next questions to ask are, naturally, how the extra term (in contrast to the standard wave equation) $-\big[\mathbf{A} \cdot \nabla_H(\eta)\big]$ will affect η and how the effects look like after being translated back to physical coordinates (x, y, z, t).

Chapter 7

Numerical Methods for Hyperbolic Equations

The adapted model turned out to result in a second-order hyperbolic equation with non-constant coefficients. It is therefore expected that numerical methods for general hyperbolic equations shall be used for its numerical solution. In this chapter, both the analytical and numerical aspects of one-dimensional general hyperbolic partial differential equations will be introduced. At the end of chapter, the generalisation to two-dimensional equation will be outlined.

7.1 Basic Definitions

7.1.1 Hyperbolicity

Consider a scalar field $q = q(x, t)$ which depends on spatial coordinate x and temporal coordinate t. q is governed by a second-order partial differential equation:

$$aq_{tt} + bq_{tx} + cq_{xx} + du_t + eu_x + fu = 0$$

where a, b, c, d, e and f can be functions of x and t. If the coefficients satisfy the condition:

$$b^2 - 4ac > 0$$

at all (x, t), then the second-order partial differential equation is known to be *hyperbolic*.

Consider the advection equation for scalar $q = q(x, t)$, with u being the a constant advection velocity:

$$q_t - uq_x = 0 \tag{7.1}$$

Differentiation of the above equation with respect to t gives a second-order partial differential equation. The resulted second-order partial differential equation

$$q_{tt} - uq_{xt} = 0$$

is always hyperbolic. Note that differentiating the advection equation with respect to x also yields the same conclusion.

In general, consider a vector field $\underline{q} = \underline{q}(x,t) : \Re \times \Re \to \Re^m$, where m is the dimension of the vector field, suppose $\underline{q}$ satisfies the first-order system of partial differential equation, with $A = A(x,t)$ being a m by m real matrix:

$$\underline{q}_t - A\underline{q}_x = 0$$

If a has real eigenvalues $\lambda^i, i = 1, 2, \dots m$ and correspondingly m linearly independently eigenvectors $\underline{r_i}, i = 1, 2, \dots m$ at all (x,t), then the equation can be shown to be *hyperbolic*. The details can be found from Leveque, 2002.

7.1.2 Case Studies: Adapted Wave Equation

The one-dimensional adapted wave equation (c.f. equation (8.2)) is given by

$$\eta_{tt} - \big(g_0 D_0\big)\eta_{xx} + g_0\left[D_0 \frac{d}{dx}(\frac{g'_z}{g_0}) - \frac{d}{dx}\big(D_0\big)\right]\eta_x = 0$$

The derivation has been given in chapter 6. Note that $g_0 D_0$ is strictly positive. Denote $c(x) = \sqrt{g_0 D_0}$ and $s(x) = c^2(x)\frac{d}{dx}(\frac{g'_z}{g_0}) - \frac{dc^2}{dx}(x)$. Then the one-dimensional adapted wave equation can be noted as:

$$\eta_{tt} - c^2\eta_{xx} + s\eta_x = 0 \tag{7.2}$$

and is thus a second-order hyperbolic equation. Note that when $s = 0$ at all (x,t), equation (7.2) is known as the standard (second-order) wave equation.

It is also possible to transform the second-order equation into a system of two first-order equation, by introducing two artificial variables $q^1(x,t) = \eta_t$ and $q^2(x,t) = \eta_x$, such that $\underline{q} = (q^1, q^2)^T$. It then follows:

$$q^1_t - c^2 q^2_x + sq^2 = 0 \tag{7.3}$$

If, furthermore, it is assumed that η is smooth in space and time, then the mixed derivatives should equal:

$$\begin{aligned} \eta_{xt} &= \eta_{tx} \\ \Rightarrow q^2_t &= q^1_x \end{aligned}$$

Hence a system of first-order equation for $\underline{q} = (q^1, q^2)^T$ can be formulated from the second-order hyperbolic equation:

$$q^1_t - c^2 q^2_x = -sq^2 \tag{7.4a}$$

$$q^2_t - q^1_x = 0 \tag{7.4b}$$

Denoting the 2 by 2 matrix A as

$$A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$

and the 2 by 2 matrix S as

$$S = \begin{pmatrix} 0 & -s \\ 0 & 0 \end{pmatrix}$$

will lead to the system of equation:

$$\underline{q}_t + A\underline{q}_x = S\underline{q} \tag{7.5}$$

This equation is known as the advection equation (left-hand side of the equation) with source term $S\underline{q}$ (right-hand side of the equation).

In the remaining of this chapter, the treatment for the particular case when $S = 0$ at all (x, t) will be considered.

7.2 Analytic Aspects of Hyperbolic Equation

7.2.1 Eigenvalues and Characteristics

In this section the analytic solution of the standard second-order wave equation will be examined. While the standard wave equation is a specific example of hyperbolic equations, the methods to solve it are general to all hyperbolic equations.

Recall that the first-order system corresponding to the wave equation is given by (7.5).

$$\underline{q}_t + A\underline{q}_x = 0 \tag{7.6}$$

with A equal to:

$$A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of A are $\lambda^1 = -c$ and $\lambda^2 = c$ and the corresponding eigenvectors are $\underline{r^1} = (c, 1)^T$ and $\underline{r^2} = (-c, 1)^T$ respectively, A can hence be diagonalised into $A = R\Lambda R^{-1}$, where

$$R = (\underline{r^1}, \underline{r^2}) = \begin{pmatrix} c & -c \\ 1 & 1 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}$$

Note that Λ is a diagonal matrix containing the eigenvalue of A.

Express $\underline{w}(x, t)$ by $\underline{w}(x, t) = R^{-1}\underline{q}(x, t)$, then the first-order system (7.6) is transformed into

$$\underline{w}_t + \Lambda\underline{w}_x = 0 \tag{7.7}$$

so that a set of decoupled equations is formed, with $\underline{w}(x, t) = (w^1(x, t), w^2(x, t))^T$:

$$w^1_t - cw^1_x = 0$$
$$w^2_t + cw^2_x = 0$$

Suppose that $\underline{\mathring{w}}(x) = \underline{w}(x, 0)$ is the initial condition, then it can be shown that the functions

$$w^1(x, t) = \mathring{w}^1(x - ct)$$
$$w^2(x, t) = \mathring{w}^2(x + ct)$$

satisfy equation (7.7) at all (x, t). The eigenvalues $\lambda^1 = c$ and $\lambda^2 = -c$ are known as characteristic speeds. The curves $x^i(t) = x_0 + \lambda^i t$ for arbitrary x_0 are known as *characteristic curve*.

