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Abstract

In this paper, some theoretical results on the eigenvalue analysis of the SIMPLER preconditioning for incompressible flow is presented. Some formulations have been derived to characterize the spectrum of the preconditioned matrix. These results could be helpful for the practical use of the SIMPLER preconditioning. Some numerical tests are reported.

Key words. preconditioning, SIMPLER preconditioner, eigenvalues, Schur complement.

AMS subject classification. 65N22

1 Introduction.

In this paper, we will analyze the spectrum of the SIMPLER preconditioned matrix which is resulted from the SIMPLER preconditioning collaborated with some Krylov subspace iterative methods solving the large sparse linear algebraic system

$$\begin{pmatrix} Q & G \\ G^T & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.1}$$

where $Q \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}, m \leqslant n, \det(Q) \neq 0, \operatorname{rank}(G) = m$. This large linear system is often obtained from discretization and linearization of incompressible Navier-Stokes equations, for which, $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$ in (1.1) are the velocity vector and the pressure vector respectively.

In [6, 7], Vuik et al. proposed GCR-SIMPLE and GCR-SIMPLER algorithms for solving the large linear system (1.1). The algorithm can be considered as a combination of

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the Krylov subspace method GCR [3] with the SIMPLE(R) algorithm[5]. In these combined algorithms, the inner iterations of SIMPLE and SIMPLER algorithms are preconditioners for the GCR method. In our numerical tests, we have observed that the SIMPLE and SIMPLER preconditioning are effective and competitive for practical use. Some theoretical results on the eigenvalue analysis for the SIMPLE preconditioning had been given in our former technical report [4]. We have also observed that the SIMPLER preconditioning is even more efficient than SIMPLE preconditioning in terms of the iteration number and the eigenvalue distributions.

In this paper, we are concentrated to the spectral analysis for the SIMPLER preconditioning, which might be helpful to the theoretical explanation of the observation. Several formulations are derived to describe the spectrum of the SIMPLER preconditioned matrix. Some numerical tests are reported.

In the remaining parts of this paper, the linear system (1.1) is abbreviated as $Ax = b$, where $A \in \mathbb{R}^{(n+m)\times(n+m)}, b \in \mathbb{R}^{n+m}$. Notations have the same meaning with references [7, 6] and [4]. $\sigma(A)$ represents the set of all eigenvalues of matrix A, for example. Besides, we assume that the matrix Q, its diagonal matrix $D := \mathrm{diag}(Q)$, and its Jacobi iteration matrix J ($J := D^{-1}(D - Q)$), are all nonsingular in this paper.

2 Description of the spectrum of the SIMPLER preconditioned matrix.

Consider the right preconditioning to the linear system (1.1)

$$AP^{-1}y = b, \quad x = P^{-1}y. \tag{2.1}$$

When the SIMPLER algorithm is used as preconditioning, it is equivalent to choose the preconditioner P^{-1} as[7, 8]

$$P^{-1} = B_R M_R^{-1} - B_R M_R^{-1} A M_L^{-1} B_L + M_L^{-1} B_L, \tag{2.2}$$

where,

$$B_R = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \quad M_R = \begin{pmatrix} Q & O \\ G^T & R \end{pmatrix},$$

$$B_L = \begin{pmatrix} I & O \\ -G^T D^{-1} & I \end{pmatrix}, \quad M_L = \begin{pmatrix} Q & G \\ O & R \end{pmatrix},$$

and

$$D = \mathrm{diag}(Q), R = -G^T D^{-1} G.$$

We can get easily that

$$B_R^{-1} = \begin{pmatrix} I & D^{-1}G \\ O & I \end{pmatrix}, \quad M_R^{-1} = \begin{pmatrix} Q^{-1} & O \\ -R^{-1}G^T Q^{-1} & R^{-1} \end{pmatrix},$$

$$B_L^{-1} = \begin{pmatrix} I & O \\ G^T D^{-1} & I \end{pmatrix}, \quad M_L^{-1} = \begin{pmatrix} Q^{-1} & -Q^{-1}GR^{-1} \\ O & R^{-1} \end{pmatrix}.$$

We call this preconditioning as SIMPLER preconditioning, and the preconditioner P^{-1} as SIMPLER preconditioner.

