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TRANSIENTS IN POWER SYSTEMS: A LITERATURE SURVEY

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Transients in Power Systems: A Litterature Survey

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Chapter 1

Introduction

An electrical network is formed by generators (producers), loads (consumers) and transmission lines for the connection between producers and consumers.

Time integration methods used to simulate a large electrical network, at the scale of a country, cost too much computing time. To understand this observation, we are interested especially in methods to transform an electrical network into a set of equations and in time integration methods.

This report is divided into four parts. The first part is an historical overview of electrical simulation methods. The second part describes various methods to transform an electrical network into a set of equations. The third part is a summary of time integration methods. Finally, the last part is an overview of approaches which can be used to decrease the computing time.

Chapter 2

Electrical simulation history

To understand the motivation of users of electrical software to use one particular method for the electrical simulation, it is useful to have a quick overview of what has been developed in the past and also what has been done in the last six years.

2.1 Short historical overview

From Figure 2.1, it appears that advanced mathematical tools are not used directly in software for electrical simulation. Actually, for the study of large networks for transient recovery voltage, the nodal analysis developed in the sixties by Dommel is still used (Section 3.4) (EMPT-RV, PSCAD and RTDS). The strong point of this method is the easy way and the fast computation to get the set of equations. However, other methods can be used to model an electrical network (Section 3.5 and Section 3.6). Mathematical tools developments are in expansion but few of these tools are used for solving the set of equations.

2.2 Last six years literature

The topic of my research is the fast simulation of transient recovery voltage for networks represented by a large amount of lumped elements (large networks). Two issues need to be considered:

- Modeling methods;
- Numerical integration methods.

Especially, in the modeling method world, many methods to translate an electrical network into a set of differential equations are already found. However, it will be interesting to use a Model Order Reduction (MOR) for large networks. MOR permits to reduce the set of differential equations in order to be faster during the time numerical integration (Section 5.1). Various methods are presented by Antoulas [1] in 2005. Several publications

Figure 2.1: Time line electrical simulation

in electrical paper has been published about MOR applied in power systems [13] [20] [27] [19] [22] [12] [28] [23]. Also hybrid methods of modeling electrical networks can be used [7] [26]. The principle of hybrid methods is to cut into several parts the network and use different modeling methods for each parts.

In the numerical integration world, two parts are taken into account multirate methods and the computation time. The computation time can be reduced by using fast solvers [29] in well implemented package [32]. Fast solvers are used in electricity for the calculation of the power flow for example [17]. Secondly, the multirate approach is also interesting for decreasing the computation time. This approach is based on slow and fast time component and the error. In electricity, several papers has been published about this method with is indexed in three sub-methods (Section 5.2) [8] [14] [33] [31] [30] [25].

Chapter 3

Electrical modeling

Electrical modeling is a method to convert an electrical network into a set of equations. There are three main methods and they are based on the general electrical laws and on the graph theory. However, the first step is to transform an electrical network into an electrical diagram.

3.1 Electrical diagram

Definition:

An electrical diagram is composed of lumped elements, which represents the electrical network. The current through a lumped element is denoted by $i(t)$ and measured in Ampere (A). The voltage across an element is denoted by $v(t)$ and the unit of $v(t)$ is Volt (V). The main lumped elements are:

- Ideal voltage source

  The ideal voltage source is an element of which the voltage is independent of the current delivered. The voltage supplies is denoted $e(t)$.

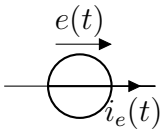

Figure 3.1: Symbol of an ideal voltage source

- Ideal current source

  The ideal current source is an element of which the current supplied is independent from its voltage. The current supplies is denoted $j(t)$.

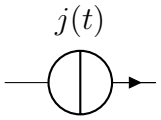

Figure 3.2: Symbol of an ideal current source

- Resistor

  The current flow depends on a property of the resistor called resistance. In 1827, Ohm observed the relation between the current through and the voltage across the resistor as a factor R such as:

$$v_R(t) = Ri_R(t) \tag{3.1}$$

  and

$$R = \frac{1}{G} \tag{3.2}$$

  where R is the resistance in Ohm (Ω) and G is the conductance in Siemens (S).

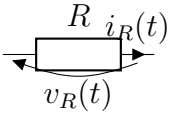

Figure 3.3: Symbol of a resistor

- Inductor

  An inductor is an energy storage device. The voltage across an inductance is proportional to the derivative of the current through it. The voltage-current relationship is given by:

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{3.3}$$

  where L is the inductance in Henry (H).

