Lagrangian Formulation of Multiclass Kinematic Wave Model

Femke van Wageningen-Kessels, Hans van Lint,
Serge P. Hoogendoorn, and Kees Vuik

The kinematic wave model is often used in simulation tools to describe dynamic traffic flow and to estimate and predict traffic states. Discretization of the model is generally based on Eulerian coordinates, which are fixed in space. However, the Lagrangian coordinate system, in which the coordinates move with the velocity of the vehicles, results in more accurate solutions. Furthermore, if the model includes multiple user classes, it describes real traffic more accurately. Such a multiclass model, in contrast to a mixed-class model, treats different types of vehicles (e.g., passenger cars and trucks or vehicles with different origins or destinations, or both) differently. The Lagrangian coordinate system is combined with a multiclass model, and a Lagrangian formulation of the kinematic wave model for multiple user classes is proposed. It is shown that the advantages of the Lagrangian formulation also apply for the multiclass model. Simulations based on the Lagrangian formulation result in more accurate solutions than simulations based on the Eulerian formulation.

Traffic flow models and simulation tools are often used for traffic state estimation and prediction. They are applied for online and real-time predictions such as those used in traffic control systems and for dynamic traffic assignment. For many applications, in particular real-time applications, fast and accurate computing is required. One of the main drawbacks of the current traffic flow simulation tools is their long computation time, especially if the user needs accurate results for a large road network. It takes so long to make an accurate prediction that the information is no longer valuable for real-time applications by the time it becomes available. Computing time is also important for evaluating dynamic traffic-management scenarios, especially if an iterative optimization technique is used. Alternative model formulations and numerical implementations of the model are studied to improve the accuracy and the computing time of the simulation. This paper focuses on solving the model equations accurately.

F. van Wageningen-Kessels, H. van Lint, and S. P. Hoogendoorn, Faculty of Civil Engineering and Geosciences, Delft University of Technology, Stevinweg 1, P.O. Box 5048, 2600 GA Delft, Netherlands. K. Vuik, Faculty of Electrical Engineering, Computer Science and Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, Netherlands. Corresponding author: F. van Wageningen-Kessels, f.l.m.vanwageningen-kessels@tudelft.nl.

Transportation Research Record: Journal of the Transportation Research Board, No. 2188, Transportation Research Board of the National Academies, Washington, D.C., 2010, pp. 29–36.
DOI: 10.3141/2188-04

LAGRANGIAN COORDINATES AND MULTICLASS MODELS

The kinematic wave model, also known as the Lighthill–Whitham–Richards, or LWR, model (*1, 2*), is used to describe dynamic traffic flow as if it were a continuum flow. Discretization and simulation of the model are generally based on Eulerian coordinates, with the coordinate system fixed in space. However, recent studies show the advantages of the Lagrangian formulation (*3–6*). In this formulation the coordinate system moves with the same velocity as the vehicles. Simulations based on the Lagrangian formulation lead to more accurate results with less numerical diffusion (*3, 6*).

Van Lint et al. introduced a kinematic wave model with multiple user classes (*7*). They show qualitatively that this multiclass representation (referred to as Fastlane) describes traffic flow more accurately than mixed-class (or single-class) models. As in most multiclass models, the speed is class-specific. In the Fastlane model the total density is a weighted summation of all class-specific densities. One of the key characteristics of the model is the dependence of the weight (which is equal to the passenger-car equivalent [PCE] value) on the current speed and consequently on the total density. Therefore, it represents that in congestion a truck takes relatively more space, when compared to a car, than under free-flow conditions.

Leclercq and Laval propose a multiclass model with Lagrangian coordinates using a variational formulation, which restricts the shape of the fundamental diagram to be triangular or piecewise linear (*8*). Moreover, user classes are only introduced after discretization. Therefore, the multiclass model is not truly continuous. Zhang et al. show that certain multiclass models are hyperbolic (*9*). Hyperbolicity is an important condition for fast and accurate simulations in Lagrangian coordinates. However, this analysis is limited to multiclass models in which only the speed is class-specific. Moreover, the speed of a certain class is always a certain fixed percentage lower than that of the fastest class. This is especially unrealistic in congestion: then one would expect the same speed for all user classes. Furthermore, the total density is an unweighted summation over all class-specific densities.

The main contribution of the present study is the development of a Lagrangian formulation of a more generic multiclass traffic flow model, including fundamental diagrams of any (realistic) shape. Moreover, the fundamental diagrams can have different shapes for each class, and the total density is a weighted summation over all class-specific densities with the weight depending on the total density. In the next section such a generic multiclass model is derived. The derivation is partly based on the multiclass model in Eulerian formulation and partly based on the conservation of vehicles law. In the third section the mathematical properties of the Lagrangian formulation of the multiclass model are analyzed, and it is shown that under certain mild modeling conditions the same advantages hold as for the Lagrangian formulation of the mixed-class model. The Lagrangian formulation of the Fastlane model is derived, and it is shown that the conditions hold for the two-class version of the Fastlane model in the Lagrangian formulation. In the fourth section both the Lagrangian and Eulerian formulations of the Fastlane model are applied to a simple test problem. Simulations show that the Lagrangian formulation gives more accurate results than the Eulerian formulation.