It is not very difficult to verify that

$$AB_R M_R^{-1} = \begin{pmatrix} I - (I - QD^{-1})GR^{-1}G^T Q^{-1} & (I - QD^{-1})GR^{-1} \\ O & I \end{pmatrix}, \tag{2.3}$$

$$AM_L^{-1} B_L = \begin{pmatrix} I & O \\ G^T Q^{-1}(I + GR^{-1}G^T D^{-1}) & -G^T Q^{-1}GR^{-1} \end{pmatrix},$$

and hence

$$AB_R M_R^{-1} AM_L^{-1} B_L = \begin{pmatrix} I + (I - QD^{-1})GR^{-1}G^T Q^{-1}GR^{-1}G^T D^{-1} & -(I - QD^{-1})GR^{-1}G^T Q^{-1}GR^{-1} \\ G^T Q^{-1} + G^T Q^{-1}GR^{-1}G^T D^{-1} & -G^T Q^{-1}GR^{-1} \end{pmatrix}.$$

It follows that the SIMPLER preconditioned matrix is

$$\widetilde{A} := AP^{-1} = AB_R M_R^{-1} - AB_R M_R^{-1} AM_L^{-1} B_L + AM_L^{-1} B_L = \begin{pmatrix} I - (I - QD^{-1})GR^{-1}G^T Q^{-1}(I + GR^{-1}G^T D^{-1}) & (I - QD^{-1})GR^{-1}(I + G^T Q^{-1}GR^{-1}) \\ O & I \end{pmatrix}. \tag{2.4}$$

For SIMPLER preconditioning, we get a result concerning with its spectrum:

Proposition 2.1. *For the SIMPLER preconditioned matrix $\widetilde{A}$,*

1. *1 is an eigenvalue with multiplicity at least of m, and*
2. *the remaining eigenvalues are $1 - \mu_i,\ i = 1, 2, \cdots, n$, where*

$$\mu_i \in \sigma(C), C := (I - QD^{-1})GR^{-1}G^T Q^{-1}(I + GR^{-1}G^T D^{-1}). \tag{2.5}$$

Remark 2.1. The matrix $(I - QD^{-1})GR^{-1}G^T Q^{-1}$ in equation (2.5) is closely related with the SIMPLE preconditioned matrix. We refer to the equation (2.3) in [4] for reference.

For nonsingular matrices A and P, $P^{-1}A$ has the same spectrum with AP^{-1} (see reference [1]). So, we can derive another formulation for the spectrum of the SIMPLER preconditioned matrix $\widetilde{A}$. Since that

$$B_R M_R^{-1} A = \begin{pmatrix} I & (I + D^{-1}GR^{-1}G^T)Q^{-1}G \\ O & -R^{-1}G^T Q^{-1}G \end{pmatrix}, \tag{2.6}$$

$$M_L^{-1} B_L A = \begin{pmatrix} I + Q^{-1}GR^{-1}G^T(D^{-1}Q - I) & O \\ R^{-1}G^T(I - D^{-1}Q) & I \end{pmatrix},$$

and

$$
\begin{array}{l}
B_R M_R^{-1} A M_L^{-1} B_L A = \\
\begin{pmatrix} I + D^{-1}GR^{-1}G^TQ^{-1}GR^{-1}G^T(I - D^{-1}Q) & (I + D^{-1}GR^{-1}G^T)Q^{-1}G \\ R^{-1}G^TQ^{-1}GR^{-1}G^T(D^{-1}Q - I) & -R^{-1}G^TQ^{-1}G \end{pmatrix}.
\end{array}
$$