7.2.2 General Solution

A general solution to the wave equation (7.6) is now considered.

Given an initial condition $\underline{\mathring{q}}(x)$, the solution to the hyperbolic equation (7.6) can be found by, firstly, transforming the initial condition $\underline{\mathring{q}}(x)$ to $\underline{\mathring{w}}(x)$ by $\underline{\mathring{w}}(x) = R^{-1}\underline{\mathring{q}}(x)$; secondly, evaluating $w^1(x,t)$ and $w^2(x,t)$ through the characteristic curves $x^i(t)$; and finally, transforming $w^1(x,t)$ and $w^2(x,t)$ back to $q^1(x,t)$ and $q^2(x,t)$ through $\underline{q}(x,t) = R\underline{w}(x,t)$, the solution $\underline{q}(x,t)$ can be obtained.

Rigourously, denote the rows of matrix R^{-1} by $\vec{l^1}$ and $\vec{l^2}$, which is called the left eigenvectors of matrix A, then the i-th component of $\underline{w}(x,t)$, which has been denoted as $w^i(x,t)$, can be given by the inner product $\vec{l^i}\underline{q}$.

Hence summarising the procedures, the general solution to (7.6) is given by

$$\underline{q}(x,t) = [\vec{l^1}\underline{\mathring{q}}(x-ct)]\underline{r^1} + [\vec{l^2}\underline{\mathring{q}}(x+ct)]\underline{r^2} \tag{7.8}$$

In the general case with a system of m hyperbolic equation $\underline{q}_t + A\underline{q}_x = 0$, the general solution is given by:

$$\underline{q}(x,t) = \sum_{i=1}^{m}[\vec{l^i}\underline{\mathring{q}}(x-\lambda^i t)]\underline{r^i} \tag{7.9}$$

where $\vec{l^i}$ and $\underline{r^i}$ are the left eigenvetors and eigenvectors of matrix A, and λ_i are the eigenvalues of A. The details can be found in chapter 3 of Leveque 2002.

7.2.3 Domain of Dependence and Range of Influence

From equation (7.8), it can be seen that the solution at $\underline{q}(X,T)$ depends only the initial condition $\underline{\mathring{q}}(x)$ at $x = X - cT$ and $x = X + cT$.

In general, with a system of m hyperbolic equation $\underline{q}_t + A\underline{q}_x = 0$, the set of points:

$$\mathrm{D}(X,T) = \{X - \lambda^i T : i = 1, \dots m\} \tag{7.10}$$

is known as the *Domain of Dependence*. Note that the Domain of Dependence is always a bounded set, which indicates the 'information' of $\underline{q}$ at all (x,t) has roots from limited source of initial data.

On the other hand, consider the point $(x,t) = (x_0, 0)$ at intial time $t = 0$, the initial condition $\underline{\mathring{q}}(x_0)$ only affect $\underline{q}(x,t)$ at later time along the characteristic curves $x^i(t) = x_0 + \lambda^i t$. In other words, given a time interval t, 'information' from the initial condition $\underline{\mathring{q}}(x)$ only propagates within bounded region spatially. The collection of such points:

$$\mathrm{RI}(x_0) = \{x_0 + \lambda^i t : i = 1, \dots m\} \tag{7.11}$$

is known as *Range of Influence.*

7.2.4 Riemann Problem

The *Riemann Problem* refers to the particular problem that a system of hyperbolic equation is equipped with a special discontinuous initial condition:

$$\underline{\mathring{q}}(x) = \begin{cases} \underline{q_l} & \text{for } x < 0 \\ \underline{q_r} & \text{for } x > 0 \end{cases}$$

where $\underline{q_l}$ and $\underline{q_l}$ are some constants vectors. In other words, Riemann Problem is a system of hyperbolic equation with a piecewise constant initial conditions.

Particularly for the standard wave equation (7.5) discussed in the previous section, performing the transformations $\underline{w_l}(x,t) = R^{-1}\underline{q_l}(x,t)$ and $\underline{w_r}(x,t) = R^{-1}\underline{q_r}(x,t)$ one can obtain the initial data for $\underline{u}(x,t)$:

$$\underline{\mathring{w}}(x) = \begin{cases} \underline{w_l} & \text{for } x < 0 \\ \underline{w_r} & \text{for } x > 0 \end{cases}$$

such that

$$\underline{q_l}(x) = w_l^1 \underline{r^1} + w_l^2 \underline{r^2}$$
$$\underline{q_r}(x) = w_r^1 \underline{r^1} + w_r^2 \underline{r^2}$$

Since w_l^1 and w_r^1 propagate with velocity $-c$, while w_l^2 and w_r^2 propagate with velocity c, there are 2 scenarios for the each of the resulted $w^1(x,t)$ and $w^2(x,t)$:

$$w^1(x,t) = \begin{cases} w_l^1 & \text{for } x < -ct \\ w_r^1 & \text{for } x > -ct \end{cases}$$
$$w^2(x,t) = \begin{cases} w_l^2 & \text{for } x < ct \\ w_r^2 & \text{for } x > ct \end{cases}$$

Hence transforming $\underline{w}$ back to $\underline{q}$ by $\underline{q}(x,t) = R\underline{w}(x,t)$ gives:

$$\underline{q}(x,t) = w^1 \underline{r^1} + w^2 \underline{r^2} \tag{7.12}$$
$$= \begin{cases} w_1^1 \underline{r^1} + w_l^2 \underline{r^2} & \text{for } x < -ct \\ w_r^1 \underline{r^1} + w_l^2 \underline{r^2} & \text{for } -ct < x < ct \\ w_r^1 \underline{r^1} + w_r^2 \underline{r^2} & \text{for } x > ct \end{cases} \tag{7.13}$$