LAGRANGIAN FORMULATION OF MULTICLASS MODEL

In multiclass models the difference in characteristics of classes is taken into account. Classes can be origin- or destination-specific, or both, based on differences in, for example, speed and length. In the latter case the model usually makes a distinction between passenger cars and trucks; sometimes these classes are subdivided, for example, into light and heavy trucks. This study considers neither origin- nor destination-specific classes. However, the approach and the analysis can be extended to this. The multiclass model in the Lagrangian formulation can be derived partly from the model in the Eulerian formulation. Therefore, the model in its traditional Eulerian formulation is first introduced, followed by the introduction and derivation of the Lagrangian formulation.

Multiclass Kinematic Wave Model

The multiclass kinematic wave model is based on the conservation of vehicles. The conservation equation holds for each user class, each user class has its own fundamental relation, and the total density is a weighted summation over all class-specific densities:

Class-specific conservation equation:

$$\frac{\partial \rho_u}{\partial t} + \frac{\partial q_u}{\partial x} = 0 \qquad \forall u \tag{1}$$

Class-specific fundamental relation:

$$q_u = Q_u(\rho) \qquad \forall u \tag{2}$$

Total density:

$$\rho = \sum_u \eta_u \rho_u \tag{3}$$

where

- t = time coordinate;
- x = space coordinate;
- ρ_u = average class-specific density, i.e., average number of vehicles of class u per meter;
- ρ = average total (or effective) density (PCE veh/m);
- $q_u = \rho_u v_u$ = average flow (veh/s) of class u;
- v_u = average speed (m/s) of class u;
- $Q_u(\rho)$ = class-specific fundamental relation denoting the class-specific equilibrium flow as a function of total density; and
- η_u = user class–specific PCE value.

Values for η_u can be either constant (static) or depend on the total density (dynamic). In the Fastlane model the PCE values are dynamic.

Class 1 is usually considered as the reference class. In most practical models Class 1 represents passenger cars and its PCE value is one. The PCE value of other classes is related to the PCE value of the reference class. If the PCE value is dynamic it usually is an implicit function: the PCE value depends (indirectly) on the total density and vice versa. For example, in Fastlane the PCE value of trucks depends on the current speed of both cars and trucks, which depends on the current density, which in turn depends on the PCE value of trucks.

More generic versions of the multiclass kinematic wave model can be defined. However, this model is much more generic than the ones studied in other research (*8, 9*). One such model (*9*) has the fundamental relation $Q_u(\rho)$ = cu$Q1(\rho)$ with cu a class-specific constant and $Q1(\rho)$ the fundamental relation of the reference class. Furthermore, the summation is unweighted: $\eta_u = 1, \forall u$. In this study the generalized formulation is used as described above. A more detailed discussion of this multiclass Eulerian model is available elsewhere (*7*).

Lagrangian Formulation

The multiclass model can also be formulated in Lagrangian coordinates. The model equations are first introduced, and the derivation is shown below:

Conservation Class 1:

$$\frac{Ds_1}{Dt} + \frac{\partial v_1}{\partial n} = 0 \tag{4}$$

Other conservation classes:

$$\frac{D}{Dt}\left(\frac{s_1}{s_u}\right) + \frac{\partial}{\partial n}\left(\frac{v_1 - v_u}{s_u}\right) = 0 \qquad \forall u \neq 1 \tag{5}$$

Class-specific fundamental relation:

$$v_u = V_u(s) \qquad \forall u \tag{6}$$

Average vehicle spacing:

$$s = \frac{1}{\sum_u \frac{\eta_u}{s_u}} \tag{7}$$

where $s_u = 1/\rho_u$ is the average vehicle spacing (m/veh) of class u, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} \tag{8}$$

is the Lagrangian time derivative. Vehicles are numbered opposite to the driving direction. The vehicle number of User Class 1 is denoted by n; vehicles of other user classes are not numbered (see Figure 1). The model in the Lagrangian formulation is a continuum model, like the original kinematic wave model, and consequently n can take any real value: it is not integer. The average vehicle spacing of all vehicles in meters per PCE vehicle is denoted by $s = 1/\rho$. $V_u(s)$ is the class-specific fundamental relation in Lagrangian formulation, denoting the class-specific equilibrium vehicle speed as a function of average spacing. For readability reasons in the remainder of the paper, Class 1 is referred to as (passenger) cars, and it is assumed that there is only one other class: trucks. The derivation and most of the analysis of the model can be generalized to more or other classes straightforwardly.

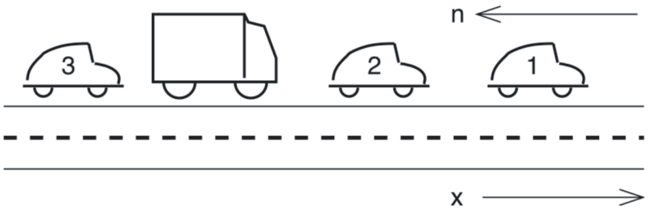

FIGURE 1 Vehicles of User Class 1 (usually passenger cars) are numbered opposite to driving direction.

Derivation of the Lagrangian Formulation

With the definitions

$$q_u = \rho_u v_u$$

$$s_u = \frac{1}{\rho_u}$$

and

$$s = \frac{1}{\rho} \quad (9)$$

the derivation of the Lagrangian fundamental relation (Equation 6) from the Eulerian fundamental relation (Equation 2) is straightforward. Any fundamental diagram that is valid in the Eulerian formulation is also valid in the Lagrangian formulation, and there is no restriction on the shape as in Leclercq and Laval (*8*). Using the definitions from Equation 9, the average vehicle spacing (Equation 7) can be derived from the average density (Equation 3).