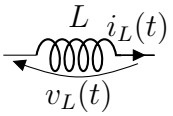

Figure 3.4: Symbol of an inductor

- Capacitor

  Like an inductor, a capacitor is an energy storage device. The capacitor is the opposite of the inductor, the current is proportional to the derivative of the voltage across it. The fundamental equation of a capacitor is:

$$i_C(t) = C\frac{dv_C(t)}{dt} \tag{3.4}$$

  where C is the capacitance in Farad (F) .

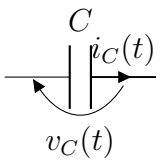

Figure 3.5: Symbol of a capacitor

- Earth

  It is the reference potential for the voltages.

Figure 3.6: Symbol of the earth

Two type of connections are used:

- series connection

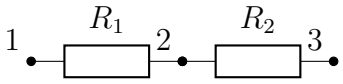

Figure 3.7: series connection

  In Figure 3.7, resistance R_1 and R_2 are in serie because the node 2 is common to these two elements.

- parallel connection

  In Figure 3.8, resistance R_1 and R_2 are in parallel because all extremities elements are connected together.

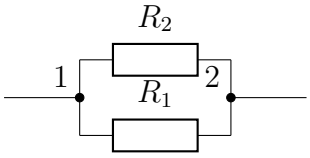

Figure 3.8: parallel connection

3.2 General electrical law and theorem

3.2.1 Kirchhoff's laws

Kirchhoff's current law (KCL)

Kirchhoff's current law states that at every node of the electrical diagram, the currents entering a node are equal to the currents leaving from this node.[9].

$$\sum_{currents\ entering} i_e(t) = \sum_{currents\ leaving} i_l(t) \tag{3.5}$$

Properties:

- Kirchhoff's current law forbids series connection of current sources;
- An ideal current source supplying a current of zero ampere is considered as an open circuit.

Kirchhoff's voltage law (KVL)

Kirchhoff's voltage law states that the sum of all voltage differences in a closed loop in an electrical diagram is equal to zero volt [9].

$$\sum v_{loop}(t) = 0 \tag{3.6}$$

Properties:

- Kirchhoff's voltage law forbids parallel connection of voltage sources;
- An ideal voltage source supplying a voltage of zero volt is considering as a short-circuit.

3.2.2 Thévenin-Norton theorem

Thévenin-Norton theorem enables to convert a non-ideal voltage source into a non-ideal current source and vice-versa [9].

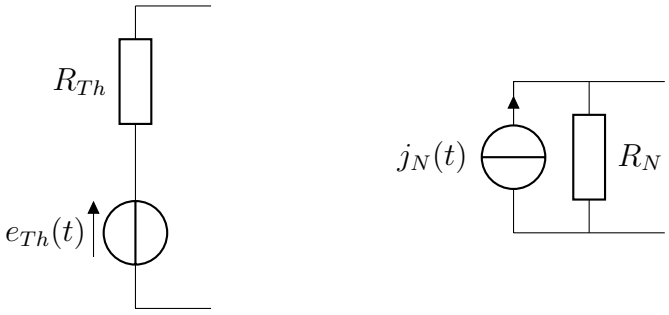

Figure 3.9: Thevenin-Norton Transformation

The mathematical formulation of this transformation is : $R_N = R_{Th}$ and $j_N(t) = \frac{e_{Th}(t)}{R_{Th}}$

Remark:

If a same load is connecting to the non-ideal voltage source or the non-ideal current source modeling by this theorem, the current through the load and the voltage across the load will be the same in both cases.

3.2.3 Millman's theorem

Millman's theorem permits to calculate the voltage between two nodes [24]. In the following diagram (Figure 3.10), we want to calculate the voltage $v(t)$.

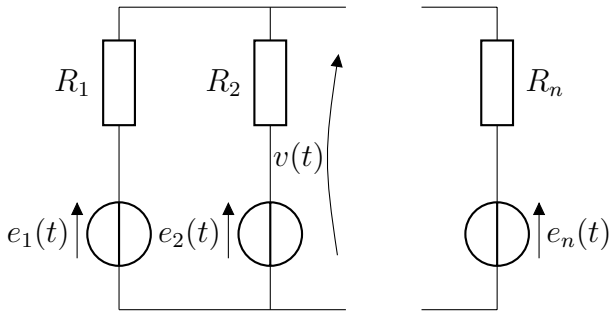

Figure 3.10: Millman's theorem

The Millman's theorem states that:

$$v(t) = \frac{\sum_{i=1}^{n} \frac{e_i(t)}{R_i}}{\sum_{i=1}^{n} \frac{1}{R_i}} \tag{3.7}$$

where n is the number of branches. Here, a branch is a voltage source and a resistance in series.