It follows that

$$
\begin{array}{l}
P^{-1}A = B_R M_R^{-1} A - B_R M_R^{-1} A M_L^{-1} B_L A + M_L^{-1} B_L A = \\
\begin{pmatrix} I + (I + D^{-1}GR^{-1}G^T)Q^{-1}GR^{-1}G^T(D^{-1}Q - I) & O \\ R^{-1}G^T(I + Q^{-1}GR^{-1}G^T)(I - D^{-1}Q) & I \end{pmatrix}.
\end{array} \tag{2.7}
$$

Remark 2.2. In the equation (2.6), the matrix $B_R M_R^{-1} A = P_S^{-1} A$ is the (left) SIMPLE preconditioned matrix. It has the same spectrum as $AP_S^{-1} := \widetilde{A_S}$. From equation (2.6), we get to know that

$$
\sigma(\widetilde{A_S}) = \{1\} \cup \sigma(R^{-1}S), \tag{2.8}
$$

in which, $S = -G^TQ^{-1}G$ is the Schur complement of the original matrix A. This result had been obtained in [4] by a different derivation.

3 More Formulations for the the spectrum of the SIMPLER Preconditioned Matrix $\widetilde{A}$

In this section, we will give some different formulations of the spectrum of $\widetilde{A}$ by using the singular value decomposition.

For matrix $D^{-\frac{1}{2}}G \in \mathbb{R}^{n\times m}$, making the singular value decomposition for it, we have

$$
D^{-\frac{1}{2}}G = U\Sigma V^T, \tag{3.1}
$$

in which,$U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times m}$ are unitary matrices, and

$$
\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \\ & O & & \end{pmatrix} \in \mathbb{R}^{n\times m},
$$

$\sigma_i, i = 1, 2, \cdots, m$, are the singular values of the matrix $D^{-\frac{1}{2}}G$, which are all positive numbers since $\mathrm{rank}(D^{-\frac{1}{2}}G) = m$. So,

$$
\begin{aligned}
G &= D^{\frac{1}{2}}U\Sigma V^T, \\
R &= -(U\Sigma V^T)^T(U\Sigma V^T) = -V\Sigma^T\Sigma V^T, \\
R^{-1} &= -V(\Sigma^T\Sigma)^{-1}V^T, \\
GR^{-1}G^T &= (D^{\frac{1}{2}}U\Sigma V^T)(-V\Sigma^T\Sigma V^T)^{-1}(D^{\frac{1}{2}}U\Sigma V^T)^T \\
&= -D^{\frac{1}{2}}U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^TU^TD^{\frac{1}{2}}.
\end{aligned}
$$

It follows that matrix C as given in (2.5) can be written as

$$\begin{aligned} C &= (I - QD^{-1})GR^{-1}G^TQ^{-1}(I + GR^{-1}G^TD^{-1}) \\ &= (I - QD^{-1})(-D^{\frac{1}{2}}U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^TU^TD^{\frac{1}{2}})Q^{-1}(I - D^{\frac{1}{2}}U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^TU^TD^{-\frac{1}{2}}) \end{aligned} \tag{3.2}$$

If we take the notation $N := \Sigma(\Sigma^T\Sigma)^{-1}$, then

$$\begin{aligned} N &= \Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T \\ &= \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \\ & O & & \end{pmatrix}_{n\times m} \begin{pmatrix} \frac{1}{\sigma_1^2} & & & \\ & \frac{1}{\sigma_2^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_m^2} \end{pmatrix}_{m\times m} \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & O \\ & & & \sigma_m & \end{pmatrix}_{m\times n} \\ &= \begin{pmatrix} 1 & & & & \\ & 1 & & O & \\ & & \ddots & & \\ & & & 1 & \\ & O & & O & \end{pmatrix}_{n\times n} = \begin{pmatrix} I_m & O \\ O & O \end{pmatrix} \in \mathbb{R}^{n\times n}. \end{aligned}$$

So, the matrix C can be simplified as

$$C = (I - QD^{-1})(-D^{\frac{1}{2}}UNU^TD^{\frac{1}{2}})Q^{-1}(I - D^{\frac{1}{2}}UNU^TD^{-\frac{1}{2}}). \tag{3.3}$$