Equation 4, the conservation equation for cars, can be derived similarly to the mixed-class Lagrangian conservation equation as described previously by the authors (*5*). This method boils down to following a platoon of vehicles over a time period. The new length of the platoon equals the old length plus the distance traveled by the first vehicle minus the distance traveled by the last vehicle. By taking an infinite platoon and following it over an infinite time period, the Lagrangian formulation of the conservation equation in the mixed-class model is found. In the multiclass model only members of the reference class (cars) are numbered, and the procedure as described above and earlier (*5*) can be followed to derive the conservation equation for cars in the Lagrangian formulation (Equation 4).

In Figure 2 an intuitive derivation of the Lagrangian formulation of the conservation equation for trucks (Equation 5) is illustrated. Vehicle trajectories are drawn in the t, x-plane. The gray box is a control volume that indicates a platoon of cars with length dn; the platoon is followed over time dt. Recall that vehicle flow is considered as a continuum, as in the Eulerian formulation. Therefore, every class is assumed to be present at every location at every time. However, the classes have different characteristics and may travel at different speeds.

All trucks that enter the control volume will also leave it (conservation of vehicles). Trucks enter the box either from the left (i.e., they are initially within the platoon) or from left and above (i.e., they are overtaken by cars). The number of trucks that are initially in the platoon is the initial length of the platoon divided by their vehicle spacing: $dns_{1,\text{initial}}/s_{u,\text{initial}}$. The trucks that are overtaken are the same trucks that pass line segment l. The length of this line segment is the length of the time period times the speed difference: $(v_1 - v_u)dt$. The number of trucks entering the control volume from left and above is thus $dt(v_{1,\text{first}} - v_{u,\text{first}})/s_{u,\text{initial}}$. The number of trucks leaving the control volume is similarly $dns_{1,\text{end}}/s_{u,\text{end}} + dt(v_{1,\text{last}} - v_{u,\text{last}})/s_{u,\text{end}}$. With the conservation of vehicles, Equation 10 is found:

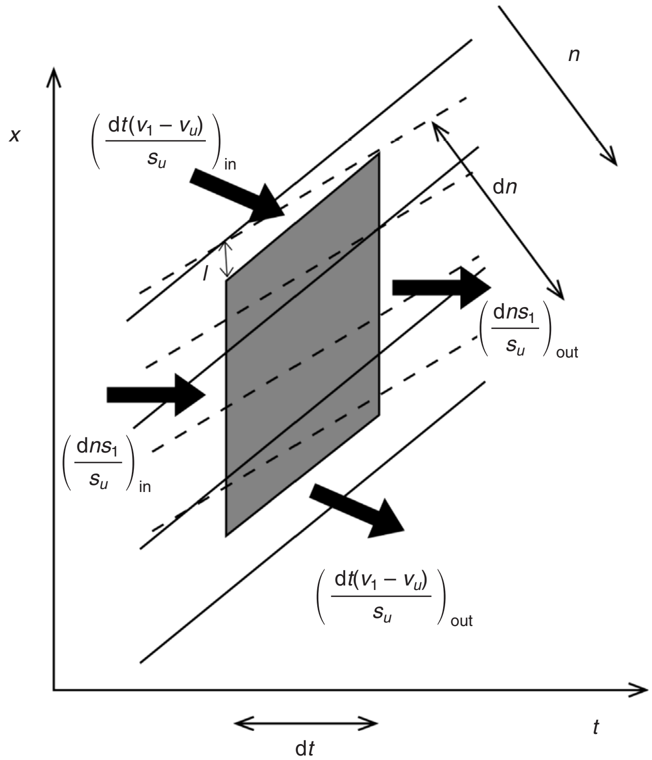

FIGURE 2 Diagram showing vehicle trajectories in *t*, *x*-plane (solid and broken lines represent trajectories of cars (Class 1, fast) and trucks (class *u*, slow), respectively; number of trucks in control volume (gray box) is conserved).

$$dn\frac{s_{1,\text{end}}}{s_{u,\text{end}}} + dt\frac{\left(v_{1,\text{last}} - v_{u,\text{last}}\right)}{s_{u,\text{end}}} = dn\frac{s_{1,\text{initial}}}{s_{u,\text{initial}}} + dt\left(\frac{v_{1,\text{first}} - v_{u,\text{first}}}{s_{u,\text{initial}}}\right) \quad (10)$$

Rearrangement gives

$$\frac{1}{dt}\left(\frac{s_{1,\text{end}}}{s_{u,\text{end}}} - \frac{s_{1,\text{initial}}}{s_{u,\text{initial}}}\right) + \frac{1}{dn}\left(\frac{v_{1,\text{last}} - v_{u,\text{last}}}{s_{u,\text{end}}} - \frac{v_{1,\text{first}} - v_{u,\text{first}}}{s_{u,\text{initial}}}\right) = 0 \quad (11)$$