3.3 Graph theory applied to an electrical diagram

Graph theory is useful to pass from an electrical diagram into a graph. From this graph, it is possible to complete the edge node incidence matrix $T \in \mathbb{N}^{nbel \times n}$ where $nbel$ is the number of element and n is the number of node and the elements matrix $D \in \mathbb{R}^{nbel \times nbel}$ [18][35]. We can build the matrix T as:

$$T_{ij} = \begin{cases} +1 & \text{if the direction of the branch } j \text{ leaves the node i} \\ -1 & \text{if the direction of the branch } j \text{ enters the node i} \\ 0 & \text{else} \end{cases}$$

We can build also the incidence matrix T_G for conductances,T_L for inductances, T_C for capacitances, T_e for voltage sources and T_j for current sources. For this only branches with this type of element is taken into account. After that, we can build the matrix D as:

$$D_{jj} = \begin{cases} G & \text{if the branch } j \text{ represents a conductance} \\ L & \text{if the branch } j \text{ represents an inductance} \\ C & \text{if the branch } j \text{ represents a capacitance} \\ e(t) & \text{if the branch } j \text{ represents a voltage source} \\ j(t) & \text{if the branch } j \text{ represents a current source} \end{cases}$$

We can build also the element matrix of D_G for conductances,D_L for inductances, D_C for capacitances, $D_e(t)$ for voltage sources and $D_j(t)$ for current sources. For this only branches with this type of element is taken into account.

3.4 Nodal analysis

The Nodal analysis method is based on KCL (Section 3.2.1) and on the Thévenin-Norton theorem (Section 3.2.2). If the electrical diagram is only resistive and with current sources, the general equation can be written from the graph theory (Section 3.3)[21][34]:

$$TD_GT^T\hat{v}(t) = T_J\text{diag}(D_j(t)) \tag{3.8}$$

We can denote this equation by:

$$Y\hat{v}(t) = \hat{j}(t) \tag{3.9}$$

where $Y = TD_GT^T$ and $\hat{j}(t) = T_J\text{diag}(D_j(t))$.

- Y is the nodal admittance matrix (this matrix is only composed by conductance of resistances);
- $\hat{v}(t)$ is the vector with the unknown voltage;
- $\hat{j}(t)$ is the vector with the current sources.

Using inductances or capacitances is not possible with the previous method because they impose differential equations. However, Dommel's method transforms differential equations of the network into algebraic equations. The principle of transformation can be

derived from the Trapezoidal rule. After this transformation, inductances or capacitances can be modeled as a control current source with a resistance in parallel[11]. From Dommel's method, we can write:

- An inductance is replaced by a resistance $R_{L\Delta t}$ in parallel with a control current source $j_{L\Delta t}(t)$.
- A capacitance is replaced by a resistance $R_{C\Delta t}$ in parallel with a control current source $j_{C\Delta t}(t)$.

It is only after this transformation that graph theory can be applied. The new formulation of the method is:

$$TD_{G\Delta t}T^T\hat{v}(t) = T_J\text{diag}(j_{\Delta t}(t)) \tag{3.10}$$

We can denote this equation by:

$$Y_{\Delta t}\hat{v}(t) = \hat{j_{\Delta t}}(t) \tag{3.11}$$

where $Y_{\Delta t} = TD_{G\Delta t}T^T$ and $\hat{j_{\Delta t}}(t) = T_J\text{diag}(j_{\Delta t}(t))$.

At each step, it is needed to recalculate the current of all inductances and capacitances for the next step. The advantages of Dommel's method are:

- Simplicity: the network is reduced to a number of current sources and resistances of which Y is easy to construct ;
- Robustness: the integration method is the Trapezoidal rule, which is a numerically stable and a robust integration routine.

However, the method has some disadvantages too:

- An ideal voltage source poses a problem, the Thévenin-Norton theorem can not be applied;
- It is difficult to change the computational step size dynamically during the calculation. The resistance values should be recomputed at each change. This implies that Y has to be inverted every time step. This is time-consuming for large networks.

3.5 Modified nodal analysis

The Modified Nodal Analysis (MNA)[15] was developed to decrease the time of simulation. This method is developed from the KCL(Section 3.2.1) and KVL(Section 3.2.1). The MNA method has several forms and the result is always a Differential Algebraic Equation (DAE). A DAE has an index to identified it nature. From the definition of [2] the DEA index is the minimum number of differentiation to arrive to an Ordinary Differential Equation (ODE) (index 0). The simple method was developed by Prof. L. van der Sluis [3] [4] and a general method was developed for complex electrical diagram [31]. The DAE index is for the simplest formulation is 1 and for the second formulation is 2. Implicit integration methods are recommended to solve DAE.

