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# An Eigenvalue Analysis of Nonassociated Plasticity

H. van der Veen
Faculty of Civil Engineering, Delft University of Technology
P.O. Box 5048, 2600 GA Delft, The Netherlands

C. Vuik
Faculty of Technical Mathematics and Informatics, Delft University of Technology
P.O. Box 5048, 2600 GA Delft, The Netherlands

R. de Borst
Faculty of Civil Engineering, Delft University of Technology
P.O. Box 5048, 2600 GA Delft, The Netherlands

**Abstract**—Boundary value problems with operators that are not self-adjoint are a direct consequence of the use a nonassociated plasticity model. As a result, the material stiffness matrix, and therefore also, the ensuing structural stiffness matrix become nonsymmetric, and complex eigenvalues are possible. In practice, however, these are not encountered for the structural stiffness matrix. We present a mathematical analysis of the eigenvalues characterizing the elasto-plastic material stiffness matrix with a Drucker-Prager yield function, for orthotropic and isotropic materials. We confine ourselves to plane-strain and stress conditions. All possible stress distributions are considered showing possible complex eigenvalues in case of orthotropy but none for isotropy. Finally, a numerical analysis is performed to gain insight into the eigenvalues of the structural stiffness matrix. 

**Keywords**—Eigenvalues, Elasto-plasticity, Drucker-Prager, Soil.

## 1. INTRODUCTION

While self-adjoint operators are predominant in boundary value problems in solid mechanics, there also exist some practically important cases where self-adjointness is lost. Indeed, almost any model in which frictional effects in the material play a role, eventually leads to a loss of self-adjointness. On the other hand, in the absence of frictional effects, with the simplest case being elasticity, the constitutive operator together with the equilibrium equations and the kinematic relations leads to a set of differential equations that results in self-adjoint boundary value problems.

Material models in which frictional effects become important normally result in constitutive models where the tangential stiffness tensor that sets the relation between the stress rate tensor and the strain rate tensor becomes nonsymmetric. This nonsymmetry in the constitutive relation directly leads to a loss of self-adjointness of the boundary value problem. As a result, the structural tangential stiffness matrix that is obtained after discretization of the boundary value problem, becomes nonsymmetric. In principle, nonsymmetric matrices allow for complex eigenvalues. However, on physical grounds it is hard to imagine that complex eigenvalues would arise.

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Indeed, since the structural stiffness matrix is still filled by real numbers, the complex eigenvalues, if they exist, must be complex conjugate. This in turn would require the eigenvectors to be complex conjugate and it is hard to attach any physical significance to a complex displacement vector under static loading conditions [1].

In this article, we shall adopt a particular constitutive model that gives rise to a nonsymmetric constitutive operator, namely a nonassociated plasticity model. Such models typically arise when describing sand behavior. Pressure-dependent yield functions like those of Drucker-Prager or Mohr-Coulomb are needed to describe the basic features of the frictional character of the resistance in sands, while a so-called nonassociated flow rule is needed to capture the inelastic volume effects (dilatancy and contraction) properly. As we shall briefly recapitulate, this nonassociated flow rule causes nonsymmetry in the constitutive operator matrix.

We shall examine a relatively simple model, namely a Drucker-Prager yield function with a nonassociated flow rule. Anisotropy is introduced in the elastic part of the elasto-plastic model. Then, it is demonstrated that, for this fairly simple case, complex eigenvalues arise for the (nonsymmetric) constitutive matrix. This is shown analytically for plane-stress conditions and numerically for plane-strain conditions. It is demonstrated that, for vanishing anisotropy, the complex eigenvalues disappear. Finally, it is shown that after combination of the constitutive operator with the equilibrium and the kinematic relations, and after discretization of the set of equations which has thus arisen, the resulting (nonsymmetric) tangential stiffness matrix does not possess any complex eigenvalues, in spite of the fact that, for some stress states, the underlying material stiffness matrix does feature complex eigenvalues.