The definition of the derivative is used to find

$$\lim_{dt\to 0}\frac{1}{dt}\left(\frac{s_{1,\text{end}}}{s_{u,\text{end}}} - \frac{s_{1,\text{initial}}}{s_{u,\text{initial}}}\right) = \lim_{dt\to 0}\frac{1}{dt}\left(\frac{s_1(n,t+dt)}{s_u(n,t+dt)} - \frac{s_1(n,t)}{s_u(n,t)}\right) = \frac{D}{Dt}\left(\frac{s_1(n,t)}{s_u(n,t)}\right),$$

$$\lim_{dn\to 0}\frac{1}{dn}\left(\frac{v_{1,\text{last}} - v_{u,\text{last}}}{s_{u,\text{end}}} - \frac{v_{1,\text{first}} - v_{u,\text{first}}}{s_{u,\text{initial}}}\right) = \lim_{dn\to 0}\frac{1}{dn}\left(\frac{v_1(n+dn,t) - v_u(n+dn,t)}{s_u(n+dn,t)} - \frac{v_1(n,t) - v_u(n,t)}{s_u(n,t)}\right) = \frac{\partial}{\partial n}\left(\frac{v_1(n,t) - v_u(n,t)}{s_u(n,t)}\right) \quad (12)$$

An infinite control volume is taken (i.e., let d$t \to 0$ and d$n \to 0$), Equation 12 is applied, and the conservation of vehicles equation for trucks is found (Equation 5).

Alternatively, the Lagrangian multiclass conservation equations (Equations 4 and 5) can be derived from the Eulerian multiclass conservation equation (Equation 1) by using the definitions in Equation 9 and the Lagrangian time derivative (Equation 8) (*6*). However, the derivation presented above is more intuitive and is, in the authors' opinion, easier to understand.

ADVANTAGES OF LAGRANGIAN FORMULATION AND ITS APPLICATION TO FASTLANE

In this section the advantages of the multiclass model in the Lagrangian formulation in comparison to the Eulerian formulation are studied. Both the Lagrangian formulation in general and the Lagrangian formulation of the Fastlane model are analyzed.

Numerical Advantages

The Godunov scheme (also known as the minimum supply–demand scheme) is widely used for spatial discretization of the kinematic wave model (*10*). The scheme uses information either from downstream (in congestion) or upstream (in free-flow conditions) in the spatial discretization. Especially when the traffic state changes from free flow to congestion, or vice versa, this is computationally demanding (*11*).

The main advantage of the Lagrangian coordinate system is that the direction of the information flow is always the same, independent of the traffic state. Traffic is anisotropic: information never travels faster than traffic (*12*). Since the coordinate system travels at the same velocity as the vehicles, information does not travel faster than the coordinates. This implies that information only flows downstream with respect to the coordinate system, that is, in the direction of increasing vehicle number n, and not necessarily downstream with respect to the road. This can also be understood in terms of the reaction of drivers: drivers only react to vehicles in front of them, not to vehicles behind. Therefore, information only travels from one vehicle to its follower. Because information only flows downstream, the Godunov scheme reduces to an upwind scheme (*4*). The upwind scheme is computationally much less demanding and results in more accurate solutions with less numerical diffusion. Numerical diffusion occurs in solutions computed with the Godunov scheme whenever the Courant–Friedrichs–Lewy condition ($\Delta t/\Delta x\, v_{\max} \leq 1$) is not satisfied exactly (i.e., if $\Delta t/\Delta x\, v_{\max} < 1$). Because the maximum velocity in multi-class models is class-dependent, the Courant–Friedrichs–Lewy condition cannot be satisfied for all classes, and there will be diffusion for classes slower than the fastest class if the Eulerian formulation is used. In Lagrangian coordinates combined with an upwind scheme, there is less diffusion. This has been shown theoretically and numerically for the mixed-class model (*3*). In the fourth section it is shown numerically that this also holds for multiclass models.

However, some mild modeling conditions need to be satisfied to guarantee that information only flows downstream, and as a consequence an upwind scheme can be used to solve the equations. For the two-class model it can be proven (*6*) that information only flows downstream if the following four conditions hold:

1. Class 1 is the reference class with a constant PCE value not larger than the PCE value of Class 2: $1 = \eta_1(\rho) \leq \eta_2(\rho)$, $\forall \rho \in (0, \rho\text{max})$.
2. Class 1 is the fastest class: $v_1(\rho) \geq v_2(\rho)$, $\forall \rho \in (0, \rho\text{max})$.
3. The speed is a nonincreasing function of the density: $dv_u/d\rho \leq 0$, $u = 1, 2$, $\forall \rho \in [0, \rho_{\max}]$.
4. The total density is an increasing function of the class-specific density of Class 2: $\partial\rho/\partial\rho_2 > 0$, $\forall \rho \in [0, \rho_{\max}]$.

In practice these conditions are very mild and are expected to hold for most commonly used models. In the next subsection it is verified that these conditions hold for the Fastlane model.

