Block Preconditioners for the Incompressible Stokes Problem

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Abstract. This paper discusses the solution of the Stokes problem using block preconditioned iterative methods. Block preconditioners are based on the block factorization of the discretized problem. We focus on two specific types: SIMPLE-type preconditioners and the LSC preconditioner. Both methods use scaling to improve their performance. We test convergence of GCR in combination with these preconditioners both for a constant and a non-constant viscosity Stokes problem.

1 Introduction

Solution of the Stokes problem is a hot topic in the research community nowadays. Discretization of Stokes results in a so-called saddle point type problem. Saddle point problems appear not only in fluid dynamics but also in elasticity problems and some other fields. An iterative method that is developed for one type of saddle point can be applied in other areas as well [1]. Our work is focused on solving the saddle point problem that arises from the finite element discretization of the Stokes problem. In case of stable discretization we can formulate the problem as:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{1}$$

F corresponds to the viscos part, B^T is the discretized gradient operator, and B the divergence operator. We define n as the number of velocity unknowns and m the number of pressure unknowns ($n \geq m$). The system is symmetric positive semi-indefinite.

A Krylov subspace method is employed to solve the incompressible Stokes problem. Convergence of Krylov methods depends on the eigenvalue spectrum. A preconditioning technique is used to improve the convergence. Instead of solving $\mathcal{A}x = b$, one solves a system $P^{-1}\mathcal{A}x = P^{-1}b$, where P is the preconditioner. A good preconditioner should lead to fast convergence and the system of the form $Pz = r$ should be easy to solve. Since system (1) is symmetric indefinite , in the literature [2], preconditioned MINRES [3] is frequently used to solve the Stokes problem. However, one of the requirement of MINRES is that the preconditioner should be symmetric positive definite. This restricts the choice to a

I. Lirkov, S. Margenov, and J. Waśniewski (Eds.): LSSC 2009, LNCS 5910, pp. 829–836, 2010.

block diagonal preconditioner. Since we are using block triangular preconditioners which are not SPD, it is impossible to use MINRES. Therefore, we use GCR [4,7] to solve the Stokes problem. GCR also allows variable preconditioners so we can use inaccurate solvers for the subsystems.

We compare preconditioners that use scaling based on the velocity matrix and the velocity mass matrix. The preconditioners that uses scaling with the velocity mass matrix perform better than the rest of the preconditioners even in the variable viscosity Stokes problem.

1.1 Block Preconditioners

Block preconditioners are based on factorization of formulation (1) and can be written as:

$$\mathcal{A}_s = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b, \tag{2}$$

where $\mathcal{A}_s$ is the system matrix, $\mathcal{L}_b$ the lower block, $\mathcal{D}_b$ the block diagonal and $\mathcal{U}_b$ the upper block matrices represented as:

$$\mathcal{L}_b = \begin{bmatrix} I & 0 \\ M_l^{-1} B^T & I \end{bmatrix}, \ \mathcal{D}_b = \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \text{ and } \mathcal{U}_b = \begin{bmatrix} I & M_u^{-1} B^T \\ 0 & I \end{bmatrix}, \tag{3}$$

where $M_l = M_u = F$ and $S = -BF^{-1}B^T$ is known as the Schur complement matrix.

Since the computation of F^{-1} is not practical for large problem it is necessary to approximate the Schur complement. This is the basis for all block preconditioners. The generalized form of the Schur complement matrix can be written as $\hat{S} = BM^{-1}B^T$, where M^{-1} is a suitable approximation of F^{-1} which is cheap to compute. We define two extra block matrices

$$\mathcal{L}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \tag{4}$$

and

$$\mathcal{U}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}. \tag{5}$$

We will discuss preconditioners based on these definitions in the next section.

