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THE TERMINATION PRINCIPLE OF MULTIPLE LIMIT CYCLES FOR THE KUKLES CUBIC SYSTEM

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Abstract

We carry out the global bifurcation analysis of the Kukles system representing a planar polynomial dynamical system with arbitrary linear and cubic right-hand sides and having an anti-saddle at the origin. Using the Wintner–Perko termination principle of multiple limit cycles, we solve the problem on the maximum number and distribution of limit cycles in this system. Numerical experiments are done to illustrate the obtained results.

Key words

Kukles cubic system; Wintner–Perko termination principle; field rotation parameter; bifurcation; limit cycle.

1 Introduction

In this paper, we continue studying the Kukles cubic system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + \delta y + a_1x^2 + a_2xy + a_3y^2 \\ &\quad + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3.\end{aligned} \tag{1}$$

I. S. Kukles was the first who began to study (1) solving the center-focus problem for this system in 1944: he gave the necessary and sufficient conditions for $O(0,0)$ to be a center for (1) with $a_7 = 0$ [Kukles, 1944]. Later, system (1) was studied by many mathematicians. In [Lloyd and Pearson, 1992], for example, the necessary and sufficient center conditions for arbitrary system (1), when $a_7 \neq 0$, were conjectured. In [Rousseau et al., 1995], global qualitative pictures and bifurcation diagrams of a reduced Kukles system ($a_7 = 0$) with a center were given. In [Wu at al., 1999], the global analysis of system (1) with two weak foci was carried out. In [Ye and Ye, 2001], the number of singular points under the conditions of a center or a weak focus for (1) was investigated. In [Zang at al., 2008], new distributions of limit cycle for the Kukles system were obtained. In [Robanal, 2014], an accurate bound of the maximum number of limit cycles in a class of Kukles type systems was provided.

A new impulse to the study of limit cycles was given by ideas and methods from bifurcation theory; see [Gaiko, 2003].

There are three principal bifurcations of limit cycles.

1) The *Andronov–Hopf bifurcation* from a singular point of the center or focus type (Figure 1).

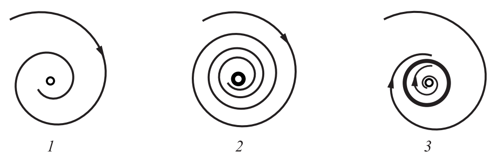

Figure 1. The Andronov–Hopf bifurcation.

2) The *separatrix cycle bifurcation* from a singular closed trajectory (Figure 2).

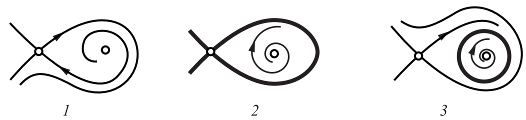

Figure 2. The separatrix cycle bifurcation.

3) The *multiple limit cycle bifurcation* (Figure 3).

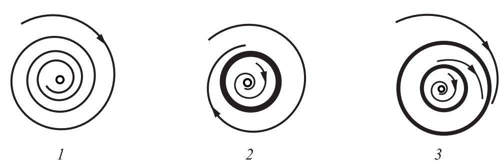

Figure 3. The multiple limit cycle bifurcation.

The first bifurcation has been studied completely only for quadratic systems: N. N. Bautin has proved that the maximum number of limit cycles appearing from a singular point under quadratic perturbations is equal to three (this number is called the *cyclicity of a singular point*). For cubic systems, as has been shown by H. Żołądek, the cyclicity of a singular point is at least eleven. The second bifurcation has been intensively studying by F. Dumortier, R. Roussarie, C. Rousseau and other mathematicians. Now we have a classification of separatrix cycles and know the cyclicity of the most of them. The last bifurcation is the most complicated. Multiple limit cycles have been considered, for instance, by L. M. Perko.

Unfortunately, all these bifurcations of limit cycle are local bifurcations: we consider only a sufficiently small neighborhood of either a singular point or a separatrix cycle, or a multiple limit cycle and the corresponding sufficiently small neighborhood of the parameter space.