## 2. YIELD SURFACE

We consider a standard elasto-plastic material under small deformations, so that the strain can be split into an elastic and a plastic part. The elastic strains are related to the stresses by Hooke's law and the plastic strain rates obey a nonassociated flow rule. We assume that the Drucker-Prager yield function signals the onset of plastic deformation [2,3]

$$f_{\mathrm{DP}} = \left(I_1^2 - 3I_2\right)^{1/2} + \frac{\alpha I_1}{3} - k = 0, \tag{1}$$

with

$$\alpha = \frac{6\sin\phi}{3-\sin\phi}, \qquad k = \frac{6c\cos\phi}{3-\sin\phi}. \tag{2}$$

Parameter $c$ denotes the cohesion and $\phi$ is the internal friction angle. The plastic flow direction $\mathbf{m}$ is described by $\frac{\partial \tilde{f}_{DP}}{\partial \sigma}$ with

$$\tilde{f}_{\mathrm{DP}} = \left(I_1^2 - 3I_2\right)^{1/2} + \frac{\beta I_1}{3} - k, \qquad \beta = \frac{6\sin\psi}{3-\sin\psi}. \tag{3}$$

Normally, $\psi$ is smaller than $\phi$ and can be negative.

Two material simplifications are considered, namely plane-stress and plane-strain conditions. In case of plane-strain, the gradient vector $\mathbf{n}$ to the yield surface $f_{\mathrm{DP}}$ becomes

$$\mathbf{n} = \begin{bmatrix} \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{xx}} \\ \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{yy}} \\ \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{zz}} \\ \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{xy}} \end{bmatrix} = \frac{1}{2\left(I_1^2 - 3I_2\right)^{1/2}} \begin{bmatrix} 2\sigma_{xx} - \sigma_{yy} - \sigma_{zz} \\ 2\sigma_{yy} - \sigma_{xx} - \sigma_{zz} \\ 2\sigma_{zz} - \sigma_{xx} - \sigma_{yy} \\ 6\sigma_{xy} \end{bmatrix} + \frac{\alpha}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{4}$$

while the flow direction is given by

$$\mathbf{m} = \mathbf{n} + \frac{(\beta - \alpha)}{3}[1,1,1,0]^t. \tag{5}$$

As we will see later, the only variables that influence the eigenvalues are the elements of $\mathbf{n}$ and $\mathbf{m}$.

The simplification for plane-stress implies that the stress tensor has three different elements and only two eigenvalues (principal stresses). Vectors $\mathbf{n}$ and $\mathbf{m}$ now become

$$\mathbf{n} = \begin{bmatrix} \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{xx}} \\ \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{yy}} \\ \dfrac{\partial f_{\mathrm{DP}}}{\partial \sigma_{xy}} \end{bmatrix} = \frac{1}{2\left(I_1^2 - 3I_3\right)^{1/2}} \begin{bmatrix} 2\sigma_{xx} - \sigma_{yy} \\ 2\sigma_{yy} - \sigma_{xx} \\ 6\sigma_{xy} \end{bmatrix} + \frac{\alpha}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{6}$$

and

$$\mathbf{m} = \mathbf{n} + \frac{(\beta - \alpha)}{3}[1,1,0]^t, \tag{7}$$

respectively. The yield surface is now an ellipsoid, Figure 1. Next, we will analyze the eigenvalues on the positive $\sigma_{xy}$ part of the ellipsoid.

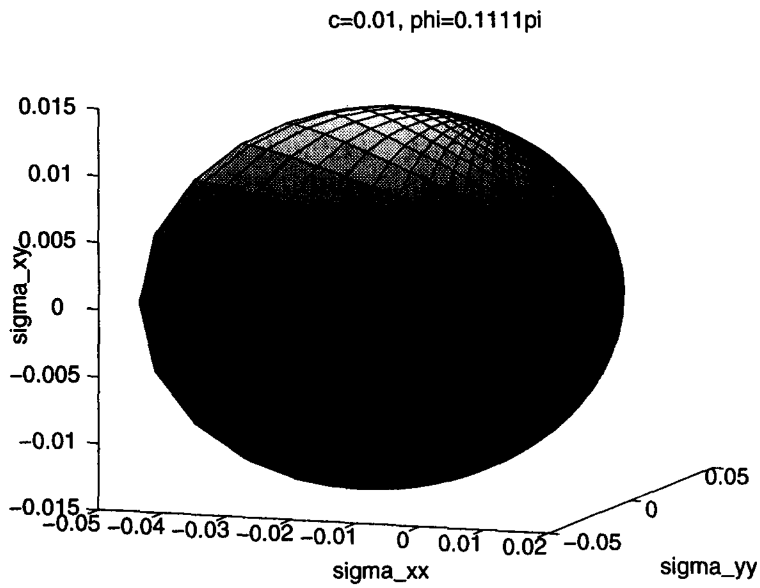


Figure 1. Drucker-Prager yield surface for plane-stress conditions.