The matrix C is spectral equivalent to

$$\begin{aligned} F: &= U^TD^{-\frac{1}{2}}CD^{\frac{1}{2}}U \\ &= -NU^TD^{\frac{1}{2}}Q^{-1}D^{\frac{1}{2}}U(I - N) + U^TD^{-\frac{1}{2}}QD^{-\frac{1}{2}}UNU^TD^{\frac{1}{2}}Q^{-1}D^{\frac{1}{2}}U(I - N). \end{aligned} \tag{3.4}$$

If we denote

$$M := U^TD^{-\frac{1}{2}}QD^{-\frac{1}{2}}U \in \mathbb{R}^{n\times n},$$

then

$$M^{-1} = U^TD^{\frac{1}{2}}Q^{-1}D^{\frac{1}{2}}U \in \mathbb{R}^{n\times n}.$$

So,

$$F := U^TD^{-\frac{1}{2}}CD^{\frac{1}{2}}U = -NM^{-1}(I - N) + MNM^{-1}(I - N),$$

that is

$$F = (M - I)NM^{-1}(I - N). \tag{3.5}$$

We partition the matrix M according to the structure of N to the sub-block form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $M_{11} \in \mathbb{R}^{m\times m}$ and $M_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ are square matrices, and denote its inverse M^{-1} with the same structure

$$M^{-1} := \widehat{M} = \begin{pmatrix} \widehat{M_{11}} & \widehat{M_{12}} \\ \widehat{M_{21}} & \widehat{M_{22}} \end{pmatrix},$$

where $\widehat{M_{11}} \in \mathbb{R}^{m\times m}$ and $\widehat{M_{22}} \in \mathbb{R}^{(n-m)\times(n-m)}$ are also square matrices, then, we get

$$NM^{-1}(I-N) = \begin{pmatrix} I_m & O \\ O & O \end{pmatrix} \begin{pmatrix} \widehat{M_{11}} & \widehat{M_{12}} \\ \widehat{M_{21}} & \widehat{M_{22}} \end{pmatrix} \begin{pmatrix} O & O \\ O & I_{n-m} \end{pmatrix} = \begin{pmatrix} O & \widehat{M_{12}} \\ O & O \end{pmatrix}.$$

So,

$$F = \begin{pmatrix} M_{11} - I_m & M_{12} \\ M_{21} & M_{22} - I_{n-m} \end{pmatrix} \begin{pmatrix} O & \widehat{M_{12}} \\ O & O \end{pmatrix} = \begin{pmatrix} O & (M_{11} - I_m)\widehat{M_{12}} \\ O & M_{21}\widehat{M_{12}} \end{pmatrix} \tag{3.6}$$

Notice that the relation

$$M_{21}\widehat{M_{12}} + M_{22}\widehat{M_{22}} = I_{n-m}$$

holds since that $MM^{-1} = I$. Finally, we have derived the following expression for F:

$$F = \begin{pmatrix} O & (M_{11} - I_m)\widehat{M_{12}} \\ O & I_{n-m} - M_{22}\widehat{M_{22}} \end{pmatrix}. \tag{3.7}$$

The characteristic polynomial of F is

$$\det(\lambda I - F) = \lambda^m \det(\lambda I_{n-m} - I_{n-m} + M_{22}\widehat{M_{22}}). \tag{3.8}$$

This equation together with proposition 2.1 lead to the following proposition concerning with the spectrum of the SIMPLER preconditioned matrix $\widetilde{A}$:

Proposition 3.1. *For the SIMPLER preconditioned matrix $\widetilde{A}$,*

1. *1 is an eigenvalue with multiplicity of $2m$, and*

2. *the remaining $n-m$ eigenvalues are $1-\mu_i$, $i = 1, 2, \cdots, n-m$, where μ_i , $i = 1, 2, \cdots, n-m$, are the roots of the polynomial defined by (3.8).*