To complete the study of limit cycles, it requires a qualitative investigation on the whole (both on the whole phase plane and on the whole parameter space), i. e., it requires a global bifurcation theory. This idea was introduced for the first time by N. P. Erugin. Then, we have to understand how to control the limit cycle bifurcations. The best way to do it is to use field rotation parameters, the possibility of application of which for the study of limit cycles was substantiated by G. F. D. Duff. And finally, we should connect all limit cycle bifurcations. This idea came from theory of higher-dimensional dynamical systems, being the essence of Wintner's principle of natural termination, and it was later used by L. M. Perko for the study of the global behavior of multiple limit cycles in the two-dimensional case. See [Gaiko, 2003] for more detail.

In [Gaiko and van Horssen, 2004], we constructed a canonical cubic dynamical system of Kukles type and carried out the global qualitative analysis of a special case of the Kukles system corresponding to a generalized cubic Liénard equation. In particular, it was shown that the foci of such a Liénard system could be at most of second order and that such system could have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system were studied. As a result, a classification of all possible types of separatrix cycles for the generalized cubic Liénard system was obtained and all possible distributions of its limit cycles were found.

In [Gaiko, 2003; Gaiko, 2008], we also presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3 : 1)$ is their only possible distribution. We established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems as well. In [Gaiko, 2012a], e. g., we presented a solution of Smale's thirteenth problem [Smale, 1998] proving that the classical Liénard system with a polynomial of degree $2k + 1$ could have at most k limit cycles and we could conclude that our results agree with the conjecture of [Lins et al., 1977] on the maximum number of limit cycles for the classical Liénard polynomial system. In [Gaiko, 2012b], under some assumptions on the parameters, we found the maximum number of limit cycles and their possible distribution for the general Liénard polynomial system. In [Gaiko, 2011], we studied multiple limit cycle bifurcations in the well-known FitzHugh–Nagumo neuronal model. In [Broer and Gaiko, 2010; Gaiko, 2016], we completed the global qualitative analysis of quartic dynamical systems which model the dynamics of the populations of predators and their prey in a given ecological system.

System (1) can be considered as a generalized Liénard cubic system. There are many examples in the natural sciences and technology in which such and related systems are applied; see [Gaiko, 2012b]. Such systems are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems with damping and restoring (stiffness), when modeling wind rock phenomena and surge in jet engines. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently, e. g., a Liénard system has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator. There are also some examples of using Liénard type systems in ecology and epidemiology [Gaiko, 2012b]. To control natural processes occurring in such systems, especially related to periodicity and oscillations, we use so-called field rotation parameter; see [Gaiko, 2003].

In this paper, we will use the obtained results and our bifurcational geometric approach for studying limit cycle bifurcations of Kukles cubic system (1). In Section 2, we construct new canonical systems with field rotation parameters for studying global bifurcations of limit cycles of (1). In Section 3, using the Wintner–Perko termination principle of multiple limit cycles, we give a solution of the problem on the maximum number and distribution of limit cycles for Kukles system (1). This is related to the solution of Hilbert's sixteenth problem on the maximum number and distribution of limit cycles in planar polynomial dynamical systems [Gaiko, 2003]. Numerical experiments are also done to illustrate the theoretical results.

2 Canonical Systems

Applying Erugin's two-isocline method [Gaiko, 2003] and studying the rotation properties [Bautin and Leontovich, 1990; Gaiko, 2003; Perko, 2002] of all parameters of (1), we prove the following theorem.

Theorem 2.1. *Kukles system* (1) *with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= q(x) + (\alpha_0 - \beta + \gamma + \beta\, x + \alpha_2\, x^2)\, y \\ &\quad + (c + dx)\, y^2 + \gamma\, y^3 \equiv Q(x,y), \end{aligned} \tag{2}$$

where
1) $q(x) = -x + (1+1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$
or
2) $q(x) = -x + b\, x^3,\ b = 0, -1,$
or
3) $q(x) = -x + x^2;$
$\alpha_0, \alpha_2, \gamma$ *are field rotation parameters and* β *is a semi-rotation parameter.*

Proof. System (1) has two basic isoclines: the cubic curve

$$-x + \delta y + a_1 x^2 + a_2 xy + a_3 y^2$$
$$+ a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3 = 0$$

as the isocline of "zero" and the straight line $y = 0$ as the isocline of "infinity".