Therefore, it can be seen that at $x = -ct$ and $x = ct$, there are discontinuities of $(w_r^1 - w_l^1)\underline{r^1}$ and $(w_r^2 - w_l^2)\underline{r^2}$ respectively. These two discontinuities can be denoted by $\underline{W^1} = \alpha^1 \underline{r^1}$ and $\underline{W^2} = \alpha^2 \underline{r^2}$ such that:

$$\alpha^1 = (w_r^1 - w_l^1) \tag{7.14}$$
$$\alpha^2 = (w_r^2 - w_l^2) \tag{7.15}$$
$$\underline{W^1} = \alpha^1 \underline{r^1} \tag{7.16}$$
$$\underline{W^2} = \alpha^2 \underline{r^2} \tag{7.17}$$

Note that $\underline{W^1}$ and $\underline{W^2}$ are each parallel to the eigenvectors $\underline{r^1}$ and $\underline{r^2}$. Hence that the discontinuity in $\underline{q}(x,0)$, given by $\underline{q_r} - \underline{q_l}$ can be expressed by the discontinuity in $\underline{r^1}$ and $\underline{r^2}$:

$$\underline{q_r} - \underline{q_l} = \alpha^1 \underline{r^1} + \alpha^2 \underline{r^2} \tag{7.18}$$

Hence $\underline{\alpha} = (\alpha^1, \alpha^2)^T$ can be determined also by solving $R\underline{\alpha} = \underline{q_r} - \underline{q_l}$.

In addition, it is noted the solution to the Riemman Problem of (7.12) can also be expressed using the Heaviside function $H(x)$:

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \tag{7.19}$$

such that

$$\underline{q}(x,t) = \underline{q_l} + H(x+ct)\underline{W^1} + H(x-ct)\underline{W^2} \tag{7.20}$$

In the general case with a system of m hyperbolic equations, the above treatments can be applied similarly. The general solution to the Riemann Problem is thus given by:

$$\underline{q}(x,t) = \sum_{i:\lambda^i t > x} w_l^i \underline{r^i} + \sum_{i:\lambda^i t < x} w_r^i \underline{r^i} \tag{7.21}$$

$$= \underline{q_l} + \sum_{i=1}^{m} H(x - \lambda^i t)\underline{W^i} \tag{7.22}$$

where λ^i is the i-th characteristic speed, $\underline{r^i}$ is the i-th eigenvector, $\underline{w_l}$ and $\underline{w_r}$ are the transformed Riemann initial data, and $\underline{W_i}$ is the discontinuity of i-th characteristic. The details can be found from Chapter 3 of Leveque 2002.

Solving the Riemann Problem is an important issue in the numerical aspects of solving Hyperbolic equations.

7.3 Numerical Aspects of Hyperbolic Equation

Typically, to numerically solve a system of partial differential equation, several numerical methods can be used. These include the *Finite Difference Method*, *Finite Element Method* and *Finite Volume Method*. For hyperbolic equations, especially when written in conservative form, usually a Finite Volume Method is used.

In the remaining chapter, only the numerical methods for the simple case of a first-order one-dimensional hyperbolic scalar equation will be examined.

7.3.1 Finite Volume Methods

Recall that a first-order hyperbolic scalar equation is given by:

$$q_t - uq_x = 0 \tag{7.23}$$

where u is a scalar function that is referred to be the characteristic speed. Consider the particular case that $u(x,t) = u > 0$ is constant in time and space, such that the 'information' of the scalar function $q(x)$ always propagates in the positive x direction with speed u. Denote $f(q(x,t)) = uq(x,t)$, the above equation can be expressed in the *conservative* form:

$$q_t - [f(q)]_x = 0 \tag{7.24}$$

Subdividing the one-dimensional domain into intervals called *grid cells*, the i-th grid cell C_i is given by:

$$C_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \tag{7.25}$$

For simplicity, the grid cells are assumed to be uniform in size. Spatially integrating (7.24) over the grid cell C_i yields the *integral form* of conservative equation:

$$\frac{d}{dt}\int_{C_i} q(x,t)dx = f(q(x_{i+\frac{1}{2}},t)) - f(q(x_{i-\frac{1}{2}},t)) \tag{7.26}$$

Temporally integrating (7.26) from some time interval $t = t_n$ to t_{n+1} gives:

$$\int_{C_i} q(x,t_{n+1})dx - \int_{C_i} q(x,t_n)dx = \int_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i-\frac{1}{2}},t))dt \tag{7.27}$$

Qualitatively speaking, the Finite Volume Methods deals with approximation of these integral terms and seeks the solution to the integral equation.

Denote $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $\Delta t = t_{n+1} - t_n$. Then, denote, at time $t = t_n$, an approximation to $\int_{C_i} q(x,t_n)dx$ by $(\Delta x)Q_i^n$ and the approximation to $\int_{t_n}^{t_{n+1}} f[q(x_{i+\frac{1}{2}},t)]dt$ by $(\Delta t)F_{i+\frac{1}{2}}^n$, that is:

$$Q_i^n \approx \frac{1}{\Delta x}\int_{C_i} q(x,t_n)dx$$

$$F_{i+\frac{1}{2}}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i+\frac{1}{2}},t))dt$$

Note that $F_{i+\frac{1}{2}}^n$ can be interpreted as the average flux crossing the boundary of grid cell during time interval Δt at $x = x_{i+\frac{1}{2}}$. Rearranging the terms in equation (7.26) gives the general form of numerical method for a conservative equation:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{7.28}$$

Hence to what order of accuracy the numerical Q_i^n and the flux $F_{i-\frac{1}{2}}^n$ approximate the exact quantities controls the actual accuracy of numerical solution.

7.3.2 Time Integrator

The form of (7.28) has already indicated that explicit time integration is commonly used in hyperbolic equation. The rationale behind, other than an efficient implementation, is also to do with the nature of hyperbolic equation.

Recall in section 7.2.3, the Domain of Dependence and Range of Influence were discussed. It was shown that both Domain of Dependence and Range of Influence are bounded set, suggesting that the 'information' of exactly soluation $q(x,t)$ propagates only with limited spatial influence for a fixed time interval. Therefore, implicit time integration, in which all grid cells in the computational domain are coupled, is not necessary. Explicit schemes with finite coupling between grid cells is already enough for hyperbolic equation. The detailed discussion and explanation can be found from section 4.4 of Leveque 2002.

