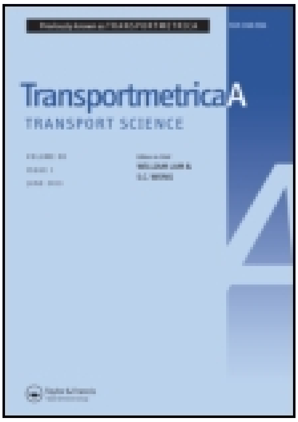

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Anisotropy in generic multi-class traffic flow models

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Anisotropy in generic multi-class traffic flow models

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Traffic flow models and simulation tools are often used for traffic state estimation and prediction. Recently several multi-class models based on the kinematic wave traffic flow model have been introduced. These multi-class models take into account the heterogeneity of both vehicles and drivers. We analyse two important properties of these models: hyperbolicity and anisotropy. Both properties relate to the propagation speed of disturbances, as can be observed in real traffic. We discuss the importance of traffic flow models to be hyperbolic and anisotropic. Moreover, we develop a framework to analyse whether traffic flow models have these properties. Therefore, we derive a generic formulation of multi-class kinematic wave traffic flow models, rewrite it in the Lagrangian formulation and apply eigenvalue analysis to the resulting system of equations. Our analysis shows that most multi-class kinematic wave traffic flow models are indeed hyperbolic and anisotropic under certain modelling conditions.

Keywords: traffic flow; continuum models; multi-class; traffic anisotropy

1. Introduction

Traffic flow models and simulation tools are often used for traffic state prediction. These predictions are used for (long-term) planning purposes, traffic state estimation and short-term prediction. We study macroscopic traffic flow models which describe dynamic traffic flow as if it were a continuum. They are mostly applied for simulation of freeway traffic flow. Predictions based on these simulations are, for example, used for short-term traffic information for road users. Road authorities may use the simulation results for traffic control such as route guidance, and (integrated) network management in which one or more measures are applied to keep the traffic flowing and to reduce the impact of congestion.

Many other modelling approaches have been suggested in the literature. Examples are microscopic models where the individual behaviour of vehicles is modelled, cellular automata models and gas-kinetic models. For an overview, see Hoogendoorn and Bovy (2001). Our analysis is based on a macroscopic traffic flow model. These models are simple and relatively easy to apply, yet the results do show many phenomena that can be observed in real traffic. Kerner (2004b, 2009) argues that macroscopic traffic flow models based on

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the fundamental diagram approach cannot adequately describe most important traffic phenomena. He suggests to use three-phase models instead. However, Treiber *et al*. (2010) show that two-phase (macroscopic) models can represent the same phenomena as three-phase models.

1.1. *Multi-class models*

The kinematic wave (traffic flow) model, also known as the LWR-model, has been used to describe traffic flow since the 1950s (Lighthill and Whitham 1955, Richards 1956). It uses a continuum approach and is therefore regarded as a macroscopic traffic flow model. This implies that vehicles are not considered individually, but only the average number of vehicles per space and time unit is considered. Many extensions of the kinematic wave model have been introduced since. Recently much attention goes to multi-class models. Instead of considering traffic flow as a homogenous flow with homogeneous vehicles and drivers (mixed class), the heterogeneity of vehicles and drivers is taken into account (Wong and Wong 2002a, Bagnerini and Rascle 2003, Benzoni-Gavage and Colombo 2003, Chanut and Buisson 2003, Zhang *et al*. 2006a, Ngoduy and Liu 2007, Logghe and Immers 2008, van Lint *et al*. 2008). Multi-class models are able to describe real-world phenomena such as capacity drop and hysteresis better than mixed-class models (see, e.g. Wong and Wong 2002a, Benzoni-Gavage and Colombo 2003, Ngoduy and Liu 2007, Ngoduy 2010 and references therein). The same phenomena are expected to be described by the so-called higher order models (Payne 1971, Aw and Rascle 2000). In higher order models vehicles are assumed to need some time for acceleration or deceleration to an equilibrium velocity. This is in contrast to first-order models in which vehicles always drive at this equilibrium velocity and therefore can have infinite accelerations and decelerations. Also in higher-order models vehicle and driver heterogeneity has been included (Hoogendoorn and Bovy 2000, Hoogendoorn *et al*. 2002, Bagnerini and Rascle 2003, Gupta and Katiyar 2007).