These isoclines intersect at least at one point: at the origin which is an anti-saddle (a center, a focus or a node). Besides, (1) can have two more finite singularities (two saddles or a saddle and an anti-saddle) or one additional finite singular point (a saddle or a saddle-node), or no other finite singularities at all. All these singular points lie on the x-axis ($y = 0$), and their coordinates are defined by the equation

$$q(x) \equiv -x + a_1 x^2 + a_4 x^3 = 0 \tag{3}$$

depending just on the parameters a_1 and a_4.

Without loss of generality, $q(x)$ as given by (3) can be written in the following forms:
1) $q(x) \equiv -(1/a)x(x-1)(x-a)$
$= -x + (1+1/a)\, x^2 - (1/a)\, x^3,\ a = \pm 1, \pm 2$ or
2) $q(x) \equiv -x(1 - bx^2) = -x + b\, x^3,\ b = 0, -1,$ or
3) $q(x) \equiv -x(1 - x) = -x + x^2.$

It means that, together with the anti-saddle in $(0, 0)$, we can have also:
1) two saddles: at $(1, 0)$ and $(-2, 0)$ for $a = -2$ or at $(1, 0)$ and $(-1, 0)$ for $a = -1$; or a saddle at $(1, 0)$ and an anti-saddle at $(2, 0)$ for $a = 2$; or a saddle-node at $(1, 0)$ for $a = 1$;
2) no other finite singularities;
3) one saddle at $(1, 0)$.

At infinity, system (1) has at most four singular points: one of them is in the vertical direction and the others are defined by the equation

$$a_7 u^3 + a_6 u^2 + a_5 u + a_4 = 0,\ u = y/x. \tag{4}$$

Instead of the parameters $\delta,\ a_2,\ a_3,\ a_5,\ a_6,\ a_7,$ also without loss of generality, we can introduce some new parameters $c, d, \alpha_0, \alpha_2, \beta, \gamma$:

$$\delta = \alpha_0 - \beta + \gamma;\ a_2 = \beta;\ a_3 = c;$$
$$a_5 = \alpha_2;\ a_6 = d;\ a_7 = \gamma$$

to have more regular rotation of the vector field of (1) [Gaiko, 2003].

Then, taking into account $q(x)$, equation (4) is written in the form

$$\begin{aligned} \gamma\, u^3 + d\, u^2 + \alpha_2\, u + s = 0, \\ u = y/x,\ s = -1/a, b. \end{aligned} \tag{5}$$

Thus, we have reduced (1) to canonical system (2).

If $c = d = \alpha_0 = \alpha_2 = \beta = \gamma = 0$, we obtain the following Hamiltonian systems:

$$\begin{aligned} \dot{x} = y,\ \dot{y} = -x + (1 + 1/a)\, x^2 - (1/a)\, x^3, \\ a = \pm 1, \pm 2; \end{aligned} \tag{6}$$

$$\dot{x} = y,\ \dot{y} = -x + b\, x^3,\ b = 0, -1; \tag{7}$$

$$\dot{x} = y,\ \dot{y} = -x + x^2. \tag{8}$$

All their vector fields are symmetric with respect to the x-axis, and, besides, the fields of system (6) with $a = 2, -1$ and system (7) with $b = 0, -1$ are symmetric with respect to the straight line $x = 1$ and centrally symmetric with respect to the point $(1, 0)$. Systems (6)–(8) have the following Hamiltonians, respectively:

$$H(x,y) = x^2 - (2/3)\,(1+1/a)\, x^3 + (1/(2a))\, x^4 + y^2,$$
$$a = \pm 1, \pm 2;$$

$$H(x,y) = x^2 - (b/2)\, x^4 + y^2,\ b = 0, -1;$$

$$H(x,y) = x^2 - (2/3)\, x^3 + y^2.$$

If $\alpha_0 = \alpha_2 = \beta = \gamma = 0$, we will have the system

$$\dot{x} = y, \quad \dot{y} = q(x) + (c + dx)\, y^2 \tag{9}$$

and the corresponding equation

$$\frac{dy}{dx} = \frac{q(x) + (c + dx)\, y^2}{y} \equiv F(x,y). \tag{10}$$

Since $F(x, -y) = -F(x, y)$, the direction field of (10) (and the vector field of (9) as well) is symmetric with respect to the x-axis. It follows that system (9) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, the parameters c and d in system (2) can be fixed.