## 3. ELASTO-PLASTIC TANGENT STIFFNESS MATRIX

Standard elasto-plasticity leads to a linear relation between the stress rate $\dot{\sigma}$ and the strain rate $\dot{\varepsilon} : \dot{\sigma} = \mathbf{D}_{\mathrm{ep}}\dot{\varepsilon}$ with $\mathbf{D}_{\mathrm{ep}}$ the elasto-plastic tangent stiffness matrix. This matrix consists of an elastic and a plastic part: $\mathbf{D}_{\mathrm{ep}} = \mathbf{D}_{\mathrm{e}} - \mathbf{D}_{\mathrm{p}}$. We will denote the elastic part by $\mathbf{D}_{\mathrm{e}}$ (symmetric) and the plastic part by $\mathbf{D}_{\mathrm{p}}$. Matrix $\mathbf{D}_{\mathrm{e}}$ is defined through Hooke's law while matrix $\mathbf{D}_{\mathrm{p}}$ is of the form

$$\mathbf{D}_{\mathrm{p}} = \frac{\mathbf{D}_{\mathrm{e}}\mathbf{m}\mathbf{n}^t\mathbf{D}_{\mathrm{e}}}{\mathbf{m}^t\mathbf{D}_{\mathrm{e}}\mathbf{n}}. \tag{8}$$

Accordingly, $\mathbf{D}_{\mathrm{ep}}$ reads

$$\mathbf{D}_{\mathrm{ep}} = \mathbf{D}_{\mathrm{e}} - \frac{\mathbf{D}_{\mathrm{e}}\mathbf{m}\mathbf{n}^t\mathbf{D}_{\mathrm{e}}}{\mathbf{m}^t\mathbf{D}_{\mathrm{e}}\mathbf{n}}. \tag{9}$$

If $\mathbf{m}$ differs from $\mathbf{n}$, the plasticity is nonassociated and $\mathbf{D}_{ep}$ will be nonsymmetric. As was observed by Wedderburn [4] as early as in 1934 (see [5]), this matrix has a rank that is one less than the rank of $\mathbf{D}_e$. More specifically: vector $\mathbf{m}$ is a right eigenvector, and $\mathbf{n}$ a left one, belonging to a zero eigenvalue. This means that for plane-strain conditions, the nonzero eigenvalues are the roots of a polynomial of degree three and for plane-stress conditions of degree two. For plane-stress conditions, the eigenvalues of the elasto-plastic matrix will thus be real if the discriminant is positive

$$A\lambda^2 + B\lambda + C, \qquad \lambda \text{ real } \Leftrightarrow B^2 - 4AC > 0. \tag{10}$$

For plane-strain conditions, we have a third degree polynomial $\lambda^3 + A\lambda^2 + B\lambda + C$. Now let

$$q = \frac{B}{3} - \frac{A^2}{9}, \qquad r = \frac{AB}{6} - \frac{C}{2} - \frac{A^3}{27}, \tag{11}$$

then all roots will be real if $q^3 + r^2 < 0$ [6].

## 4. ORTHOTROPIC MATERIALS

In orthotropic materials, the stiffness is different in orthogonal directions. To not over complicate matters, we assume no contraction effects. The elasticity matrix $\mathbf{D}_e$ is diagonal. Consider now the following lemma.