1.2 SIMPLE-Type Preconditioners

In this section, we discuss variants of the SIMPLE preconditioner [8]. The SIMPLE preconditioner can be written as

$$P_S = \mathcal{L}_{bt} \mathcal{U}_b, \tag{6}$$

where $M = M_u = D$ with D the diagonal of the velocity matrix. The preconditioner consists of one velocity solve and one pressure solve. It appears from our experiments that the number of iterations increases with an increase in problem size (number of unknowns). However, this preconditioner shows robust convergence with rough accuracy for the subsystems. A variant of SIMPLE, SIMPLER, uses the same approximation for the Schur complement. To solve $P_{SR}Z_{up} = r_{up}$, where P_{SR} is the SIMPLER preconditioner and r_{up} the residual, SIMPLER performs the following steps:

$$z_{up} = \mathcal{U}_{bt}^{-1}\mathcal{L}_{b}^{-1}r_{up}, \ M_l = D, \tag{7}$$

$$Z_{up} = z_{up} + \mathcal{U}_{b}^{-1}\mathcal{L}_{bt}^{-1}r_{up}(r_{up} - \mathcal{A}_s z_{up}), \ M_u = D. \tag{8}$$

The above formulation shows that SIMPLER consists of two velocity solves and two pressure solves. Usually SIMPLER is used with one velocity solve and two pressure solves, it appears that the second velocity solve in (8) can be skipped without any effect on the convergence.

Lemma 1. *In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical when the subsystems are solved by a direct solution method.*

For a proof see [9].

In this paper, we will use hSIMPLER instead of SIMPLER because we have observed that for Stokes, hSIMPLER performs better than SIMPLER. In hSIMPLER (P_{hSR}) we use SIMPLE in the first iteration, whereas SIMPLER is employed afterwards. Another variant of SIMPLE known as MSIMPLER (Modified SIMPLER) uses the same formulation. However, instead of using D, the diagonal of the velocity mass matrix Q is used as scaling matrix. Q is defined by:

$$Q_{i,j} = \int_{\Omega} \phi_i \phi_j d\Omega, \tag{9}$$

where ϕ_j and ϕ_i are the velocity basis functions. MSIMPLER (P_{MSR}) uses $M_l = M = M_u = diag(Q)$ as scaling. This preconditioner gives better convergence than other variants of SIMPLE. Convergence of MSIMPLER is almost independent of the inner accuracy.

1.3 LSC Preconditioner

Next we discuss the diagonally scaled LSC preconditioner. A block triangular preconditioner is used with a special approximation of the Schur complement, defined by:

$$P_{LSC} = \mathcal{L}_{bt}, \tag{10}$$

where $\hat{S}$ is approximated by a least squares commutator approach [10].

The approximation is based on commutator of the convection diffusion operator (on the velocity and pressure space) with the gradient operator. The Schur complement approximation is given as:

$$\hat{S}^{-1} \approx -F_p(BM_1^{-1}B^T)^{-1}, \tag{11}$$

and

$$F_p = (BM_2^{-1}B^T)^{-1}(BM_2^{-1}FM_1^{-1}B^T),$$

where M_1 and M_2 are scaling matrices.

With $M_1 = M_2 = diag(Q)$. The preconditioner shows nice convergence in solving Stokes and Navier-Stokes. Since it also involves two Poisson and one velocity solve, it construction and cost per iteration is comparable to MSIMPLER.

In general the convergence of LSC depends both on the mesh size and the Reynolds number. According to [2] sometimes there is no h-dependency. For LSC some results with stabilized elements have been published.

In this paper, we use scaling based on the diagonal of the velocity matrix ($M_1 = M_2 = D$), which we call LSC_D. The same type of scaling is also used in [11]. In the next section, we compare all these preconditioners with two types of scaling for an isoviscos and a varying viscosity problem. For the subsystems solves, we use both ICCG(0) [5] and AMG preconditioned CG from PETSc (Portable, Extensible Toolkit for Scientific Computation) [6].

2 Numerical Experiments

We divide this section into two parts. In the first part, we solve the Stokes problem with constant viscosity and in the second part, a variable viscosity Stokes problem is solved. The Stokes problem is solved up to an accuracy of 10^{-6}. The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol$, where r^k is the residual at the kth step of Krylov subspace method, b is the right - hand side, and tol is the desired tolerance value. In all experiments, the Stokes equation discretized with Taylor-Hood Q2-Q1 elements.

2.1 Isoviscos Problem

In this section, we solve the driven cavity Stokes problem by employing preconditioners discussed in this paper. The definition of region and boundary conditions is given on the left-hand side of Fig. 1.

The velocity subsystem is solved with AMG preconditioned CG and the pressure subsystem with ICCG(0). One V(1,1) cycle of AMG is used per preconditioning step in CG. In SIMPLE, MSIMPLER and LSC we compute the residual with accuracy 10^{-1} for velocity and 10^{-2} for the pressure subsystems. In hSIMPLER and LSC_D we keep the inner accuracy 10^{-6} for both subsystems. This is done since the convergence of both preconditioners depends on the inner accuracy.