In this article, we focus on first-order multi-class traffic flow models. Some of them are very simple extensions of the kinematic wave model and only incorporate different velocities (Wong and Wong 2002a, Zhang *et al*. 2006a). Others have a term or equation representing that larger vehicles (trucks) take more space than smaller vehicles (passenger cars) (Benzoni-Gavage and Colombo 2003, Chanut and Buisson 2003, Ngoduy and Liu 2007, Logghe and Immers 2008). In most cases, the ratio between the contribution of other vehicle types and passenger cars (the passenger car equivalent (pce) value) is taken constant. Only the Fastlane model represents that in congestion a truck takes relatively more space than in free flow (van Lint *et al*. 2008) and the pce-value is a function of the traffic state. Furthermore, some models use a simple velocity-density relationship (Greenshields or Drake fundamental diagram) (Wong and Wong 2002a, Benzoni-Gavage and Colombo 2003), whereas others take a more realistic velocity-density relationship (Daganzo or Smulders fundamental diagram) (Chanut and Buisson 2003, Ngoduy and Liu 2007, Logghe and Immers 2008), see Figure 1, or apply a more generic formulation of the relationship between effective density and velocity (Zhang *et al*. 2006a).

In general, the less realistic models have been analysed thoroughly and their mathematical properties are well understood. However, there is only little known about important properties of the more realistic models. We now first discuss the properties and argue why they are important.

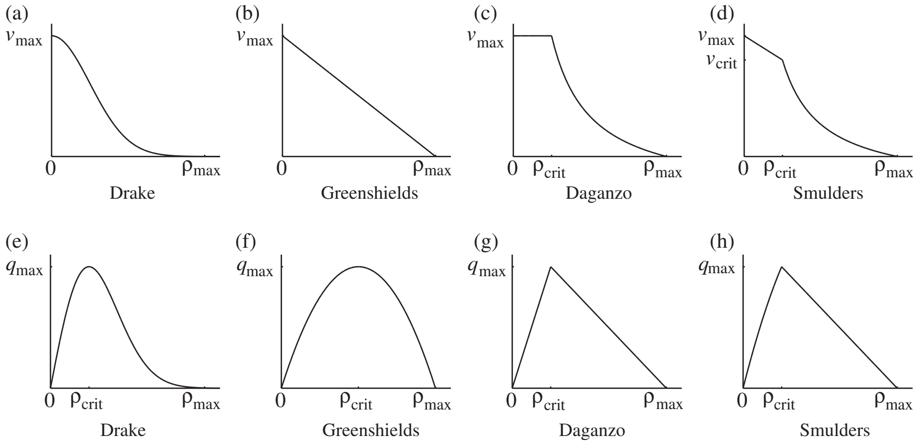

Figure 1. Fundamental diagrams. (a)–(d) velocity as a function of density, (e)–(h) flow as a function of density.

1.2. *Anisotropy*

It is known from practice that drivers mainly react on vehicles in front of them, not on vehicles behind them. In traffic flow theory this is referred to as anisotropy. The fact that drivers do look in their rear view mirrors and occasionally react on a vehicle that follows very closely (bumper tailing) or an emergency vehicle is neglected here. In a traffic flow model this means that the velocity of characteristics is never higher than the highest vehicle velocity. These characteristics are curves along which small disturbances propagate. Daganzo (1995) initiated an ongoing debate on whether or not higher-order traffic flow models represent anisotropy and whether it is necessary that they do so. The main argument that anisotropy should be represented by a traffic flow model is that traffic itself is anisotropic. A counter argument is given in Zhang (2003): on a multi-lane road in free flow all vehicles do not have the same velocity and information can travel with the velocity of the fastest vehicle and therefore faster than the average vehicle velocity. As argued in that paper, the problem can be resolved, at least partly, by a multi-lane and multi-class approach. Closely related to anisotropy is the nonnegativity of vehicle velocity: that is, if vehicle velocities are predicted to be negative, for example at the upstream end of a queue, the model is not anisotropic. Nonnegativity of vehicle velocity was proven in Garavello and Piccoli (2009) for some macroscopic traffic flow models among which the multi-class models by Benzoni-Gavage and Colombo (2003) and Wong and Wong (2002a).

A second reason why one would want to know whether a given traffic flow model is anisotropic is related to computational efficiency. If a model is anisotropic and the Lagrangian coordinate system (Section 3) is applied, more efficient computational methods can be applied (Daganzo 2006, Leclercq *et al.* 2007, 2008, van Wageningen-Kessels *et al.* 2009a, 2009b, 2009c). This will reduce computational time and/or improve accuracy of the simulation results.

