# Projection acceleration of Krylov solvers

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In many applications it appears that the initial convergence of preconditioned Krylov solvers is slow. The reason for this is that a number of small eigenvalues are present. After these bad eigenvector components are approximated, the fast superlinear convergence sets in. A way to have fast convergence from the start is to remove these components by a projection. In this paper we give a comparison of some of these projection operators.



## 1 Introduction

To predict the presence of oil and natural gas in a reservoir it is important to know the fluid pressure in the rock formations. A mathematical model for the prediction of fluid pressure is given by a time-dependent diffusion equation. Application of the finite element method leads to systems of linear equations. A complication is that the underground consists of layers with very large contrasts in permeability. This implies that the symmetric and positive definite coefficient matrix has a very large condition number. Bad convergence behavior of the ICCG method has been observed, and a classical termination criterion is not valid in this problem.

It appears that the number of small eigenvalues of the IC preconditioned matrix is equal to the number of high permeable domains, which are not connected to a Dirichlet boundary. The Deflated ICCG method annihilates the effect of these small eigenvalues. Problems with large jumps in the coefficients can also be solved with a suitable (additive) coarse grid correction. In this paper we compare deflation method with the additive coarse grid correction and the abstract balancing preconditioner. For more details of these methods and comparisons we refer to [3].

## 2 The projection methods

We consider solution methods for the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD). We give an analysis of the Preconditioned Conjugate Gradient (PCG) method with preconditioning matrix $M \in \mathbb{R}^{n\times n}$. As projection vectors we use the column vectors of the matrix $Z \in \mathbb{R}^{n\times r}$ which is of full rank and $r < n$. Furthermore, we define the following matrices: $E = Z^T AZ$, $Q = ZE^{-1}Z^T$, and $P = I - AQ$.

In this paper we consider the additive coarse grid preconditioner (AD), the abstract balancing preconditioner (BNN) and two deflation methods, (DEF1) proposed in [5, 1], and (DEF2) proposed by Nicolaides [2]. Furthermore, we consider a more robust variant of deflation (ADEF) and a faster variant of abstract balancing (R-BNN2) preconditioner.

For the analysis we consider the preconditioners, which are used in the various methods, see Table 1. Note that some methods have the same preconditioner. We distinguish them, because they are derived from different methods and their implementation is different. Note that projection methods are used in various types of solvers: preconditioned Krylov subspace methods, multigrid methods, and domain decomposition methods [4].

| method | AD | DEF1 | DEF2 | ADEF | BNN | R-BNN |
|---|---|---|---|---|---|---|
| preconditioner | $M^{-1} + Q$ | $M^{-1}P$ | $P^TM^{-1}$ | $P^TM^{-1} + Q$ | $P^TM^{-1}P + Q$ | $P^TM^{-1}$ |

**Table 1** The operators used in the projection methods

## 3 Theoretical comparison

In this section we give a spectral analysis of the projection methods. It appears that there are two classes of methods. In the first class the bad eigenvalues are projected to the zero eigenvalue:

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**Theorem 3.1** *The operators of DEF1, DEF2, and R-BNN have the same condition numbers:*

$$\sigma\left(M^{-1}PA\right) = \sigma\left(P^T M^{-1} A\right) = \sigma\left(P^T M^{-1} PA\right) = \{0, 0, \ldots, 0, \lambda_{r+1}, \ldots, \lambda_n\}.$$

In the second class the bad eigenvalues are projected to the eigenvalue 1:

**Theorem 3.2** *The operators of BNN and A-DEF have the same condition numbers:*

$$\sigma\left((P^T M^{-1} P + Q)A\right) = \sigma\left((P^T M^{-1} + Q)A\right) = \{1, 1, \ldots, 1, \lambda_{r+1}, \ldots, \lambda_n\}.$$

The methods of the first class have the smallest effective condition number.

## 4 Comparison of the implementation

For the amount of work and memory of the various methods we refer to [3]. Here we only give some results about the robustness of the methods with respect to rounding errors.

**Theorem 4.1** *The spectrum $\sigma$ after perturbing $E^{-1}$ with a small matrix $\epsilon R$, where $R \in \mathbb{R}^{k \times k}$ is a symmetric random matrix, are given by the following expressions:*

- *The operators of DEF1, DEF2, and R-BNN:* $\sigma \approx \{\mathcal{O}(\epsilon), \ldots, \mathcal{O}(\epsilon), \lambda_{r+1}, \ldots, \lambda_n\}$
- *The operators of BNN and A-DEF:* $\sigma \approx \{1 + \mathcal{O}(\epsilon), 1 + \mathcal{O}(\epsilon), \ldots, 1 + \mathcal{O}(\epsilon), \lambda_{r+1}, \ldots, \lambda_n\}$

This implies that the methods from the first class can be sensitive to rounding errors because eigenvalues of the order $\epsilon$ spoil the convergence of the projection methods. This is in contrast with the methods from the second class where the perturbed eigenvalues are close to eigenvalue 1, so the effect on the convergence is negligible.

In [3] a number of experiments are reported that illustrate the theoretical results. From these experiments and the theory we derive a number of conclusions, which are given in the next section.

## 5 Conclusions

In this paper we give an eigenvalue analysis for various projection methods with and without the effect of rounding errors. From this comparison (for full details we refer to [3]) we conclude the following:

- BNN is the most robust method, however it can be 3 times more expensive than the DEF1 method.
- BNN and A-DEF are the most robust methods with respect to rounding errors and perturbations.
- A-DEF is a good compromise with respect to robustness and efficiency for the projection methods considered in this paper.

## References


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