LEMMA 1. *Let $\mathbf{A}$ be an $n \times n$ diagonal matrix. The nonsymmetric rank-1 update $\mathbf{A} + k\mathbf{u}\mathbf{v}^t$ is similar to a symmetric matrix, if the products $u_i v_i$ are all positive.*

PROOF. We wish to prove that the characteristic polynomial of $\mathbf{A} + k\mathbf{u}\mathbf{v}^t$ consists of terms in which either both $u_i$ and $v_i$ are present as a product for some $i$, or neither one is. Because then, if all $u_i v_i > 0$, $\mathbf{A} + k\mathbf{u}\mathbf{v}^t$ would be similar to $\mathbf{A} + k\mathbf{z}\mathbf{z}^t$ with

$$\mathbf{z} = \left[|u_1 v_1|^{1/2}, \dots, |u_n v_n|^{1/2}\right]^t,$$

for $z_i z_i = |u_i v_i|^{1/2} |u_i v_i|^{1/2} = u_i v_i$.

The characteristic polynomial of $\mathbf{A} + k\mathbf{u}\mathbf{v}^t$ is equal to the determinant of $\mathbf{B} = \mathbf{A} + k\mathbf{u}\mathbf{v}^t - \lambda\mathbf{I}$. For the computation of a determinant, we can use the following formula [7]:

$$\det(\mathbf{B}) = \sum_{\pi} \prod_{i=1}^{n} b_{i\pi_i} \det(\mathbf{P}_\pi). \tag{12}$$

The columns of $\mathbf{P}_\pi$ are a permutation of the columns of the identity matrix with the ordering according to the permutation $\pi = (\pi_1, \dots, \pi_n)$ of the numbers $(1, \dots, n)$. The sum is taken over all $n!$ permutations $\pi$.

A term $b_{i\pi_i}$ is either a diagonal element of the form $a_{ii} - \lambda + k u_i v_i$ if $i = \pi_i$, or an off-diagonal element of the form $k u_i v_{\pi_i}$, $i \neq \pi_i$. Therefore, we can rewrite equation (12) as

$$\det(\mathbf{B}) = \sum_{\pi} \left\{ \prod_{i \in \omega_1(\pi)} (a_{ii} - \lambda + k u_i v_i) \prod_{j \in \omega_2(\pi)} k u_j v_{\pi_j} \right\} \det(\mathbf{P}_\pi). \tag{13}$$

In this expression, $\pi_j \neq j$. The set $\omega_1(\pi)$ consists of those $i \in \{1, \dots, n\}$ for which $i = \pi_i$. The numbers $j$ in $\{1, \dots, n\}$ for which $j \neq \pi_j$ are gathered in $\omega_2(\pi)$. Therefore, $\omega_1$ and $\omega_2$ are two complementary sets.

The first product of (13) already consists of products $u_i v_i$. To show that this holds also for the second product, we will make use of the fact that every first and second index of $b$ in (12) appears *exactly* once. Suppose $1 \in \omega_2$. Then $1 \notin \omega_1$ because $\omega_1$ and $\omega_2$ are disjoint. But then, 1 must be supplied as a second index by $\pi_j$ because $\omega_1$ and $\omega_2$ are complementary. Thus, both $u_1$

and $v_1$ will be present in the second product of equation (13). Naturally, this reasoning holds for any $j \in \omega_2$. Consequently, after rearranging the second product of equation (13), we obtain

$$\det(\mathbf{B}) = \sum_{\pi} \left\{ \prod_{i\in\omega_1(\pi)} (a_{ii} - \lambda + ku_iv_i) \prod_{j\in\omega_2(\pi)} ku_jv_j \right\} \det(\mathbf{P}_\pi). \tag{14}$$ ∎

This lemma is helpful because it relates directly to the elasto-plastic matrix $\mathbf{D}_{\mathrm{ep}}$. Matrix $\mathbf{D}_{\mathrm{ep}}$ is a rank-1 update of the elasticity matrix $\mathbf{D}_{\mathrm{e}}$, that is diagonal in case of orthotropy. So, $\mathbf{A} = \mathbf{D}_{\mathrm{e}}$ and $k\mathbf{u}\mathbf{v}^t = -\mathbf{D}_{\mathrm{e}}\mathbf{m}\mathbf{n}^t\mathbf{D}_{\mathrm{e}}/\mathbf{m}^t\mathbf{D}_{\mathrm{e}}\mathbf{n}$. Because all elements of $\mathbf{D}_{\mathrm{e}}$ are positive, the condition on the products $u_iv_i$ implies similarity of $\mathbf{D}_{\mathrm{ep}}$ to a symmetric matrix in case of positive products $m_in_i$.