Courant-Friedrichs-Lewy(CLF) Condition

A necessary condition to the stability of numerical methods is the Courant-Friedrichs-Lewy Condition, which states: *the actual domain of dependence of* *the differential equation must be included in the numerical domain of dependence* in order to have a stable numerical method.

The numerical domain of dependence can be interpreted as how far discrete 'information' Q_i in the discretised domain can propagates after a time step. If the actual 'information' $q(x)$ travels too 'fast' and reaches a grid point C_j that is not coupled to the original grid C_i, then numerically the 'information' from Q_i cannot be passed successfully to C_j and be used to update Q_j. Hence the numerical scheme break down.

In one-dimension first-order hyperbolic scalar equation, define the quantity Courant number $C = \frac{u\Delta t}{\Delta x}$, the CFL condition for explicit time integrator can be summarised as:

$$C = \frac{u\Delta t}{\Delta x} < 1 \tag{7.29}$$

In other words, the time step Δt should be chosen such that the 'information' can only propagate at distance $u\Delta t$ within the distance Δx between C_i neighbouring two cells $C_{i\pm1}$. The detailed discussion of a more general CFL condition can be found from section 4.4 and 4.12 of Leveque 2002.

7.3.3 Typical Flux Approximation

Recall that $F^n_{i+\frac{1}{2}}$ is the flux at $x = x_{i+\frac{1}{2}}$ lying on the boundary of grid cells. Hence it is expected $F^n_{i+\frac{1}{2}}$ will be approximated numerically using the value of Q^n_i at the neighbouring grid cells. That is, denote $\mathcal{F}$ to be the numerical flux function:

$$F^n_{i+\frac{1}{2}} = \mathcal{F}(Q^n_i, Q^n_{i+1}) \tag{7.30}$$

Classical Lax-Friedrichs Method

By choosing

$$\mathcal{F}(Q^n_i, Q^n_{i+1}) = \frac{1}{2}[f(Q^n_{i+1}) + f(Q^n_i)] - \frac{\Delta x}{2\Delta t}(Q^n_{i+1} - Q^n_i), \tag{7.31}$$

Then the resulted numerical scheme (7.28) is given by:

$$Q^{n+1}_i = \frac{1}{2}(Q^n_{i-1} - Q^n_{i+1}) - \frac{\Delta x}{2\Delta t}[f(Q^n_{i+1}) - f(Q^n_{i-1})]$$

which can be shown to be *consistent* with the exact solution with order of $(\Delta x)^2$. When Δt and Δx satisfies the CFL condition, this method is stable. However, the drawback of this method is that this method induces artificial diffusion to the numerical solution. Hence Classical Lax-Friedrichs Method is used usually when very fine grid cells are used. The detailed discussion and interpretation of the method can found in section 4.6 of Leveque 2002.

Upwind method

Note that in the scalar advection equation (7.23), the advection speed u is assumed to be positive, which means the 'information' of the scalar function q at any x always propagates in the positive x direction. Hence it is proposed to choose the numerical flux function $\mathcal{F}$:

$$\mathcal{F}(Q^n_i, Q^n_{i+1}) = u^- Q^n_{i+1} + u^+ Q^n_i \tag{7.32}$$

where

$$u^- = \min(u, 0) \text{ and } u^+ = \max(u, 0) \tag{7.33}$$

In particular, if $u > 0$ at all x, then u^- is always 0. The numerical flux is hence always one-sided, in the sense that only Q_i^n, from which 'information' propagates, is involed in defining the flux $\mathcal{F}(Q_i^n, Q_{i+1}^n)$ at $x_{i+\frac{1}{2}}$. Hence the upwind scheme can be interpreted as a scheme with flux approximated along the 'upwind' direction of 'information' flow.

Denoting $W_{i+\frac{1}{2}} = Q_{i+1}^n - Q_i^n$ as the jump of Q_i^n across the grid cell boundary at $x_{i+\frac{1}{2}}$. Then the resulted numerical scheme (c.f equation (7.28)) is given by:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(u^+ W_{i-\frac{1}{2}} + u^- W_{i+\frac{1}{2}})$$

which is *consistent* to the order of $(\Delta x)^1$. When Δt and Δx is chosen to satisfy the CFL Condition, this method is also stable. The particular advantage of the Upwind scheme is that several features of the monotonicity in the scalar field $q(x, t)$ can largely be maintained at time= $t + \Delta t$ also, although at the expense of a lower order of consistency compared with the Classical Lax-Friedrichs Method.

REMARK: The upwind scheme is a special case of the more general Godunov's Method. The detailed discussion and interpretation of the upwind scheme and Godunov's Method can found in section 4.9 and 4.12 of Leveque 2002.

7.3.4 Limiters

In the previous section, two numerical flux functions were introduced. The Classical Lax-Friedrichs Method is a second-order method and works well when the solution is smooth, while the Upwind method is a first-order method yet preserves better the monotonicity in the solution.

The *limiter* can be interpreted as correction term to the numerical flux functions, with actual effects depending on the smoothness of numerical solution Q_i^n. In smooth region, the limiters will lead to an effectively higher-order (usually second order) flux function, while at location with sharp change or even discontinuity of evaluated quantities, the limiters will result in a first-order one-sided flux function, so that the discontinuity will not lead to unphysical oscillations in the numerical solution. Detailed explanation and discussion of the oscillations can be found in section 6.6 of Leveque 2002.

There are multiple ways to interpret or construct the limiters. From an algebraic point of view, the discrete solution Q_i^n can be used to construct piecewise linear functions in each grid cell. The *slope limiters* are posed to limit the slope of the piecewise linear function, so that unphysical oscillations will not take place when the advected piecewise linear function is averaged out to obtain the discrete solution Q_i^{n+1} in next time step. Detailed discussion can be found in section 6.4 to 6.9 in Leveque 2002.

Another way to interpret the limiters is to limit the numerical flux function, called *flux limiters*.