Application to the Fastlane Model

Fastlane is a multiclass model, based on the kinematic wave model in Eulerian coordinates. The model equations are Equations 1 through 3 together with the fundamental diagram:

$$Q_u(\rho, \rho_u) = \begin{cases} \rho_u \left(v_{u,\max} - \dfrac{\rho}{\rho_{\text{crit}}} \left(v_{u,\max} - v_{\text{crit}} \right) \right) & \text{for } 0 \leq \rho < \rho_{\text{crit}} \text{ (free flow)} \\ \rho_u \dfrac{v_{\text{crit}} \rho_{\text{crit}}}{\rho} \dfrac{\rho_{\max} - \rho}{\rho_{\max} - \rho_{\text{crit}}} & \text{for } \rho_{\text{crit}} \leq \rho < \rho_{\max} \text{ (congestion)} \end{cases} \tag{13}$$

The class-specific flow is the class-specific density multiplied by the class-specific velocity. In free flow the velocity is a linear function of the total density; in congestion the flow is a linear function of the total density (see Figure 3). In free flow the class-specific velocity is really class-specific and depends on the maximum speed of the class, while in congestion the class-specific velocity is equal for each class, and thus not really class-specific. Fastlane uses the PCE function

$$\eta_u(x, t) = \frac{L_u + T_u v_u(x, t)}{L_1 + T_1 v_1(x, t)} \tag{14}$$

where L_u is the average vehicle length of class u, and T_u is the average minimum time headway of class u. The PCE function implies that Class 1 is the reference class with $\eta_1 = 1$. Equations 1 and 3 can be transformed into the Lagrangian formulation (Equations 4, 5, and 7) as described above. The PCE function (Equation 14) can be applied in either the Eulerian or Lagrangian formulation. In the Lagrangian formulation the Fastlane fundamental diagram becomes

$$v_u = \frac{Q_u(\rho, \rho_u)}{\rho_u} = \begin{cases} v_{u,\max} - \dfrac{s_{\text{crit}}}{s} \left(v_{u,\max} - v_{\text{crit}} \right) & \text{for } s > s_{\text{crit}} \text{ (free flow)} \\ v_{\text{crit}} \dfrac{s - s_{\min}}{s_{\text{crit}} - s_{\min}} & \text{for } s_{\min} \leq s \leq s_{\text{crit}} \text{ (congestion)} \end{cases} \tag{15}$$

as shown in Figure 3. It can easily be verified that a two-class version of this model formulation satisfies Conditions 1 through 3 above if

- Vehicles of Class 1 are not slower than vehicles of Class 2 ($v_{1,\max} \geq v_{2,\max}$),
- Vehicles of Class 1 are not longer than vehicles of Class 2 ($L_1 \leq L_2$), and
- Vehicles of Class 1 do not have a longer time headway than vehicles of Class 2 ($T_1 \leq T_2$).

In practice these conditions are satisfied if passenger cars are defined as Class 1 and trucks as Class 2, because cars are faster and

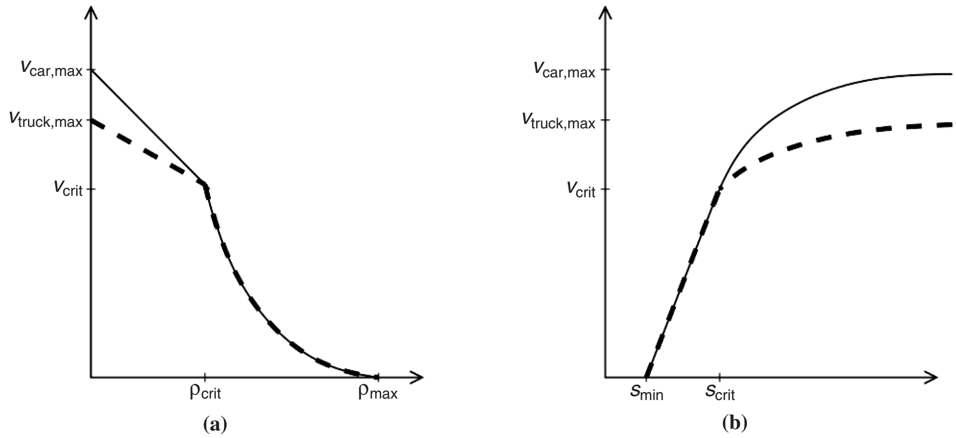

FIGURE 3 Multiclass fundamental diagram in (*a*) Eulerian and (*b*) Lagrangian formulations [in congestion ($\rho_{crit} \leq \rho \leq \rho_{max}$, $s_{min} \leq s \leq s_{crit}$), velocities are equal for both classes; in free flow ($0 \leq \rho < \rho_{crit}$, $s > s_{crit}$), velocity is class-specific].

shorter than trucks and have a lower time headway (*13*). Condition 4 guarantees that if the number of trucks increases, the total density increases. Obviously, this holds for any physically meaningful model. Analysis of the condition shows that it is sufficient, but not necessary, if the PCE function is constant or convex. However, the condition is not satisfied if the PCE value of Class 2 increases too fast if the total density increases (*6*). Whether the condition is satisfied for the Fastlane model depends on the parameter settings in the PCE function (Equation 14). It was verified that Condition 4 is satisfied for a wide range of realistic parameters, including the parameters used in the test problem in the next section.

TESTS AND RESULTS

Both the Eulerian and the Lagrangian formulation of the Fastlane model are applied on a simple test problem. The simulation results are compared on their accuracy.