3.5.1 Simple method

The mathematical formulation of this method is [34]:

$$\dot{x_1}(t) = \tilde{A}x_2(t) \tag{3.12}$$

$$\tilde{B}(t)x_2(t) = \tilde{C}x_1 + \tilde{D}source(t) \tag{3.13}$$

n	$\in \mathbb{N}$ is the number of differential variables,
m	$\in \mathbb{N}$ is the number of algebraic variables,
p	$\in \mathbb{N}$ is the number of sources,
x_1	$\in \mathbb{R}^n$ of differential variables,
x_2	$\in \mathbb{R}^m$ of algebraic variables,
$\tilde{A}$	$\in \mathbb{R}^{n\times m}$ representing the linear relation between $\dot{x_1}$ and x_2,
$\tilde{B}(t)$	$\in \mathbb{R}^{m\times m}$, it is the MNA matrix which may contain time-dependent elements,
$\tilde{C}$	$\in \mathbb{R}^{m\times n}$ representing the linear relation between x_1 and x_2,
$\tilde{D}$	$\in \mathbb{R}^{m\times p}$ representing the linear relation between $E(t)$ and x_2,
$source(t)$	$\in \mathbb{R}^n$ of time dependent contributions to the right-hand side vector from voltage and current sources.

Vector x_1 is composed of all currents through inductances (i_{l_n}) and all voltages across capacitances(v_{c_n}) where l_n represents the n^{th} inductance and c_n represents the n^{th} capacitance of the electrical diagram. Vector x_2 is composed of all voltage between all nodes and the reference voltage of the electrical diagram v_{n0} where n =1 to N and N is the number of node and all currents through capacitances (i_{c_n}) and all currents through voltage sources (i_{e_n}) where c_n represents the n^{th} capacitance and e_n represents the n^{th} voltage source of the electrical network. The matrices $\tilde{A}$, $\tilde{B}(t)$, $\tilde{C}$ and $\tilde{D}$ define the electrical network topology.

To complete all matrices and vectors, there are two possibilities, the first is a mathematical approach (based on the graph theory) and the second is a logical method.

3.5.2 General method

The mathematical formulation of this method is[31]:

$$\begin{bmatrix} TD_CT^T & 0 & 0 \\ 0 & D_L & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} TD_GT^T & T_L & T_e \\ -T_L^T & 0 & 0 \\ -T_e^T & 0 & 0 \end{bmatrix} x + \begin{bmatrix} T_j\text{diag}(D_j(t)) \\ 0 \\ T_e\text{diag}(D_e(t)) \end{bmatrix} = 0 \tag{3.14}$$

If all elements of the i^{th} row of Equation (3.14) are equal to 0, the i^{th} column and i^{th} row of the first and second matrix can be deleted and the i^{th} row of the vector.

After this transformation, the vector x is composed of differential variables and algebraic variables. x is composed of all voltages between all nodes and the reference voltage of the electrical diagram v_{n0} where n =1 to N and N is the number of node, of all currents through inductances (i_{l_n}) and of all currents through voltage sources (i_{e_n}) where l_n represents the n^{th} inductance and e_n represents the n^{th} voltage source of the electrical network. The

differential variables correspond to all voltages around a capacitance (if a capacitance is between the node 2 and 3 the differential variables are v_{20} and v_{30}, Equation (3.4)) and to all current through inductance (Equation (3.3)).

Remarks:

- T_e^T is not invertible;
- $TD_CT^Tu(t) + TD_GT^Tu(t) + T_Li_l(t) + T_ei_e(t) + T_j\text{diag}(D_j(t)) = 0$ corresponds to the KCL equation. The KCL equation contains differential variables and algebraic variables (Equation (3.4));
- $D_Li_l(t) + T_e^Tu(t) = 0$, this equation corresponds to the calculation of the derivative of the current thought all inductances (Equation (3.3)).