Remark 3.1. If the matrix $M_{22}\widehat{M_{22}}$ is close to I_{n-m} in some sense, say, $M_{22}\widehat{M_{22}} = I_{n-m}+E$, where $\|E\|_2 < \delta$, then according the matrix perturbation theory, see reference [2, pp.97] and [9], we have

$$\max_i |1-(1-\mu_i)| = \max_i |\mu_i| \leqslant \alpha(n-m)^{\frac{1}{n-m}}(2\beta)^{\frac{n-m-1}{n-m}}\delta^{\frac{1}{n-m}}, \tag{3.9}$$

where, $\beta = \max(1, \|I_{n-m} + E\|), \alpha = n-m$ or $n-m-1$. We have seen frequently that the eigenvalues of the SIMPLER preconditioned matrices are clustered in a quite small region around 1 in the complex plane.

Remark 3.2. From the definition of matrix M, it can be observed that $D^{-\frac{1}{2}}QD^{-\frac{1}{2}}$ is just the symmetric diagonal scaling of the matrix Q, and that the matrix M is unitary equivalent to this scaled matrix. It is obvious that both M and M^{-1} are symmetric if the matrix Q is symmetric.

Next, we derive another formulation by only using the sub-blocks of the matrix M (without using any blocks of M^{-1}).

We assume that the block matrix M_{11} is nonsingular. Then the Schur complement of M with respect to M_{11} is defined as

$$S_M := M_{22} - M_{21}M_{11}^{-1}M_{12}. \tag{3.10}$$

We can verify that (see [1, pp.93])

$$\begin{aligned}\widehat{M} = M^{-1} &= \begin{pmatrix} I & -M_{11}^{-1}M_{12} \\ O & I \end{pmatrix} \begin{pmatrix} M_{11}^{-1} & O \\ O & S_M^{-1} \end{pmatrix} \begin{pmatrix} I & O \\ -M_{21}M_{11}^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}S_M^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}S_M^{-1} \\ -S_M^{-1}M_{21}M_{11}^{-1} & S_M^{-1} \end{pmatrix}.\end{aligned}$$

Comparing this expression with that of the matrix $\widehat{M}$, we see that

$$\widehat{M_{22}} = S_M^{-1}.$$

It follows that

$$I_{n-m} - M_{22}\widehat{M_{22}} = I_{n-m} - M_{22}S_M^{-1} = (S_M - M_{22})S_M^{-1}.$$

So, the non-zero eigenvalues of the matrix F is the solution of the following generalized eigenvalue problem

$$(S_M - M_{22})v = \lambda S_M v, \tag{3.11}$$

which is equivalent to

$$M_{22}v = (1-\lambda)S_M v. \tag{3.12}$$

Remark 3.3. If matrix Q is symmetric, then both the matrix M (see remark 3.2) and its Schur complement S_M are also symmetric [1]. So, the eigenvalues of the SIMPLER preconditioned matrix are the solution of a symmetric generalized eigenvalue problem (3.11). This observation might be helpful to obtain some more practical eigenvalue bounds for Stokes equations.

4 Numerical Tests.

We report some numerical test results here.

Example 4.1. *In this example, the coefficient matrix is taken from a discretized Navier-Stokes equations on a* 24×24 *grid [7](lengthy* $= 2$, $\nu = 1$*).The dimensions are* $n = 1200, m = 576,$ *and* $n + m = 1776$. $A \in \mathbb{R}^{1776\times 1776}$ *is a nonsymmetric matrix.*

The eigenvalues of the preconditioned matrix $\widetilde{A}$ were computed by both proposition 2.1 and proposition 3.1. The computing results were the same, which coincided with the theoretical analysis. Spectra of A and $\widetilde{A}$ are plotted in Fig. 4.1 and Fig. 4.2, and some comparisons for the numerical performance of GCR (without any preconditioning), GCR-SIMPLE, and GCR-SIMPLER are listed in Table 4.1.