The general form, in the one-dimensional scalar advection equation, for the numerical flux function equipped with a flux limiter is given by:

$$\mathcal{F}(Q_i^n, Q_{i+1}^n) = u^- Q_{i+1}^n + u^+ Q_i^n + \frac{1}{2}|u|(1 - |\frac{u\Delta t}{\Delta x}|)\phi(\theta_{i+\frac{1}{2}}^n) W_{i+\frac{1}{2}} \tag{7.34}$$

where ϕ is some real function called *flux-limiter function*, and $\theta^n_{i+\frac{1}{2}}$ is a measure of smoothness of the numerical solution Q^n_i which is given by:

$$\theta^n_{i+\frac{1}{2}} = \frac{W_{I+\frac{1}{2}}}{W_{i+\frac{1}{2}}}$$

and

$$I = \begin{cases} i+1, \text{ if } u > 0 \\ i-1, \text{ if } u < 0 \end{cases}$$

so that I represents the upwind side of $x_{i+\frac{1}{2}}$. Recall that $W_{i+\frac{1}{2}} = Q^n_{i+1} - Q^n_i$ is the jump of Q^n_i defined in last section. Hence it can be interpreted as the numerical flux of upwind scheme (7.32) with an addition term $\frac{1}{2}|u|(1 - |\frac{u\Delta t}{\Delta x}|)\phi(\theta^n_{i-\frac{1}{2}})W_{i+\frac{1}{2}}$.

There are several possible choices of flux-limiter function. A very popular one is the *Monotonised central-difference limiter* (MC limiter):

$$\phi(\theta) = \max(0, \min(\frac{1+\theta}{2}, 2, 2\theta)) \tag{7.35}$$

It turns out that the MC limiters will lead to a second-order flux approximation similar to the classical Lax-Friedrichs Method at region with smooth Q^n_i and a first-order flux approximation which suppresses numerical oscillation in region with unsmooth Q^n_i. Other possible choices of flux limiters and their rationales are suggested in section 6.10 to 6.12 in Leveque 2002. Interested readers may refer to Chapter 6 of Leveque 2002 for detailed explanation and discussion of limiters.

7.4 Some Remarks about Two-Dimensional Hyperbolic Equation

Consider the simple first-order two-dimensional advection equation:

$$q_t - (uq_x + vq_y) = 0$$

where $u = u(x, y)$ and $v = v(x, y)$ are scalar fields and represent the advection speed of 'information' of q in x and y direction respectively.

A Finite Volume method can be applied to solve the equation numerically on subdivided sections, by considering rectangular grid cells $C_{i,j} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$. The procedure to reach the discrete equation for each grid cell is almost exactly the same as in one-dimension case. The biggest differences is that there are 4 boundary faces of a grid cell, and the faces are aligned in two perpendicular directions. Hence it is expected there will be two distinct flux functions corresponding to each of the directions. These are denoted by:

$$\text{Along } x \text{ direction, at } x = x_{i-\frac{1}{2}}, y = y_j : F_{i-\frac{1}{2},j}$$
$$\text{Along } y \text{ direction, at } x = x_i, y = y_{j-\frac{1}{2}} : G_{i,j-\frac{1}{2}}$$

The *donor-cell method* is a first order accuracy method that is based on the same rationale of the upwind scheme in one-dimensional problem, and is given by:

$$F_{i-\frac{1}{2},j} = u^+ Q_{i-1,j} + u^- Q_{i,j}$$
$$G_{i,j-\frac{1}{2}} = v^+ Q_{i,j-1} + v^- Q_{i,j}$$

where $u^\pm$ and $v^\pm$ are defined in the same way as equation (7.33), $Q_{i,j}$ represents the value of $q(x,t)$ at grid cell $C_{i,j}$. Higher order numerical flux function can be obtained by using the more sophisticated consideration by viewing $Q_{i,j}$ as an average of a piecewise continuous function $q(x,y)$ within grid cell $C_{i,j}$. The details can be found in Chapter 20 of Leveque 2002.

7.4.1 Implementation

It is proposed that the *CLAWPACK* software packages, developed and maintained by the University of Washington, will be used to numerically solve the the adapted wave equations (6.58). The package is well-tested. Whenever necessary, adaptations to the source code of *CLAWPACK* will be implemented. Details of the package can be found at the official website: http://www.clawpack.org/.

Chapter 8

Case Studies

In Chapter 6 an adapted wave equation in non-rotating Cartesian coordinates is derived. In this chapter, the analytic solution to the one-dimensional adapted wave equation in idealised scenarios will be presented.

8.1 One-Dimensional Adapted Wave Equation

Note that $g_z' = g_z'(x)$ and $D_0 = D_0(x)$ are time independent. In one-dimensional flow, the adapted wave equation (6.58) is written as:

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\big[g_0 D_0 \frac{\partial \eta}{\partial x}\big] = -\big[g_0 D_0 \frac{d}{dx}(\frac{g_z'}{g_0})\frac{\partial \eta}{\partial x}\big] \tag{8.1}$$

Applying the chain rule to the second term on the left-hand side gives:

$$\begin{aligned}\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\big(g_0 D_0\big)\frac{\partial \eta}{\partial x} - g_0 D_0 \frac{\partial}{\partial x}(\frac{\partial \eta}{\partial x}) &= -g_0 D_0 \frac{d}{dx}(\frac{g_z'}{g_0})\frac{\partial \eta}{\partial x} \\ \frac{\partial^2 \eta}{\partial t^2} - g_0 D_0 \frac{\partial}{\partial x}(\frac{\partial \eta}{\partial x}) &= -\Big[g_0 D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(g_0 D_0\big)\Big]\frac{\partial \eta}{\partial x}\end{aligned}$$

Hence, the solution to the linear partial differential equation:

$$\frac{\partial^2 \eta}{\partial t^2} - g_0 D_0 (\frac{\partial^2 \eta}{\partial x^2}) + g_0 \Big[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Big]\frac{\partial \eta}{\partial x} = 0 \tag{8.2}$$

will be examined.

8.2 Plane wave solution

Consider a plane wave solution to equation (8.2), that is:

$$\eta(x,t) = a(x)\exp\big(i(kx - \omega t)\big) \tag{8.3}$$

where $a(x,t)$ is the wave amplitude, which depends on position and time (x,t), k is the wave number and ω is the angular frequency. Detailed discussion on the physical meaning of these quantities can be found in chapter 5.