3.6 Black box

The Black Box (BB) representation is based on the KVL (Section 3.2.1). The mathematical formulation is a first Order Differential Equation (ODE). Each black box represents a component of the electrical network. Inside each black box, there are two matrices A_s and B_s. A black box can be connected to one or more black boxes by Millman's theorem (Section 3.2.3). The formulation of this method is:

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(t,x) = Ax(t) + Bsource(t) \tag{3.15}$$

where

- $nbx \in \mathbb{N}$ is the number of differential variables (number of inductances and capacitances);
- $nbs \in \mathbb{N}$ is the number of sources;
- $x \in \mathbb{R}^{nbx}$ is the state vector;
- $source(t) \in \mathbb{R}^{nbs}$ is the time-dependent input vector;
- $A \in \mathbb{R}^{nbx \times nbx}$ is the state matrix and is formed by all matrices A_s and the Millman's theorem;
- $B \in \mathbb{R}^{nbx \times nbs}$ is the input matrix and is formed by all matrices $B_s$and the Millman's theorem.

Vector x is composed of all currents through inductances (i_{l_n}) and all voltages across capacitances(v_{c_n}) where l_n represents the n^{th} inductance and c_n represents the n^{th} capacitance of the electrical network. Matrices A and B define the electrical network topology.

From Equation 3.15, an analytical solution can be found. However, this method is slow due to the eigenvalue computation (matrix Λ) and the transformation matrix where each column consists of eigenvector (matrix T) of the matrix A [10].

$$A = T\Lambda T^{-1} \tag{3.16}$$

If all sources are time independent, the analytical solution is [10]:

$$x(t) = Te^{\Lambda t}T^{-1}x(0) - A^{-1}\left[I + Te^{\Lambda t}T^{-1}\right]Bsource \tag{3.17}$$

If all sources are sinusoidal, the analytical solution is:

$$x(t) = Te^{\Lambda t}T^{-1}x_{const} + TQ\left[\Omega\sin(\omega t) - \Lambda\cos(\omega t)\right]T^{-1}Bsource \tag{3.18}$$

$$x_{const} = x(0) - TQ\left[\Omega\sin(0) - \Lambda\cos(0)\right]T^{-1}Bsource \tag{3.19}$$

or

$$x(t) = Te^{\Lambda t}T^{-1}x_{const} + TQ\left[\Omega\sin(\omega t + \varphi) - \Lambda\cos(\omega t + \varphi)\right]T^{-1}Bsource \tag{3.20}$$

where $source(t) = \cos(\Omega t + \varphi)source$, $Q = \left[\Lambda^2 + \Omega^2\right]^{-1}$ and $\Omega = \mathrm{diag}(\omega)$.

3.7 Comparison

	NA	MNA	BB
Δt	constant	adaptive	adaptive
Complexity of the method	Easy	Medium	Medium
exact solution	No	No(execpt for reduction to index 0)	Yes(if linear)
Non-linear elements	Yes (difficult)	Non-linear DAE	Non-linear ODE

Chapter 4

Overview of time integration methods

The general form of a system of ordinary differential equations (ODE) is:

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(x,t) \tag{4.1}$$

where $x \in \mathbb{R}^n$ and n is the number of differential variables. The initial condition of the integration method is $x(t_0) = x_0$. The solution of this differential equation is:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x,t)dt \tag{4.2}$$

A time integration method solves differential equations. There are many time integration methods. The choice of a method depends on the characteristics of the problem and on the required solution. In this chapter, some basic integration methods and approach of the time step control are described.

There are two important notations:

- Continuous function $x(t)$;
- Numerical approximation $u_n \approx x(t_n)$ where $t_n = n\Delta t$ and Δt is the time step.

4.1 Basic methods

In this section, we give three basic methods, Euler forward, Euler backward and the Trapezoidal rule [35].

- Euler forward

  The Euler forward method is based on the formulation of the derivative during one step (Δt) at the time t to find an approximation of $x(t + \Delta t)$. The remainder term is $O(\Delta t)$.

$$\dot{x}(t_0) = \frac{x(t_1) - x(t_0)}{\Delta t} + O(\Delta t) \tag{4.3}$$

so

$$x(t_1) = x(t_1) + \Delta t \dot{x}(t_0) + O(\Delta t^2) \tag{4.4}$$

The general form of Euler forward where we remove the $O(\Delta t^2)$ term is:

$$u_{n+1} = u_n + \Delta t f(u_n, t_n) \tag{4.5}$$

This method is explicit.

- Euler backward

  The Euler backward method is based on the formulation of the derivative at the time $t + \Delta t$ to find an approximation of $x(t + \Delta t)$.

$$\dot{x}(t_1) = \frac{x(t_1) - x(t_0)}{\Delta t} + O(\Delta t) \tag{4.6}$$

so

$$x(t_1) = x(t_0) + \Delta t \dot{x}(t_1) + O(\Delta t^2) \tag{4.7}$$

and the general formulation where we remove the $O(\Delta t^2) term$ is:

$$u_{n+1} = u_n + \Delta t f(u_{n+1}, t_{n+1}) \tag{4.8}$$

This method is implicit.