Note, that the $m_i$ and $n_i$ appear only in pairs. It holds, therefore, that since the last element of $\mathbf{m}$ and $\mathbf{n}$ are identical, their product will always be positive.

## 4.1. Orthotropic Plane-Stress Conditions

The stresses are now $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$. Therefore, $\mathbf{m}$ and $\mathbf{n}$ both have three elements. We denote the three diagonal elements of $\mathbf{D}_{\mathrm{e}}$ by $a, b, c$, and we set out to determine whether the eigenvalues are real for any admissible stress distribution. We must, therefore, look at the discriminant of the characteristic polynomial of $\mathbf{D}_{\mathrm{ep}}$. After some elaboration of the appropriate formulae, we can distinguish three different notations consisting of a square and an additional term

$$\{\dots\}^2 + 4ac(b-a)(b-c)\frac{m_1n_1m_3n_3}{\delta^2}, \tag{15}$$

$$\{\dots\}^2 + 4ac(b-a)(b-c)\frac{m_1n_1m_3n_3}{\delta^2}, \tag{16}$$

$$\{\dots\}^2 + 4ac(b-a)(b-c)\frac{m_1n_1m_3n_3}{\delta^2}, \tag{17}$$

with $\delta = \mathbf{m}^t\mathbf{D}_{\mathrm{e}}\mathbf{n}$. These formulations provide the following sufficient conditions:

$$\begin{aligned} &\left.\begin{matrix} b<a<c \\ \text{or } c<a<b \end{matrix}\right\} \Rightarrow m_1n_1 > 0 \quad \text{or} \quad m_2n_2 < 0, \\ &\left.\begin{matrix} a<b<c \\ \text{or } c<b<a \end{matrix}\right\} \Rightarrow m_1n_1 < 0 \quad \text{or} \quad m_2n_2 > 0. \end{aligned} \tag{18}$$

If the conditions are not fulfilled, complex eigenvalues may appear. Because of Lemma 1, possible complex eigenvalues are expected only when one of the components of $\mathbf{n}$ changes sign. The direction of the plastic flow $\mathbf{m}$ differs from $\mathbf{n}$ so that the situation may arise that $n_i$ changes sign, but $m_i$ not, and consequently $m_in_i < 0$.

For orthotropic plane-stress conditions, the yield surface has the form of an ellipsoid (Figure 1). Since $m_3$ and $n_3$ are equal and always appear together, and since they are the only $m_i, n_i$ depending on $\sigma_{xy}$ (see equation (6)), the shear stress $\sigma_{xy}$ is present only in squared form in the characteristic polynomial. Therefore, the eigenvalues of the upper and lower part of the ellipsoid are the same and only those on the upper part are computed. Figure 2 is a view of these eigenvalues. There are two nonzero eigenvalues, and the range of each is printed above the plots, where the lowest value corresponds to the darkest shade of grey, and the highest to the lightest. For the choice of $a, b$, and $c$ denoted in the figure, complex eigenvalues may only arise, according to conditions (18), in the areas where $m_1n_1 < 0$ and $m_2n_2 > 0$. This is where the first component of the normal to the oval mesh lines ($n_1$) changes sign, or put differently, where the tangent to the oval is horizontal. Some complex eigenvalues indeed appear. They are denoted by holes.