A general convergence profile of block preconditioners discussed in this paper is given in Fig. 2 that shows that the pairs of LSC and MSIMPLER and LSC_D and hSIMPLER have the same convergence pattern. Compared to other preconditioners, SIMPLE shows slow convergence.

From Table 1, it is clear that MSIMPLER and LSC outperform other preconditioners. MSIMPLER and LSC show less dependence on the grid size compared

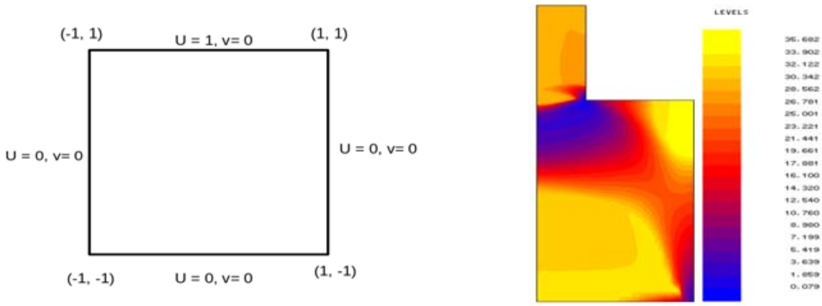

Fig. 1. Solution of the Stokes problem: (Left) Driven cavity, (Right) Die problem with varying viscosity

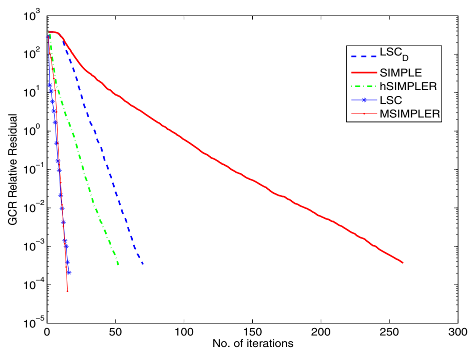

Fig. 2. Convergence pattern of various block preconditioners

Table 1. GCR iterations: Solution of the constant viscosity Stokes problem

Grid	P_S	P_{hSR}	P_{MSR}	P_{LSC_D}	P_{LSC_D}
32×32	84	25	13	19	11
64×64	162	43	16	26	17
128×128	310	80	21	40	21
256×256	705	161	28	70	27

to other preconditioners in which the number of outer iterations is almost proportional to the number of elements in each direction. If we look at the inner iterations for the velocity and the pressure in Fig. (3), we see that MSIMPLER and LSC are the best choices in terms of consumption of inner and outer iterations. SIMPLE shows robust behavior with small inner accuracy. Per iteration, hSIMPLER and LSc_D are expensive due to the demand for high inner accuracy. Therefore, even with a large number of outer iterations in SIMPLE, its inner iterations are comparable to hSIMPLER.

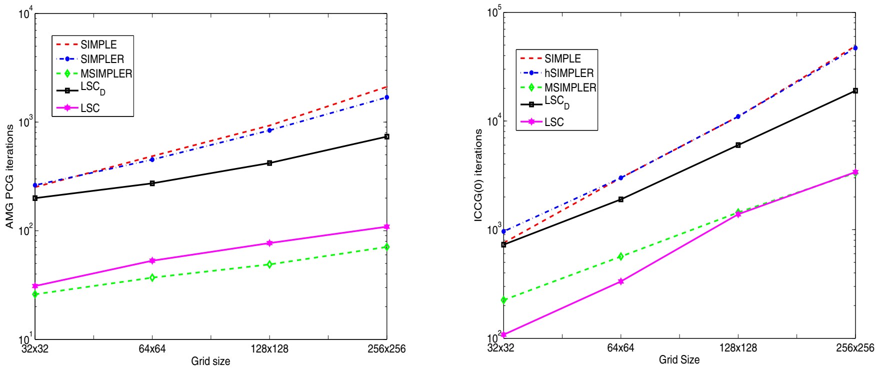

Fig. 3. Inner iterations: Solution of the driven cavity Stokes problem: Velocity subsystem(Left), Pressure subsystem (Right)