To prove that the parameters α_0, α_2, γ and β rotate the vector field of (2), let us calculate the following determinants:

$$\Delta_{\alpha_0} = P\,Q'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0,$$
$$\Delta_{\alpha_2} = P\,Q'_{\alpha_2} - QP'_{\alpha_2} = x^2 y^2 \geq 0,$$
$$\Delta_{\gamma} = P\,Q'_{\gamma} - QP'_{\gamma} = y^2(1+y^2) \geq 0,$$
$$\Delta_{\beta} = P\,Q'_{\beta} - QP'_{\beta} = (x-1)\,y^2.$$

By definition of a field rotation parameter [Bautin and Leontovich, 1990; Gaiko, 2003], for increasing each of the parameters α_0, α_2 and γ, under the fixed others, the vector field of system (2) is rotated in positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2) is rotated in negative direction (clockwise). For increasing the parameter β, under the fixed others, the vector field of system (2) is rotated in positive direction (counterclockwise) in the half-plane $x > 1$ and in negative direction (clockwise) in the half-plane $x < 1$ and vice versa for decreasing this parameter. We will call such a parameter as a semi-rotation one.

Thus, for studying limit cycle bifurcations of (1), it is sufficient to consider canonical system (2) containing the field rotation parameters α_0, α_2, γ and the semi-rotation parameter β. The theorem is proved. □

3 The Wintner–Perko Termination Principle and Global Bifurcations of Limit Cycles

By means of our bifurcational geometric approach [Gaiko and van Horssen, 2004; Gaiko, 2012a; Gaiko, 2012b; Gaiko, 2015; Gaiko, 2016], we will consider now the Kukles cubic system in the form (when $a = 2$):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(1/2)x(x-1)(x-2) \\ &\quad + (\alpha_0 - \beta + \gamma + \beta\,x + \alpha_2\,x^2)\,y \\ &\quad + (c + dx)\,y^2 + \gamma\,y^3. \end{aligned} \tag{11}$$

All other Kukles systems can be considered in a similar way.

In [Gaiko, 2003], the Wintner–Perko termination principle which connects the main bifurcations of limit cycles [Perko, 2002] was used for the global analysis of limit cycle bifurcations. Let us formulate this principle for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{12}$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 3.1 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system* (12) *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* (12) *which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (12), *which is typically a fine focus of multiplicity m, or on a* (*compound*) *separatrix cycle of* (12), *which is also typically of multiplicity m.*

The proof of the Wintner–Perko termination principle for general polynomial system (12) with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{13}$$

with a single parameter $\lambda \in \mathbf{R}$; see [Gaiko, 2003; Perko, 2002]. In particular, if λ is a field rotation parameter of (13), it is valid the following Perko's theorem on monotonic families of limit cycles.

Theorem 3.2. *If L_0 is a nonsingular multiple limit cycle of* (13) *for $\lambda = \lambda_0$, then L_0 belongs to a one-parameter family of limit cycles of* (13); *furthermore*:

1) *if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0;*

2) *if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i. e., there is a fold bifurcation at λ_0.*

Using Theorems 3.1 and 3.2, we will prove the following theorem.

Theorem 3.3. *There exists no system* (1) *having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, system* (1) *cannot have either a multiplicity-four limit cycle or four limit cycles around a singular point, and the maximum multiplicity or the maximum number of limit cycles surrounding a singular point is equal to three. Moreover, system* (1) *can have at most four limit cycles with their only possible* $(3\!:\!1)$*-distribution.*

Proof. The proof of this theorem is carried out by contradiction. Consider canonical systems (11) with three field rotation parameters α_0, α_2, γ and a semi-rotation parameter β which is also a field rotation one in the half-plane $x < 1$. Suppose this system has four limit cycles around the origin O. Then we get into some domain bounded by three fold bifurcation surfaces forming a swallow-tail bifurcation surface of multiplicity-four limit cycles in the space of the field rotation parameters α_0, α_2, γ and β. Cf. Figure 4 [Gaiko, 2003].