- Trapezoidal rule

  The Trapezoidal rule uses the average value of the Euler forward and Euler backward approximation. The following formulation is the trapezoidal rule.

$$u_{n+1} = u_n + \frac{\Delta t}{2}(f(u_n, t_n) + f(u_{n+1}, t_{n+1})) \tag{4.9}$$

The norm of the local discretization error e_n is the error made in a single step. For Euler forward and Euler backward, e_n is the same and is $e_n = \left|\frac{\Delta t}{2}\ddot{x}(\tau_n)\right|$ where $\tau_n \in [t_n\ t_{n+1}]$. For the Trapezoidal rule, e_n is $e_n = \left|\frac{\Delta t^2}{12}\dddot{x}(\tau_n)\right|$ where $\tau_n \in [t_n\ t_{n+1}]$.

To study the stability, the test equation used is:

$$\dot{x}(t) = \lambda x(t) \tag{4.10}$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. In electricity, λ is considered as the time constant of the system and the time constant can be real in the case of an RL circuit or complex in the case of an RLC circuit.

From this equation, the stability region is defined as the region of $\lambda\Delta t$ where the numerical integration method is stable:

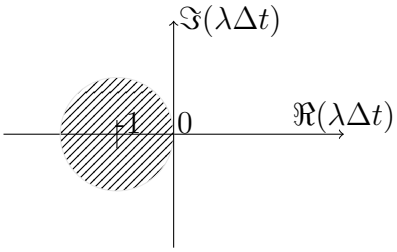

Figure 4.1: Stability region: stable inside of the circle

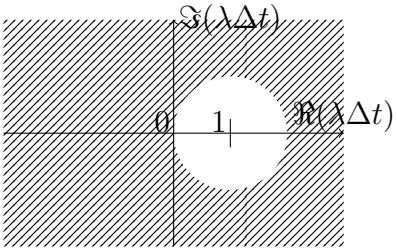

Figure 4.2: Stability region: stable outside of the circle

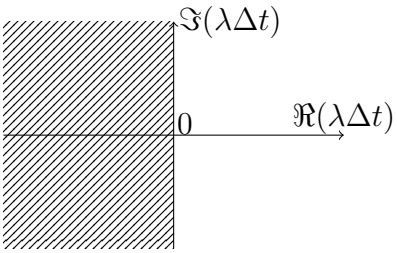

Figure 4.3: Stability region: stable in the left half plane

4.2 Runge-Kutta methods

The family of Runge-Kutta methods was developed by the German mathematicians C. Runge and M.W. Kutta in 1901. The advantages of this method are the easy implementation in a computer program, very stable, adaptive time stepping and self-starting [16].

4.2.1 Introduction

With an s-stage Runge-Kutta (RK) method, the solution of the system of differential equations (Equation (4.1)) can be written as [6]:

$$u_{n+1} = u_n + \Delta t \sum_{i=1}^{s} b_i k_i \tag{4.11}$$

$$k_i = f\left(t_n + c_i h, u_n + \Delta t \sum_{j=1}^{s} a_{ij} k_j\right) \tag{4.12}$$

All RK methods can be defined by their own Butcher tableau, which puts the coefficients of the method in the following table:

c_1	a_{11}	a_{12}	$\cdots$	a_{1s}
c_2	a_{21}	a_{22}	$\cdots$	a_{2s}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_s	a_{s1}	a_{s2}	$\cdots$	a_{ss}
	b_1	b_2	$\cdots$	b_s

Table 4.1: Butcher tableau

An RK method can be explicit if $a_{ij} = 0$ for $i \leq j$, semi-explicit if $a_{ij} = 0$ for $i < j$ and else the RK method is implicit.

4.2.2 Explicit Runge-Kutta methods

Explicit Runge-Kutta methods are for example RK2 or RK4. The coefficient k_i is only calculated from the previous k_i and $k_1 = f(t, x)$.

- RK2

  The Runge-Kutta 2 (RK2) is a second order numerical integration method. The error e_n is proportional to Δt^2. The formulation of this method is:

$$k_1 = f(t_n, u_n) \tag{4.13}$$
$$k_2 = f(t_n + \alpha\Delta t, u_n + \alpha\Delta t k_1) \tag{4.14}$$
$$u_{n+1} = u_n + \Delta t \left[\left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2\right] \tag{4.15}$$

  for some α where $0 \leq \alpha \leq 1$. If $\alpha = 1$, the formulation and the stability region is the same as the Trapezoidal rule.