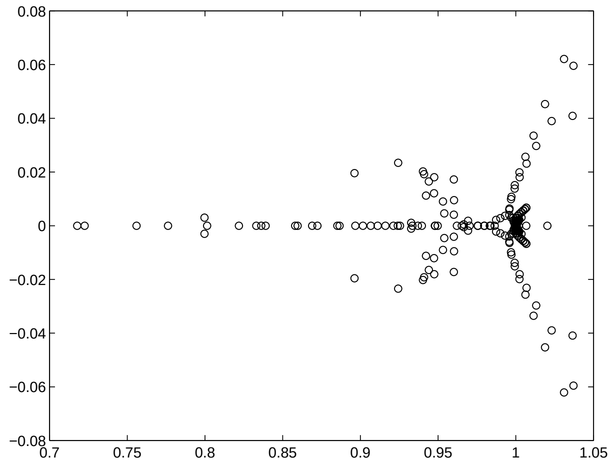

Fig4.1. Spectrum of the SIMPLER preconditioned matrix $\widetilde{A}$. 'o' for the eigenvalues of $\widetilde{A}$.

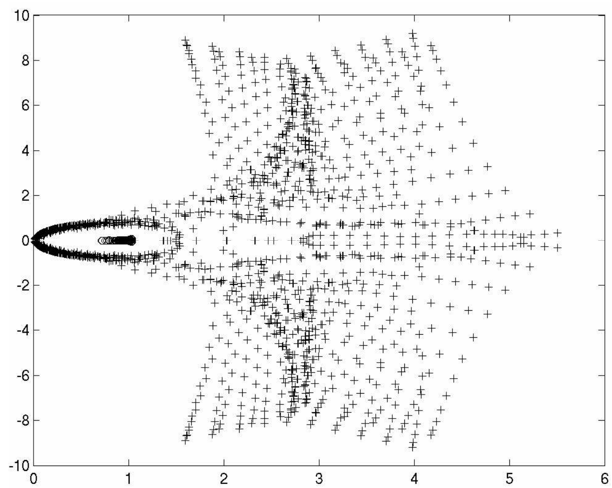

Fig4.2. Spectrum of A and $\widetilde{A}$.
The '+' represents for the eigenvalues of A, while 'o' for that of $\widetilde{A}$.

Table 4.1. Comparison of Example 4.1.

	GCR	GCR-SIMPLE	GCR-SIMPLER
iterations	907	64	10
CPU (s)	189.45	23.08	18.83

Example 4.2. *This test is taken from a discretized Stokes equation on a* 24×24 *grid (lengthy* $= 2$, $\nu = 0.0$*) by removing the Dirichlet boundary conditions. The dimensions are* $n = 1008, m = 576,$ *and* $n + m = 1584$. $A \in \mathbb{R}^{1584\times 1584}$ *is symmetric.*

All the eigenvalues of A and $\widetilde{A}$ are real. Comparisons are listed in table 4.2, and the extreme eigenvalues are listed in Table 4.3.

Table 4.2. Comparison of Example 4.2.

	GCR	GCR-SIMPLE	GCR-SIMPLER
iterations	229	37	12
CPU (s)	37.07	27.04	23.51

Table 4.3. The extreme eigenvalues of A and $\widetilde{A}$ for example 4.2.

matrix	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa(\cdot)$
A	0.0177	5.7453	324.5932
$\widetilde{A}$	1.0000	4.4162	4.4162

Both examples indicate that the spectrum could be effectively improved by using the SIMPLER preconditioner.

5 Concluding remarks.

We have derived several formulations to describe the spectrum of the SIMPLER preconditioned matrix $\widetilde{A}$. These results could be useful for practical computations using the SIMPLER preconditioning. Since that the SIMPLER iteration itself is more complicated than SIMPLE, theoretical aspects of the SIMPLER preconditioning are more difficult than those of the SIMPLE preconditioning. One attractive feature of the SIMPLER preconditioning is that the spectrum of the preconditioned matrix is clustered in a small region, which is relatively far away from the origin. This could be helpful to explain why the SIMPLER preconditioning needs less iterations in practical use.

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