Test Setup

The test problem consists of one link with periodic boundary conditions (ring road) and two vehicle classes: passenger cars and trucks. Initially the cars are uniformly distributed over the roadway stretch. The trucks are initially only on one half of the road. The total density is above the critical density only for this half of the road. A first-order upwind scheme for discretization of the Lagrangian conservation equations is applied. The Eulerian conservation equations are discretized by using the Godunov scheme. An explicit time-stepping scheme is used. With the application of the Lagrangian formulation, the following discretized equation is solved:

$$\left(s_1\right)_j^{\text{new}} = \left(s_1\right)_j^{\text{old}} - \frac{\Delta t}{\Delta n}\left(\left(v_1\right)_j^{\text{old}} - \left(v_1\right)_{j-1}^{\text{old}}\right) \tag{16}$$

$$\left(s_2\right)_j^{\text{new}} = \left(s_2\right)_j^{\text{old}} - \frac{\Delta t}{\Delta n}\left[\frac{\left(s_2\right)_j^{\text{old}}}{\left(s_2\right)_j^{\text{old}}}\left(\left(v_1\right)_j^{\text{old}} - \left(v_1\right)_{j-1}^{\text{old}}\right) + \frac{\left(v_1\right)_j^{\text{old}} - \left(v_2\right)_j^{\text{old}}}{\left(s_2\right)_j^{\text{old}}}\left(\left(s_2\right)_j^{\text{old}} - \left(s_2\right)_{j-1}^{\text{old}}\right)\right] \tag{17}$$

where

Δt = time step size,
j = vehicle group number, and
Δn = vehicle group size ($n = j\Delta n$).

With the application of the Eulerian formulation, the following discretized equation is solved:

$$\left(\rho_u\right)_i^{\text{new}} = \left(\rho\right)_i^{\text{old}} - \frac{\Delta t}{\Delta x}\left(\left(q_u\right)_{i\to i+1}^{\text{old}} - \left(q_u\right)_{i-1\to i}^{\text{old}}\right) \qquad u = 1, 2 \tag{18}$$

where i denotes the grid cell, and Δx the grid cell size ($x = i\Delta x$). The minimum supply–demand concept is used to determine the class-specific flows q_u over the cell boundaries. Therefore, first, the total demand (d) and supply (s) in each grid cell is calculated:

$$d_i = \begin{cases} q\left(\rho_i\right) \\ q_{\max} \end{cases} \qquad s_i = \begin{cases} q_{\max} & \text{if } \rho \leq \rho_{\text{crit}} \left(\text{free flow}\right) \\ q\left(\rho_i\right) & \text{if } \rho > \rho_{\text{crit}} \left(\text{congestion}\right) \end{cases} \tag{19}$$

Second, the total demand and supply is distributed over the classes

$$(d_u)_i = \frac{q_u\left(\rho_i\right)}{q\left(\rho_i\right)} d$$

and

$$(s_u)_i = \frac{q_u\left(\rho_i\right)}{q\left(\rho_i\right)} s$$

with the effective flow $q(\rho_i) = \Sigma_u\, \eta_u q_u(\rho_i)$. Third, the user class–specific flow over the cell boundary is calculated as the minimum of the demand and the supply: $(q_u)_{i\to i+1} = \min\{(d_u)_i, (s_u)_{i+1}\}$.

For fair comparison, the number of vehicle groups in the Lagrangian formulation is taken equal to the number of grid cells in the Eulerian formulation. Table 1 shows the parameters, initial conditions, and numerical settings. The PCE function is an implicit function: the PCE value of trucks depends on the total density, which depends in turn on the PCE value. Therefore, the

TABLE 1 Parameters, Initial Conditions, and Numerical Settings of the Test Case

Parameter	Value
Maximum density	133 veh/km = 0.133 veh/m
Critical density	25 veh/km = 0.025 veh/m
Maximum speed cars (v_1, max)	120 km/h = 33.3 m/s
Maximum speed trucks (v_2, max)	85 km/h = 23.6 m/s
Critical speed	85 km/h = 23.6 m/s
Initial density cars	16.7 veh/km = 0.0167 veh/m
Initial density trucks, first half	13.3 veh/km = 0.0133 veh/m
Initial density trucks, second half	0.7 veh/km = 0.0007 veh/m
Vehicle length cars (l_1)	7.5 m
Vehicle length trucks (l_2)	18 m
Minimum time headway cars (t_1)	1.2 m
Minimum time headway trucks (t_2)	1.8 s
Time horizon	300 s
Time step size	5 m
Road length	2,800 m
Number of cars per vehicle group	3.33
Number of vehicle groups	14
Grid cell size	200 m
Number of grid cells	14

PCE values of trucks are approximated by using an iterative procedure.

Test Results

Figure 4 shows simulation results based on both the Lagrangian and Eulerian formulations. For both formulations the simulation shows a platoon of trucks moving forward in space. After a short time the cars also group together, and a platoon with a high-car density appears. Finally, at the end of the simulation there is congestion on the whole roadway stretch, the total density is almost uniform over space, and the state is almost in equilibrium.

There is a remarkable difference between the results based on the Lagrangian and the Eulerian formulations: in the latter much more diffusion is seen both around the shocks starting at $x = 1{,}400$ m and at the heads and tails of the platoons. By applying shock wave theory, it is found that the model predicts two shocks starting at $x = 1{,}400$ m. These shocks are clearly visible in the results based on the Lagrangian formulation, while in the Eulerian formulation the shocks diffuse quickly. Shock wave theory also states that in an equilibrium situation with congestion everywhere and the total densities uniform over the whole roadway stretch, the characteristic velocity is equal to the vehicle velocity. Therefore, rather sharp edges at the heads and tails of the platoons toward the end of the simulation are expected. This is clearly visible with the Lagrangian formulation, while with the Eulerian formulation there is much more diffusion of the platoon.