The time and spatial derivatives of η are listed as below:

$$\frac{\partial \eta}{\partial t} = \big[\frac{\partial a}{\partial t} + i(-\omega a)\big] \exp\big(i(kx - \omega t)\big) \tag{8.4}$$

$$\frac{\partial^2 \eta}{\partial t^2} = \big[(\frac{\partial^2 a}{\partial t^2} - \omega^2 a) + i(-2\omega \frac{\partial a}{\partial t})\big] \exp\big(i(kx - \omega t)\big) \tag{8.5}$$

$$\frac{\partial \eta}{\partial x} = \big[\frac{\partial a}{\partial x} + i(ka)\big] \exp\big(i(kx - \omega t)\big) \tag{8.6}$$

$$\frac{\partial^2 \eta}{\partial x^2} = \big[(\frac{\partial^2 a}{\partial x^2} - k^2 a) + i(2k \frac{\partial a}{\partial x})\big] \exp\big(i(kx - \omega t)\big) \tag{8.7}$$

Substituting these into equation (8.2) gives:

$$\begin{aligned}\Bigg\{&\big[(\frac{\partial^2 a}{\partial t^2} - \omega^2 a) + i(-2\omega \frac{\partial a}{\partial t})\big] - g_0 D_0 \big[(\frac{\partial^2 a}{\partial x^2} - k^2 a) + i(2k \frac{\partial a}{\partial x})\big] + \\ &+ g_0 \Bigg[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Bigg] \big[\frac{\partial a}{\partial x} + i(ka)\big]\Bigg\} \exp\big(i(kx - \omega t)\big) = 0\end{aligned} \tag{8.8}$$

Considering the real part of equation (8.8) yields:

$$\begin{aligned}&\Bigg\{(\frac{\partial^2 a}{\partial t^2} - \omega^2 a) - g_0 D_0 (\frac{\partial^2 a}{\partial x^2} - k^2 a) + g_0 \Bigg[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Bigg](\frac{\partial a}{\partial x})\Bigg\} \cos(kx - \omega t) \\ &+ i^2 \Bigg\{(-2\omega \frac{\partial a}{\partial t}) + -g_0 D_0 (2k \frac{\partial a}{\partial x}) + g_0 \Bigg[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Bigg](ka)\Bigg\} \sin(kx - \omega t) = 0\end{aligned}$$

Since over the function space $\mathcal{C}^1$, $\cos(kx - \omega t)$ and $\sin(kx - \omega t)$ are linearly independent, their coefficients in above the expression vanish:

$$(\frac{\partial^2 a}{\partial t^2} - \omega^2 a) - g_0 D_0 (\frac{\partial^2 a}{\partial x^2} - k^2 a) + g_0 \Bigg[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Bigg](\frac{\partial a}{\partial x}) = 0 \tag{8.9a}$$

$$(-2\omega \frac{\partial a}{\partial t}) - g_0 D_0 (2k \frac{\partial a}{\partial x}) + g_0 \Bigg[D_0 \frac{d}{dx}(\frac{g_z'}{g_0}) - \frac{d}{dx}\big(D_0\big)\Bigg](ka) = 0 \tag{8.9b}$$

Note that by setting the gravity perturbation $g_z' = 0$ and imposing wave amplitude $a(x, t) = a$ which is independent of both space and time, (8.9a) reduces into the classical dispersion relation in shallow water (c.f. equation (5.22))) under uniform gravity:

$$\omega^2 = g_0 D_0 k^2$$

Solutions to this set of linear partial differential equations (8.9a) and (8.9b) will be studied under different scenarios.

8.3 A Test Case: Constant Depth

Test Case: Part 1 - Wave Amplitude

Consider an idealised scenario when the depth of fluid D_0 is constant, that is $D_0(x) = D_0$. It is reminded here again that the depth is measured in the Geopotential height Z-coordinates instead of physical height z. Hence $\frac{d}{dx}(D_0) = 0$ and equation (8.9b) is given by:

$$(-2\omega\frac{\partial a}{\partial t}) + -g_0 D_0(2k\frac{\partial a}{\partial x}) + g_0\left[D_0\frac{d}{dx}(\frac{g'_z}{g_0})\right](ka) = 0$$

If, furthermore, under the case when wave amplitude $a(x, t)$ varies slowly in time t, $\frac{\partial a}{\partial t}$ can be taken to 0. Hence a first order ordinary differential equation for wave amplitude $a(x)$ is obtained:

$$-g_0 D_0(2k\frac{\partial a}{\partial x}) + g_0\left[D_0\frac{d}{dx}(\frac{g'_z}{g_0})\right](ka) = 0$$

$$\frac{\partial a}{\partial x} - \left[\frac{1}{2}\frac{d}{dx}(\frac{g'_z}{g_0})\right]a = 0$$

The general solution to $a(x)$ is thus given by:

$$a(x) = \bar{a_0}\exp(-\frac{1}{2}\int(\frac{d}{dx}(\frac{g'_z}{g_0})dx)) \tag{8.10}$$

$$= a_0\exp(-\frac{1}{2}\frac{g'_z}{g_0}) \tag{8.11}$$

where $\bar{a_0}$ and a_0 are arbitrary constants.

Contrary to the classical case where gravity is uniform and wave amplitude a does not vary in uniform water depth, the adapted wave equation revealed a different story. Wave amplitude a varies spatially subject to gravity perturbation g'_z.

It is also noticed that, in uniform gravity case, wave energy E is given by $\frac{1}{2}\rho g a^2$ (c.f. equation (5.25)).

If we assume the wave energy is given by the same expression, with $g = g(x) = g_0 + g'_z(x)$ being spatially dependent, then the wave energy $E = E(x)$ at spatial coordinates x is given by:

$$E(x) = \frac{1}{2}\rho g a^2 \tag{8.12}$$

$$= \frac{1}{2}\rho(g_0 + g'_z)a_0^2\exp(-\frac{g'_z}{g_0}) \tag{8.13}$$

Express $g_0 + g'_z = g_0(1 + \frac{g'_z}{g_0})$, and note that $1 + \frac{g'_z}{g_0} \approx \exp(\frac{g'_z}{g_0})$ to first order, then the wave energy is given by

$$E(x) \approx \frac{1}{2}\rho g_0\exp(\frac{g'_z(x)}{g_0})a_0^2\exp(-\frac{g'_z(x)}{g_0}) \tag{8.14}$$

$$= \frac{1}{2}\rho g_0 a_0^2 \tag{8.15}$$

which is independent of the spatial coordinates x. In other words, the energy carried by the wave conserves (to first order approximation) when it propagates in non-uniform gravity field.