2.2 Variable Viscosity Problem

The right-hand side of Fig. 1 gives the configuration of the variable viscosity Stokes problem. The viscosity levels are plotted in color. The problem we consider is that of an aluminum round rod, which is heated and pressed through a die. In this way a prescribed shape can be constructed. In this specific example we consider the simple case of a small round rod to be constructed. The viscosity model used describes the viscosity as function of shear stress and temperature. The temperature and shear stress are the highest at the die where the aluminum is forced to flow into a much smaller region. The end rod is cut which is modelled as a free surface. Boundary conditions are prescribed velocity at the inlet and stress free at the outlet. At the container surface (boundary of thick rod), we have a no-slip condition. At the die we have friction, which is modeled as slip condition along the tangential direction and a no-flow condition in normal direction. The round boundary of the small rod satisfies free slip in tangential direction and no-flow in normal direction.

We observed that in constant viscosity problem, MSIMPLER and LSC show better convergence than the other preconditioners discussed. The main improving factor in these two preconditioners is scaling with the velocity mass matrix. Here, we solve the variable viscosity Stokes problem and we expect that since the velocity mass matrix does not contain any information about the viscosity variations, other preconditioners may perform better than these two. From Table 2, it seems that even for the variable viscosity problem, MSIMPLER and LSC perform much better than the other preconditioners. In this problem, instead of viscosity variation, grid size plays a dominant role. Although LSC_D and SIMPLE are scaled with the velocity matrix, which depends on the viscosity, their performance is poor compared to MSIMPLER and LSC. For this problem hSIMPLER is not reported since it does not show convergence even with high inner

Table 2. GCR iterations: Solution of the variable viscosity Stokes problem using block preconditioned GCR

Unknowns	P_S	P_{MSR}	P_{LSC_D}	P_{LSC}
$33k$	257	14	52	15
$66k$	419	16	64	17
$110k$	491	17	92	15
$195k$	706	17	110	19

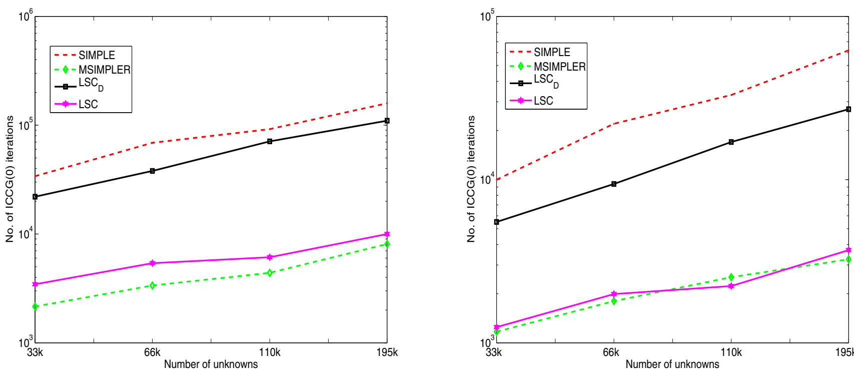

Fig. 4. Inner iterations: Solution of the variable viscosity Stokes problem: Velocity subsystem(Left), Pressure subsystem (Right)

accuracy. The inner accuracy of the preconditioners is the same that we used for constant viscosity problem except for LSC that is changed to 10^{-2} for the velocity part since it stagnates for 10^{-1}. The number of iterations consumed by the subsystems shown in Fig. 4, have the same pattern as we see in the constant viscosity Stokes problem.

3 Conclusions

In this paper, we solve constant and variable viscosity Stokes problem using block preconditioned GCR. SIMPLE and LSC-type preconditioners are used with two different scalings. Table 3 shows dependence of the preconditioners on mesh size and inner accuracy. It is observed that MSIMPLER and LSC show

Table 3. Dependence of various block preconditioners on mesh size and subsystem accuracies

Dependence on	P_S	P_{hSR}	P_{MSR}	P_{LSC_D}	P_{LSC}
Mesh size	Yes	Yes	Mild	Yes	Mild
Inner accuracy	No	Yes	No	Yes	Small

better performance than other preconditioners for both type of problems. Both use scaling with the velocity mass matrix which has no information regarding the variation of viscosity. For further research, both preconditioners will be tested for problems with large viscosity contrasts.

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