Note that both for the shocks at the start of the simulation and the sharp platoon edges at the end of the simulation the diffusion is not in the model itself, but is caused by the numerical method. The method based on the Lagrangian formulation shows less diffusion and is more accurate than the method based on the Eulerian formulation.

CONCLUSIONS

The Lagrangian formulation of the kinematic wave model with multiple user classes can be derived in an intuitive way, based on the conservation of vehicles. In this new formulation information only flows downstream with respect to the coordinate system. Therefore, a more accurate and computationally less demanding spatial discretization can be applied. Simulation of a simple test problem based on the Lagrangian formulation is more accurate than the same simulation based on the Eulerian formulation.

It is concluded that the model equations of the multiclass kinematic wave model can be solved more accurately by applying the Lagrangian formulation. This can lead to more accurate simulation tools, which can be applied in, for example, real-time traffic flow prediction and dynamic traffic-assignment problems.

The simulation results based on the Lagrangian formulation are more accurate due to less numerical diffusion. In future research the authors will study this diffusion in more detail. Part of this study was based on a two-class version of the Fastlane model. Future research includes the applicability of the proposed methods to general multiclass models with more than two user classes. The authors will study whether in more general cases information only flows downstream with respect to the Lagrangian coordinate system. Furthermore, computing times for the multiclass model in Lagrangian and Eulerian formulations will be compared and methods to reduce the computing time will be studied, such as implicit time-stepping schemes (*5*).

Another application of the Lagrangian formulation of the kinematic wave model is related to traffic state estimation. First, incorporating floating car data (such as global positioning system data) in the Eulerian formulation of the model can be complicated and computationally demanding (*14*). It would be much more straightforward to use the Lagrangian formulation to incorporate floating car data in a traffic state estimation. This approach would be especially useful when little or no data from static detectors are available. Second, and more importantly, the discretized kinematic wave model in the Lagrangian formulation is much less nonlinear then the discretized model in the Eulerian formulation. Since it is less nonlinear, simple and more efficient filters (such as an extended Kalman filter) can be used for traffic state estimation.

ACKNOWLEDGMENTS

The authors thank Bas van ’t Hof of VORtech Computing for his contributions on the analysis of the mathematical properties of the multiclass model. Femke van Wageningen-Kessels is a PhD student at the ITS Edulab, a joint collaboration between TUDelft and Rijkswaterstaat. The authors thank Henk Taale of the ITS Edulab. The authors also thank the anonymous reviewers for their useful comments.