- RK4 The Runge-Kutta 4 (RK4) is a fourth order numerical integration method. The error e_n is proportional to Δt^4. The formulation of this method is:

$$k_1 = f(t_n, u_n) \tag{4.16}$$
$$k_2 = f(t_n + \frac{\Delta t}{2}, u_n + \frac{\Delta t}{2} k_1) \tag{4.17}$$
$$k_3 = f(t_n + \frac{\Delta t}{2}, u_n + \frac{\Delta t}{2} k_2) \tag{4.18}$$
$$k_4 = f(t_n + \Delta t, u_n + \Delta t k_3) \tag{4.19}$$
$$u_{n+1} = u_n + \Delta t \left[\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right] \tag{4.20}$$

4.2.3 Semi-explicit and implicit Runge-Kutta methods

A system of equations needs to be solved in the case of a semi-explicit and an implicit method to find k_i for $i = 1$ to s. If we take the following form:

$$\dot{x}(t) = f(t, x) = Ax(t) + Bsource(t) \tag{4.21}$$

,

we can write: $k = [k_1 \dots k_s]^T$,

$\tilde{A} = \begin{bmatrix} a_{11}A & \dots & a_{1s}A \\ \vdots & \ddots & \vdots \\ a_{s1}A & \dots & a_{ss}A \end{bmatrix}$ and $m(t_n, \Delta t) = \begin{bmatrix} Au_n + Bsource\cos(\omega(t_n + c_1\Delta t) + \varphi) \\ \vdots \\ Au_n + Bsource\cos(\omega(t_n + c_s\Delta t) + \varphi) \end{bmatrix}$.

Moreover, if Equation (4.21) and Equation (4.12) are taken into account, all coefficients k_i can be written as:

$$k_i = A(u_n + \Delta t \sum_{j=1}^{s} a_{ij}k_j) + Bsource\cos(\omega(t_n + c_i\Delta t) + \varphi) \tag{4.22}$$

Consequently, the system to solve is:

$$(I - \Delta t\tilde{A})k = m(t_n, \Delta t) \tag{4.23}$$

For a semi-explicit Runge-Kutta method, the formulation of $\tilde{A}$ is:

$$\tilde{A} = \begin{bmatrix} a_{11}A & 0 & 0 \\ \vdots & \ddots & 0 \\ a_{s1}A & \dots & a_{ss}A \end{bmatrix}.$$

With these methods, it is only necessary to invert $[I - a_{ii}\Delta tA]$ instead of $[I - \Delta t\tilde{A}]$.

4.2.4 IMEX RK

The advanced IMplicit-EXplicit Runge-Kutta scheme divides the differential equation into two differential equations, the fast part and the slow part. For an ODE, the following form is used[16] [5]:

$$\dot{x} = f_E(t, x) + f_L(t, x) \tag{4.24}$$

where

- f_L is the function containing the stiff part of the ODE and it is solved by an implicit method;
- f_E is the function containing the non-stiff part of the ODE and it is solved by an explicit method.

The formulation of the IMEX RK is:

$$u_{n+1} = u_n + \Delta t \sum_{i=1}^{s} \tilde{b}_i f_E(t_n + \tilde{c}_i \Delta t, u^i) + \Delta t \sum_{i=1}^{s} b_i f_L(t_n + c_i \Delta t, u^i) \tag{4.25}$$

with internal stages given by:

$$u^i = u_n \quad + \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} f_E(t_n + \tilde{c}_i \Delta t, u^i) + \Delta t \sum_{j=1}^{i} a_{ij} f_L(t_n + c_i \Delta t, u^i) \tag{4.26}$$

$\tilde{c}_1$	$\tilde{a}_{11}$	$\tilde{a}_{12}$	$\cdots$	$\tilde{a}_{1s}$
$\tilde{c}_2$	$\tilde{a}_{21}$	$\tilde{a}_{22}$	$\cdots$	$\tilde{a}_{2s}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{c}_s$	$\tilde{a}_{s1}$	$\tilde{a}_{s2}$	$\cdots$	$\tilde{a}_{ss}$
	$\tilde{b}_1$	$\tilde{b}_2$	$\cdots$	$\tilde{b}_s$

Table 4.2: Butcher tableau for the slow part (non-stiff part)

c_1	a_{11}	a_{12}	$\cdots$	a_{1s}
c_2	a_{21}	a_{22}	$\cdots$	a_{2s}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_s	a_{s1}	a_{s2}	$\cdots$	a_{ss}
	b_1	b_2	$\cdots$	b_s

Table 4.3: Butcher tableau for the fast part (stiff part)

4.2.5 Other RK methods

Runge-Kutta methods go from the first order to higher order. The first order is equivalent to Euler forward. From this family of numerical integration, some modern methods have been developed as the Runge-Kutta-Fehlberg (RKF) in 1960 or the Runge-Kutta-Chebyshev (RKC) in 1980 [16].