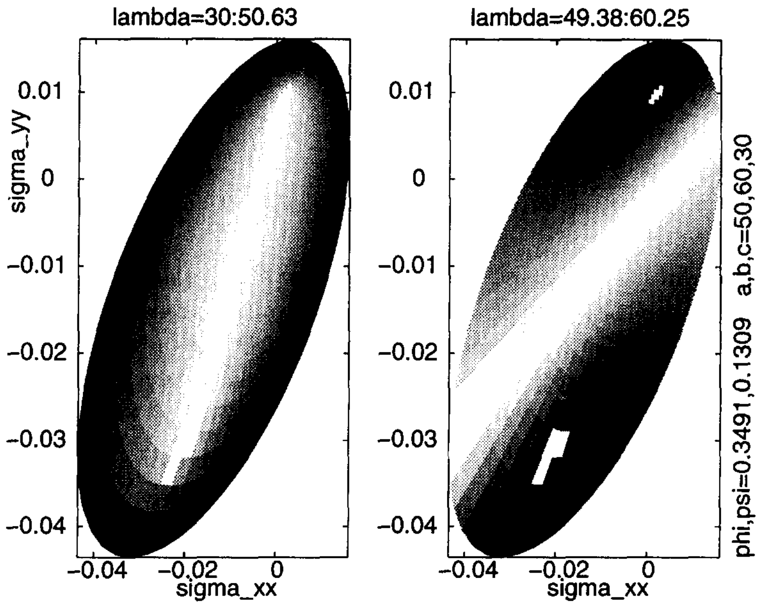


Figure 2. Eigenvalues distribution for orthotropic plane-stress conditions.

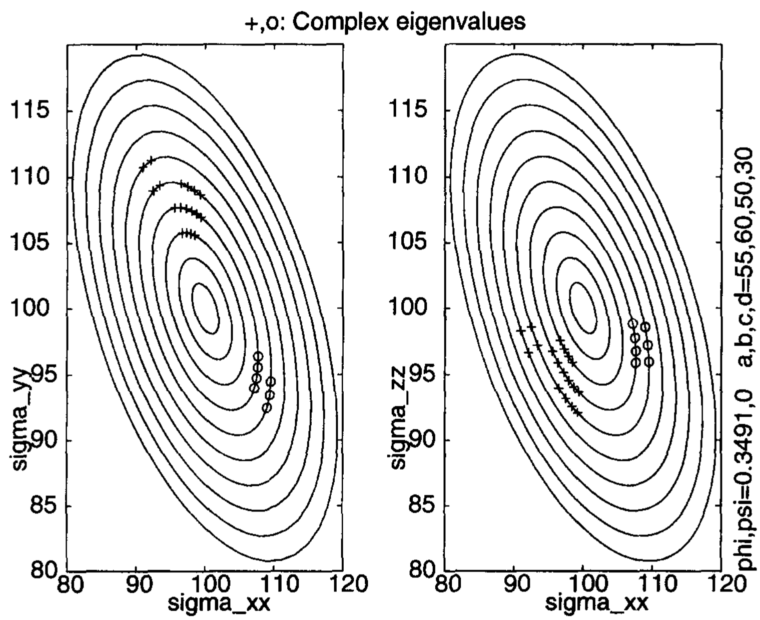


Figure 3. Complex eigenvalues for orthotropic plane-strain conditions.

### 4.2. Orthotropic Plane-Strain Conditions

There is one more stress component involved in this case: $\sigma_{zz}$. The elasticity matrix has, therefore, four columns and rows. The appropriate characteristic polynomial of the elasto-plastic polynomial is thus of degree four, with one zero root. This makes the analysis more difficult, and unfortunately to such an extent, that so far it has appeared impossible to obtain sufficient conditions such as discussed in the previous section. A numerical analysis of the eigenvalue is visualized in Figure 3. For a specific circular mesh line of the Drucker-Prager cone, the appropriate ranges of local stresses were calculated. For all these combinations of stresses, eigenvalues were computed. We have only plotted the complex eigenvalues. They appear exactly there, where Lemma 1 does not hold (products $m_i n_i$ are not positive). This is in the areas where the tangent to the mesh line is horizontal.

## 5. ISOTROPIC MATERIALS

In an isotropic materials, two parameters play a crucial role: Young's modulus $E$ and Poisson's ratio $\nu$. The elasticity matrix is composed of three different combinations of these two parameters

$$a = (1-\nu)\xi, \qquad d = \nu\xi, \qquad c = \frac{(1-2\nu)\xi}{2}, \tag{19}$$

where we have introduced

$$\xi = \frac{E}{(1+\nu)(1-2\nu)}. \tag{20}$$

### 5.1. Isotropic Plane-Stress Conditions

The elasticity matrix is of size 3 × 3

$$\mathbf{D_e} = \begin{bmatrix} a & d & 0 \\ d & a & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{21}$$