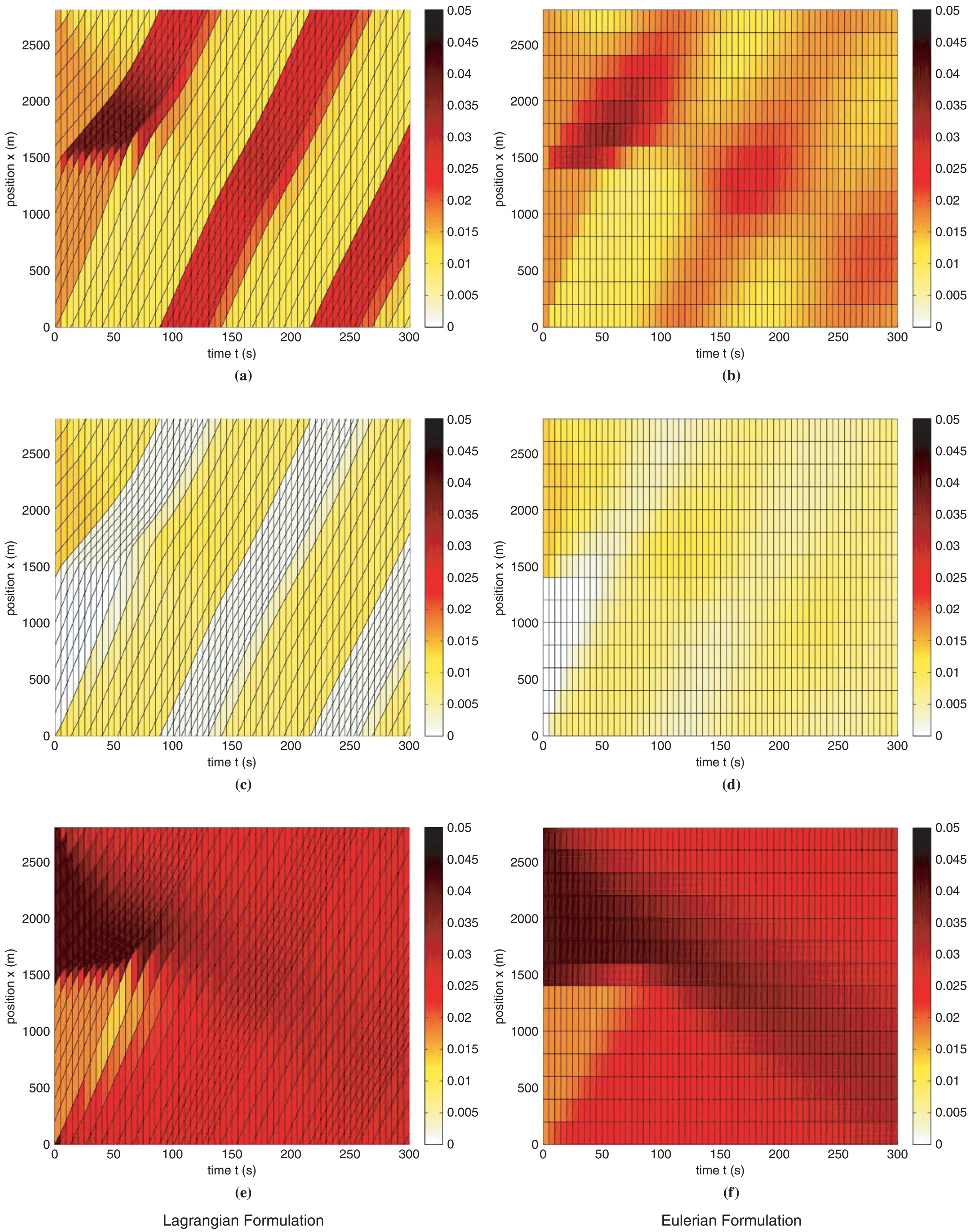

FIGURE 4 Simulation results for multiclass kinematic wave model based on Lagrangian and Eulerian formulations [densities (veh/m) are indicated as colors in *t*, *x*-plane; graphs show class-specific densities for cars (*a* and *b*) and trucks (*c* and *d*) and total densities (*e* and *f*); curved lines in Lagrangian formulations indicate trajectories of vehicle groups].

REFERENCES

1. Lighthill, M. J., and G. B. Whitham. On Kinematic Waves I: Flood Movement in Long Rivers; On Kinematic Waves II: A Theory of Traffic Flow on Long Crowded Roads. In *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences,* Vol. 229, 1955, pp. 281–345.
2. Richards, P. I. Shockwaves on the Highway. *Operations Research: Journal of the Operations Research Society of America,* Vol. 4, 1956, pp. 42–51.
3. Leclercq, L., J. Laval, and E. Chevallier. The Lagrangian Coordinates and What It Means for First Order Traffic Flow Models. In *Transportation and Traffic Theory* (Richard E. Allsop, Michael G. H. Bell, and Benjamin G. Heydecker, eds.), Elsevier, Oxford, United Kingdom, 2007, pp. 735–753.
4. Leclercq, L., J. Laval, and E. Chevallier. Lagrangian Coordinates Applied to the LWR Model. In *Hyperbolic Problems: Theory, Numerics, Applications,* Springer, Berlin, 2008, pp. 671–678.
5. van Wageningen-Kessels, F. L. M., J. W. C. van Lint, S. P. Hoogendoorn, and C. Vuik. Implicit Time Stepping Schemes Applied to the Kinematic Wave Model in Lagrangian Coordinates. In *Traffic and Granular Flow, 2009.* In press.
6. van Wageningen-Kessels, F. L. M., J. W. C. van Lint, S. P. Hoogendoorn, and C. Vuik. Multiple User Classes in the Kinematic Wave Model in Lagrangian Coordinates. In *Traffic and Granular Flow, 2009.* In press.
7. van Lint, J. W. C., S. P. Hoogendoorn, and M. Schreuder. FASTLANE: New Multiclass First-Order Traffic Flow Model. In *Transportation Research Record: Journal of the Transportation Research Board, No. 2088,* Transportation Research Board of the National Academies, Washington, D.C., 2008, pp. 177–187.
8. Leclercq, L., and J. A. Laval. A Multiclass Car-Following Rule Based on the LWR Model. In *Traffic and Granular Flow* (C. Appert-Rolland, F. Chevoir, P. Gondret, S. Lassarre, J.-P. Lebacque, and M. Schreckenberg, eds.), Springer, Berlin, 2007.
9. Zhang, P., S. C. Wong, and Z. Xu. A Hybrid Scheme for Solving a Multi-Class Traffic Flow Model with Complex Wave Breaking. *Computer Methods in Applied Mechanics and Engineering,* Vol. 197, 2008, pp. 3816–3827.
10. Lebacque, J. P. The Godunov Scheme and What It Means for First-Order Traffic Flow Models. In *Transportation and Traffic Theory: Proceedings of the 13th International Symposium on Transportation and Traffic Theory* (J. P. Lesort, ed.), Pergamon, Oxford, United Kingdom, 1996, pp. 647–677.
11. van Wageningen-Kessels, F., H. van Lint, S. Hoogendoorn, and K. Vuik. Implicit and Explicit Numerical Methods for Macroscopic Traffic Flow Models: Efficiency and Accuracy. Presented at 88th Annual Meeting of the Transportation Research Board, Washington, D.C., 2009.
12. Daganzo, C. F. Requiem for Second-Order Fluid Approximations of Traffic Flow. *Transportation Research Part B: Methodological,* Vol. 29, 1995, pp. 277–286.
13. Hoogendoorn, S. P., and P. H. L. Bovy. New Estimation Technique for Vehicle-Type-Specific Headway Distributions. In *Transportation Research Record 1646,* TRB, National Research Council, Washington, D.C., 1998, pp. 18–28.
14. Herrera, J. C., and A. Bayen. Traffic Flow Reconstruction Using Mobile Sensors and Loop Detector Data. Presented at 87th Annual Meeting of the Transportation Research Board, Washington, D.C., 2008.

The Traffic Flow Theory and Characteristics Committee peer-reviewed this paper.