4.3 Step size control

4.3.1 General idea

If we consider the interval $[t_n, t_{n+1}]$ where $t_{n+1} = t_n + \Delta t$. Let us assume that the method has an order p, the norm of the local error discretization is e_n and the tolerance Tol is specified by the user. The step size control is based on the following rule[16]:

- if $e_n \leq Tol$, Δt is accepted and the future step size will be slightly larger;

- if $e_n \geq Tol$, Δt is rejected and redone with Δt_{new} smaller. This step is used until $e_n \leq Tol$.

The choice of the time step Δt is based e_n and Tol. e_n is calculated in the majority of cases from the difference between the integration method of the order p and $p-1$. The formulation of the calculation of Δt_{new} is:

$$\Delta t_{new} = \theta \Delta t_{old} \sqrt[p]{\frac{Tol}{e_n}} \tag{4.27}$$

for some θ and p where $0 < \theta \leq 1$ and $p \geq 2$

4.3.2 Adaptive Runge-Kutta methods

The adaptive time stepping can be done by an adaptive Runge-Kutta method. This idea is if all k_i are calculated for the order p, it is possible to compute the result at the order $p-1$. These methods avoid the recalculation of a complete RK method at an order $p-1$. The Butcher tableau for an adaptive RK method is :

c_1	a_{11}	a_{12}	$\cdots$	a_{1s}
c_2	a_{21}	a_{22}	$\cdots$	a_{2s}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_s	a_{s1}	a_{s2}	$\cdots$	a_{ss}
	b_1	b_2	$\cdots$	b_s
	$\hat{b}_1$	$\hat{b}_2$	$\cdots$	$\hat{b}_s$

Table 4.4: Butcher tableau for adaptive RK

with

$$\hat{u}_{n+1} = u_n + \Delta t \sum_{i=1}^{s} \hat{b}_i k_i \tag{4.28}$$

where $\hat{u}_{n+1}$ is the computed solution with an order $p-1$. Now, we can compute the error e_n as the difference between $\|u_{n+1} - \hat{u}_{n+1}\|_\infty$. We can define the error as:

$$\begin{aligned} e_n &= \|u_{n+1} - \hat{u}_{n+1}\|_\infty && (4.29)\\ &= \left\| u_n + \Delta t \sum_{i=1}^{s} b_i k_i - u_n - \Delta t \sum_{i=1}^{s} \hat{b}_i k_i \right\|_\infty && (4.30)\\ &= \left\| \Delta t \sum_{i=1}^{s} (b_i - \hat{b}_i) k_i \right\|_\infty && (4.31) \end{aligned}$$

When e_n is found, the algorithm of Section 4.3.1 is used.

Chapter 5

Challenges

After the electrical modeling and the study of different methods of time integration for ODE and DAE set of equations, the next part of my study will be the model order reduction and the multi-rate integration method.

5.1 Model order reduction

The idea of this part is if the system is represented in a matrix formulation, it is possible to reduce the system in order to find a solution close of the solution of the non-reduce system. For example an ODE system of this form is taken [1]:

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ y = Cx + Du(t) \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{p \times m}$.

After to use an model order reduction method, the ODE representation of this system is:

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u(t) \\ y = \hat{C}\hat{x} + \hat{D}u(t) \end{cases}$$

where $\hat{A} \in \mathbb{R}^{k \times k}$, $\hat{B} \in \mathbb{R}^{p \times k}$, $\hat{C} \in \mathbb{R}^{k \times m}$, $\hat{D} \in \mathbb{R}^{p \times m}$ and $k < n$

5.2 Multirate integration method

A multi-rate method is a different approach of the time step control. Two categories of differential variables are inside of the system of equation, the slow and the fast. To do this different methods are available:

- monolithic approaches [30]
- co-simulation approaches [31]
- multivariate approaches [25]

The monolithic approaches has two methods, the first is to compute the slow part of the system and after the fast part by interpolation of the slow part or vis-versa. This method works well when different frequencies are inside of the system.

The co-simulation approaches allow individual components to be simulated by different simulation tools running simultaneously and exchanging information in a collaborative manner.

The multivariate approach is to convert the ODE system into a PDE system when the frequencies of the system are known.

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