For further simplification we define the following:

$$u = m_1n_1 + m_2n_2, \qquad v = m_1n_2 + m_2n_1, \qquad w = m_3n_3. \tag{22}$$

Again we consider the discriminant of the characteristic polynomial of the elasto-plastic matrix $\mathbf{D_{ep}}$. After some manipulations of the standard formulations, we obtain for the discriminant the following expression:

$$(1-2\nu)^2\xi^4 \frac{\{\nu^2v^2/4 - (1+\nu)\nu(w+u/2)v + (1+\nu)^2u^2/4 + \nu^2w^2 + \nu^2uw\}}{\delta^2} = (1-2\nu)^2\xi^4\frac{f(u,v,w)}{\delta^2}, \tag{23}$$

where $\delta = \mathbf{m}^t\mathbf{D_e}\mathbf{n}$. Function $f(u,v,w)$, as it appears in equation (23), can be rewritten as follows:

$$\frac{\nu(u-v)(1+\nu)(u+2w)}{2} + \frac{\nu^2(u+2w)^2}{4} + \frac{\nu^2v^2}{4} + \frac{(1+2\nu)u^2}{4}. \tag{24}$$

Elaboration of equations (6) and (7) shows that $u + w > 0$. By definition, $w > 0$ ($m_3 = n_3$).

### 5.2. Isotropic Plane-Strain Conditions

Taking into account $\sigma_{zz}$, the elasticity matrix has one column and row more than that for plane-stress conditions

$$\mathbf{D_e} = \begin{bmatrix} a & d & d & 0 \\ d & a & d & 0 \\ d & d & a & 0 \\ 0 & 0 & 0 & c \end{bmatrix}. \tag{25}$$

For further simplification, the following substitutions are made:

$$u = m_1n_1 + m_2n_2 + m_3n_3, \qquad w = m_4n_4,$$
$$v = m_1n_2 + m_1n_3 + m_2n_1 + m_2n_3 + m_3n_1 + m_3n_2.$$

Elaboration of the characteristic polynomial of $\mathbf{D_{ep}}$ supplies the ingredients for computation of $q^3 + r^2$ (see Section 3)

$$\begin{aligned} q^3 + r^2 &= \frac{-\xi^{10}\nu^2}{1728\delta^4}(1-2\nu)^6(2u-v)^2 \\ &\times \left\{(\nu v - 2\nu u - u - \nu w)^2 + 8\nu^2 w(u+w-v) - 4\nu v w\right\}, \end{aligned} \tag{26}$$

which needs to be negative to guarantee real eigenvalues. In fact, elaboration of equations (4) and (5) shows that the second part of equation (26) is always positive.

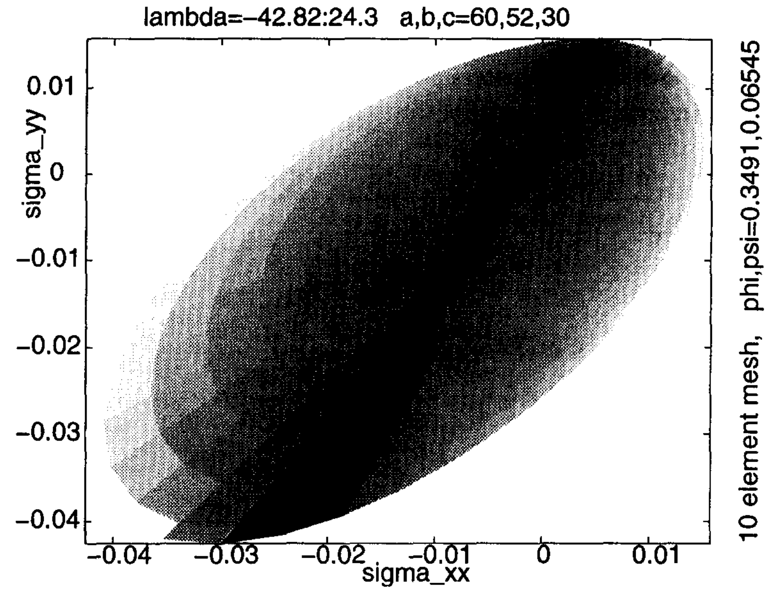


Figure 4. Smallest eigenvalue of the structural stiffness matrix for orthotropic plane-stress conditions.