However, it must be stressed here that the expression of energy in the adapted model is not yet derived. While it seems plausible that it takes the same form as equation (5.25), this should be justified rigorously and will be part of the research. If it turns out that the wave energy is not given by equation (5.25), there may be an indication that there are interactions between wave energy and (background) geopotential, which is not described in the uniform gravity model.

Test Case a: Increasing Linear Gravity Perturbation

To illustrate the actual behavior of the wave amplitude, consider a simple expression for gravity perturbation $g'_z(x) = \alpha x$, where α is some small positive constant. It is reminded here when $g'_z > 0$ implies the magnitude of downwards vertical gravity, given by $g_0 + g'_z$, is greater than the normal case g_0. It then follows that the wave amplitude $a(x)$ is given by

$$a(x) = a(0)\exp(-\frac{\alpha}{2g_0}x)$$

which is exponentially decaying in positive x direction.

It is reminded that $a(x)$ is measured in the Geopotential height Z instead of physical height z. Equation (6.10): $\frac{\partial Z}{\partial z} = 1 + \frac{g'_z}{g_0}$ suggests that when the gravity perturbation g'_z is increasing, each unit change of z coordinates will lead to greater change of units in Z coordinates.

When the wave amplitude is small, one can take the approximation $dZ = a(x)$, then the wave amplitude in physical coordinate $dz = \bar{a}(x)$ can be approximated via equation (6.10), so that $\bar{a}(x) \approx \frac{a(x)}{1+\frac{\alpha x}{g_0}} < a(x)$. This implies that, in physical coordinates z, the observed wave amplitude $z = \bar{a}(x)$ is damped at a rate even higher than that in Z coordinates.

Test Case b: Decreasing Linear Gravity Perturbation

Now the gravity perturbation $g'_z(x) = -\alpha x$, with $\alpha > 0$ is considered. This means the vertical downwards gravity reduces linearly in positive x direction. It then follows that the wave amplitude $a(x)$ is given by

$$a(x) = a(0)\exp(\frac{\alpha}{2g_0}x)$$

Performing the transformation from Z coordinates to z coordinates with the same procedures in previous section shows that in physical space wave amplitude $\bar{a}(x) \approx \frac{a(x)}{1-\frac{\alpha x}{g_0}} > a(x)$. Hence wave amplitude $\bar{a}(x)$ increases at a rate faster than exponential growth when the waves experience weaker downwards gravitational pull.

Test Case: Part 2 - Dispersion relation

In the previous subsection the (time-independent) amplitude of a gravity wave was studied. In this subsection the dispersion relation for such wave will be studied.

In equation (8.11), a general wave amplitude $a(x) = a_0 \exp(-\frac{1}{2}\frac{g'_z}{g_0})$ was obtained. Its derivatives are given by:

$$\begin{aligned}
a(x) &= a_0 \exp(-\frac{1}{2}\frac{g'_z}{g_0}) \\
\frac{da}{dx}(x) &= -\frac{a_0}{2g_0}\frac{dg'_z}{dx}\exp(-\frac{1}{2}\frac{g'_z}{g_0}) \\
&= -\frac{1}{2g_0}\frac{dg'_z}{dx}a(x) \\
\frac{d^2a}{dx^2}(x) &= -\frac{a_0}{2g_0}\Big[\frac{d^2g'_z}{dx^2} - \frac{1}{2g_0}(\frac{dg'_z}{dx})^2\Big]\exp(-\frac{1}{2}\frac{g'_z}{g_0}) \\
&= -\frac{1}{2g_0}\Big[\frac{d^2g'_z}{dx^2} - \frac{1}{2g_0}(\frac{dg'_z}{dx})^2\Big]a(x)
\end{aligned}$$

Plug these into (8.9a). Recall that $a = a(x)$ is considered to be time-independent such that $\frac{\partial a}{\partial t} = \frac{\partial^2 a}{\partial t^2} = 0$. Also the case when water depth is uniform $D_0(x) = D_0$ is still considered.

$$(0 - \omega^2 a) - g_0 D_0(\frac{\partial^2 a}{\partial x^2} - k^2 a) + g_0\left[D_0\frac{d}{dx}(\frac{g'_z}{g_0}) - 0\right](\frac{\partial a}{\partial x}) = 0$$

$$\Big\{(-\omega^2 a) - g_0 D_0(-\frac{1}{2g_0}\Big[\frac{d^2g'_z}{dx^2} - \frac{1}{2g_0}(\frac{dg'_z}{dx})^2\Big]a - k^2 a) + g_0\left[D_0\frac{d}{dx}(\frac{g'_z}{g_0})\right](-\frac{1}{2g_0}\frac{dg'_z}{dx})a\Big\} = 0$$

Eliminating a and grouping terms gives:

$$\begin{aligned}
\omega^2 &= -g_0 D_0(-\frac{1}{2g_0}\Big[\frac{d^2g'_z}{dx^2} - \frac{1}{2g_0}(\frac{dg'_z}{dx})^2\Big] - k^2) + g_0\left[D_0\frac{d}{dx}(\frac{g'_z}{g_0})\right](-\frac{1}{2g_0}\frac{dg'_z}{dx}) \\
\omega^2 &= g_0 D_0(\frac{1}{2g_0}(\frac{d^2g'_z}{dx^2}) + k^2) + (\frac{dg'_z}{dx})^2(-\frac{D_0}{2g_0} - \frac{D_0}{4g_0}) \\
\omega^2 &= g_0 D_0(\frac{1}{2g_0}(\frac{d^2g'_z}{dx^2}) + k^2) + (\frac{dg'_z}{dx})^2(-\frac{3D_0}{4g_0})
\end{aligned}$$

Note that $(\frac{dg'_z}{dx})^2$ is indeed a quadratic term of perturbation g'_z. To first order accuracy this term can be eliminated and the *adapted dispersion relation* under the case of uniform water depth is yielded:

$$\omega^2 = g_0 D_0(k^2 + \frac{1}{2g_0}(\frac{d^2g'_z}{dx^2})) \tag{8.16}$$

Naturally, when the gravity perturbation g'_z is vanishes, (8.16) reduces to the classical dispersion relation (c.f. equation (5.22)).