The corresponding maximal one-parameter family of multiplicity-four limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity five (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-five limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-five limit cycles by a field-rotation parameter, according to Theorem 3.2, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 3.1), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know at least the cyclicity of the singular point [Zang et al., 2008] which is equal to three, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate. Cf. Figure 5 [Gaiko, 2003].

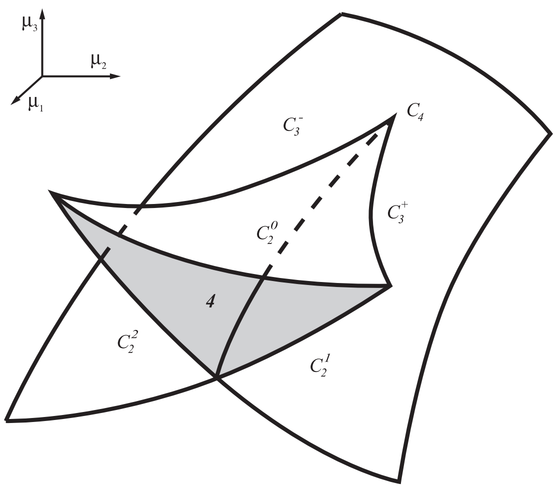

Figure 4. The swallow-tail bifurcation surface.

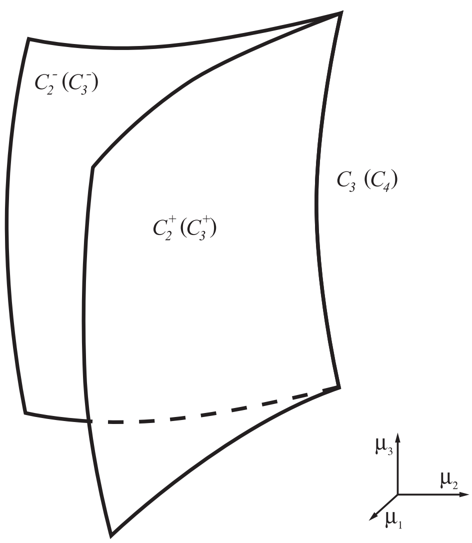

Figure 5. The bifurcation curve (one-parameter family) of multiple limit cycles.

If the maximal one-parameter family of multiplicity-four limit cycles is not cyclic, on the same principle (Theorem 3.1), this again contradicts to the result of [Zang et al., 2008] not admitting the multiplicity of limit cycles higher than three. It follows that the maximum multiplicity or the maximum number of limit cycles surrounding the origin is equal to three.

Consider other logical possibilities. For example, suppose that system (11) has for $\alpha_0 > 0$, $\alpha_2 < 0$ and $\beta > 0$ three limit cycles in the $(2:1)$-distribution: two cycles around the point O and the only one around A. Let us show impossibility of obtaining additional limit cycles around the point A by means of the parameter γ. We can suppose that a semi-stable cycle appears around A on increasing this parameter for $\gamma > 0$. Then, applying the Wintner–Perko termination principle (Theorem 3.1), we can show that the corresponding maximal one-parameter family of multiplicity-three limit cycles parameterized by another field rotation parameter, e. g., α_2, cannot terminate in the focus A, since it will be a rough one for $\gamma > 0$. The only additional limit cycle in system (11) can appear from the focus O for the set of $\alpha_0 > 0$, $\alpha_2 < 0$, $\beta > 0$ and $\gamma > 0$, when $\gamma = \beta - \alpha_0$.

All other possibilities, concerning also big limit cycles from infinity, can be considered in a similar way. It follows that system (11) can have at most four limit cycles and only in the $(3:1)$-distribution. The same conclusion can be done for system (1). The theorem is proved. □

We have done also numerical simulations supporting our results based on a Runge-Kutta method using a so-called function of limit cycles introduced in [Gaiko, 2003] which is a function of a field rotation parameter depending on a coordinate of the limit cycle and applying a flow curvature method [Ginoux, 2009] and some other numerical methods [Van 't Wout et al., 2016; Vuik et al., 2015].

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