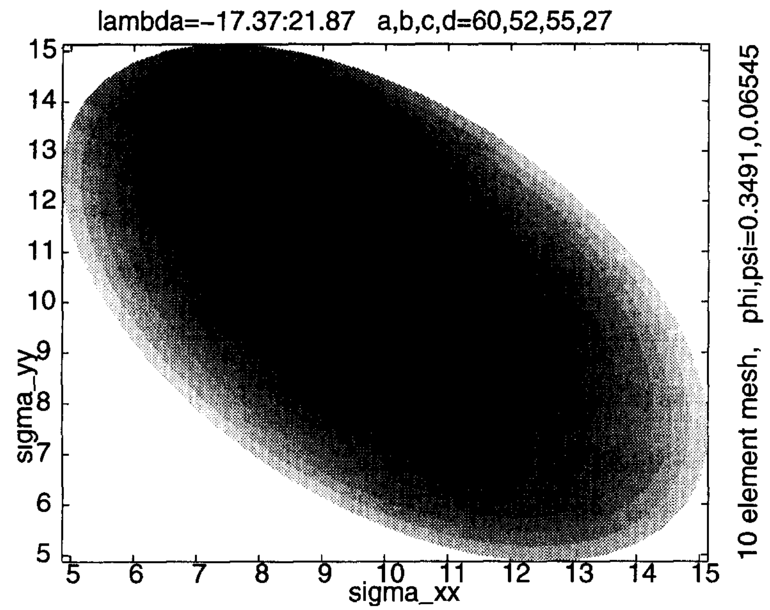


Figure 5. Smallest eigenvalue of the structural stiffness matrix for orthotropic plane-strain conditions.

## 6. SPATIAL DISCRETIZATION

To examine whether the complex eigenvalues that were computed for the Drucker-Prager plasticity model with orthotropic elastic are also found in a boundary value problem, finite element analyses were carried out. A simple rectangular domain was considered, which was discretized using ten linear elements. We recall that it was assumed that even though the material operator contains complex eigenvalues due to its nonsymmetry, the complete operator consisting also of the equilibrium conditions and the kinematic relations, would not display any complex eigenvalues. Indeed, when applying all the stress combinations on this simple (homogeneous) boundary value problem, no complex eigenvalues were computed. This holds true for both plane-stress and plane-strain conditions, for the case of orthotropic elasticity and (of course) for the isotropic case. A QR [8] method was used to extract all the eigenvalues of the discretized boundary value problem. Figures 4 and 5 show the lowest (real) eigenvalues for all stress combinations for each of the orthotropic models. The eigenvalue distribution for isotropy is very similar.

## 7. CONCLUDING REMARKS

We have investigated the eigenvalues related to the operator of an elasto-plastic constitutive model with a Drucker-Prager yield function and a nonassociated flow rule. The nonassociated flow rule leads to a nonsymmetric constitutive operator, and therefore, possibly to complex eigenvalues. For the elastic part of the elasto-plastic model, orthotropy and isotropy have been assumed, respectively.

We have shown an important role in the eigenvalue analysis is played by the difference between the normal to the yield surface $\mathbf{n}$ and the direction of the plastic flow $\mathbf{m}$. When no contraction was considered, in combination with orthotropy, we have proved that the eigenvalues of the stiffness matrix will always be real when the elements of $\mathbf{m}$ and $\mathbf{n}$ have the same sign. If they differ, we have deduced extra sufficient conditions for the plane-stress model. These enlarge the set of stress distributions for which real eigenvalues can be guaranteed. Some complex eigenvalues were computed, for those stress combinations not covered by any of the conditions.

Isotropic materials on the other hand, for which contraction was taken into account, will always produce real eigenvalues. The structure of the characteristic polynomial is such that we can guarantee real eigenvalues, for reasonable values for the elastic material parameters.

Finally, we have combined the constitutive equation with the equilibrium and kinematic relations and discretized the model with finite elements. We have shown that for the simple example chosen, the structural tangent stiffness matrix does not possess complex eigenvalues.

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