Test Case a: Increasing Linear Gravity Perturbation, Revisited

In previous section, a test case with gravity perturbation $g'_z(x) = \alpha x$, where α is some positive constant, is considered.

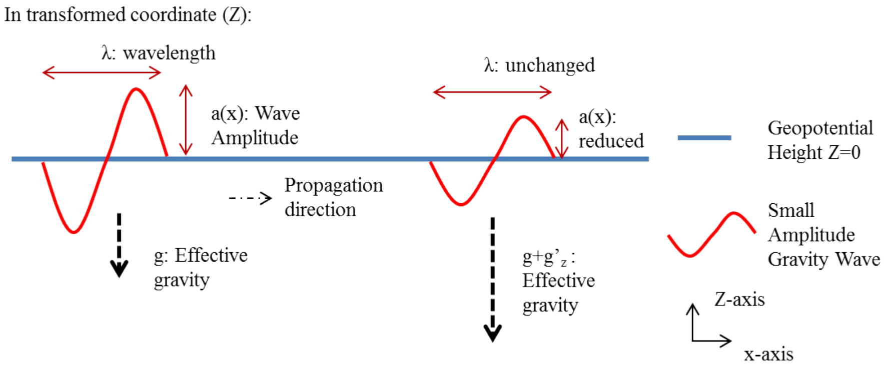
Since the second derivative of $g'_z(x) = \alpha x$ vanishes, the adapted dispersion relation (8.16) is identical to the classical case:

$$\omega^2 = g_0 D_0 k^2$$
$$\omega = \pm\sqrt{g_0 D_0} k$$

Since the water depth D_0 is considered to be uniform, this indicates that when the gravity perturbation is linear in space, both ω and k will not change in space as waves propagate.

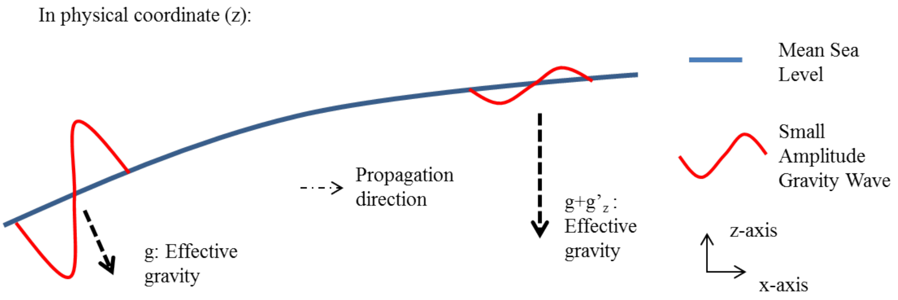
A graphical sketch of the wave propagation will now be presented. Over the Z coordinate, the test case is sketched in figure 8.1.

Figure 8.1: Wave Propagation in Z coordinate



Over the physical z coordinates, figure 8.1 is transformed to be 8.2.

Figure 8.2: Wave Propagation in z coordinate



Short Conclusions

These, altogether, seem to suggest a physical phenomenon that, when a shallow water surface gravity wave travels to a region with stronger (downwards) gravity, the wave amplitude will be damped. The damping of amplitude seems to compensate the increase in gravity, so that wave energy is conserved. In this sense, the gravitational field actually poses control over the wave amplitude, which is not described by the standard shallow water model. Detailed examination of the energy conservation should be conducted.

The result also indicates the possibility to control the gravity wave amplitude on fluid surface, which may have practical industrial applications in the future.

Chapter 9

Research Questions

9.1 Theoretical Analysis

Based on the adapted wave equation (6.57), it is noticed that there is an extra term $-\left[\mathbf{A}\cdot\nabla_H(\eta)\right]$ on the right-hand side of the generalised wave equations. We would like to firstly, identify the magnitude of the extra term $-\left[\mathbf{A}\cdot\nabla_H(\eta)\right]$, and secondly, examine analytically how this extra term will affect the wave dynamics on (x, y, Z, t) and (x, y, z, t) coordinates in terms of order of magnitude and the wave dynamics under different scenarios.

We are also interested in extending the current adapted model to include the effect of rotations: both centrifugal force and coriolis force shall be included in the model. The first natural question is whether there are simple conservative quantities in the new models, such as potential vorticity in classical shallow water model. Another question to answer is whether there are possible coupling between the gravity perturbation and rotation through the conservative quantities.

Another direction to proceed is to derive the wave model for deep water. However it is foreseeable that this will be a challenging task since the momentum equation cannot be simplified easily without the aid of hydrostatic approximation. The task may not be very rewarding also, since the dispersion relation in deep water wave does not depend on the bathymetry. In the adapted shallow water model it is revealed that the variation in gravity perturbation somewhat plays a role similar to the variation in water depth. It is thus questionable whether gravity perturbation will play any important roles in deep water.

Beyond the adapted shallow water model, it is also a natural question to ask whether it is possible to formulate the problem in an alternative way. For example, energy consideration was not carried out in the derivation of adapted shallow water model. It may be possible also to obtain a more general model through variational approach, since the gravitational force field is conservative.

9.2 Numerical Simulation

Followed by theoretical analysis, we would like to verify the analytic results through numerical simulations. If conservative quantities are found in the theoretical analysis, it may be necessary to devise a numerical scheme to preserve the quantity.

It is also proposed that complex scenarios that are incapable to be dealt with analytically, such as non-uniform bathymetry and complex geometry of modelling domain, shall be treated numerically.

9.3 Empirical Analysis

Based on altimeter data , we shall determine the spatial structure of high frequency ocean wave over an ocean sub-surface mountain or trench. Then we shall compare the observed data with numerical simulations.

Another possible empirical data may be obtained from tsunami observation platform. Tsunami wave in deep ocean, due to its long wavelength, is practically modelled by the shallow water equations. Hence this study may also contribute to forecasting of tsunami wave.

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