Hyperbolicity of a (traffic) flow model is a necessary but not sufficient condition for anisotropy. A flow model is *hyperbolic* if perturbations propagate at finite velocity. Therefore, not every point in the domain is influenced by the perturbation at once, e.g. if someone suddenly hits the brake the disturbance will only have an influence on the future state of vehicles 'close' to this vehicle, and this region of influence will grow over time. This is in contrast to elliptic and parabolic equations, where a perturbation influences the whole domain at once. This would imply that a perturbation such as sudden braking of one vehicle would immediately influence vehicles a few kilometres downstream, which would clearly be an unwanted feature of a traffic flow model. Furthermore, the model is *anisotropic* if the characteristic velocity is equal to or lower than the vehicle velocity. This implies that perturbations do not travel faster than the vehicles themselves.

Hyperbolicity and anisotropy are evident for the so-called mixed-class first-order traffic flow models on a single lane with a concave density-flow relationship, also called the fundamental relation. These models are based on the kinematic wave model (Lighthill and Whitham 1955, Richards 1956). The model consists of a hyperbolic equation $\partial\rho/\partial t + (\mathrm{d}q(\rho)/\mathrm{d})\partial\rho/\partial x = 0$ with density ρ in vehicles per metre and flow $q(\rho)$ in vehicles per second. Perturbations propagate at characteristic velocity $c(\rho) = \mathrm{d}q(\rho)/\mathrm{d}$. If the fundamental relation $q(\rho)$ is concave, the characteristic velocity is finite and smaller than or equal to the vehicle velocity $c(\rho) \leq v(\rho) = q(\rho)/\rho$.

Anisotropy is not evident for the multi-class models that were derived from the kinematic wave model as described above. In a multi-class model the single conservation equation of the LWR-model is replaced by a system of U equations, with U the number of classes. For anisotropy the system must be hyperbolic and the eigenvalues (which are equal to the characteristic velocities) cannot be larger than the vehicle velocities. It is neither trivial that this system of equations is hyperbolic, nor that the model is anisotropic.

Previously, hyperbolicity and anisotropy have been analysed for some multi-class models. Donat and Mulet (2010) propose a framework to analyse eigenvalues and strict hyperbolicity of multi-class kinematic wave models with possibly state-dependent pce-values. However, the authors focus on strict hyperbolicity, which requires the class specific vehicle velocities to be distinct. Strict hyperbolicity might be an important property for efficient computational methods, it is of less interest for anisotropy. Anisotropy (only) requires weak hyperbolicity, which, in turn, does not require vehicle velocities to be distinct. Furthermore, both Donat and Mulet (2010) and Zhang *et al.* (2006a) discuss the interlacing property $\lambda_1 < v_1 < \cdots < \lambda_U < v_U$ or $v_1 < \lambda_1 < \cdots < v_U < \lambda_U$ with λ_u and v_u the eigenvalues and class specific velocities, respectively, arranged by size ($\lambda_1 < \lambda_2 < \cdots < \lambda_U$, $v_1 < v_2 < \cdots < v_U$). However, they do not discuss the fact that if a model shows the first type of interlacing (i.e. with $\lambda_U < v_U$) the model is anisotropic. Also, Benzoni-Gavage and Colombo (2003) show that their basic model is hyperbolic and anisotropic, even though they do not use the term 'anisotropic'. Moreover, in the models discussed by Benzoni-Gavage and Colombo (2003) and Zhang *et al.* (2006a) the pce-values are constant in Benzoni-Gavage and Colombo (2003) or even 1 for all types of vehicles in Zhang *et al.* (2006a). Finally, the analysis of hyperbolicity of the basic model by Zhang *et al.* (2006a) was extended to inhomogeneous roads in Zhang *et al.* (2006b, 2008). Logghe and Immers (2008) argue that their model is anisotropic. However, we will show that depending on the choice of certain parameters the model is not anisotropic. Chanut and Buisson (2003) do not discuss hyperbolicity or anisotropy of their model. Ngoduy and Liu (2007) only discuss hyperbolicity for 2-class models and it is argued that the eigenvalues cannot be determined analytically for more than 4 user classes. Ngoduy and Liu discuss anisotropy without reference to hyperbolicity, and only for the case where all classes have distinct velocities in free flow. The Fastlane model introduced by van Lint *et al.* (2008)

incorporates more features present in real traffic than the other models and most other models can be expressed as a special form of the Fastlane model. However, little is known about the mathematical properties of the model such as hyperbolicity and anisotropy.

1.3. *Research objective and outline*

Hyperbolicity and anisotropy of the models introduced before is analysed. The analysis in Benzoni-Gavage and Colombo (2003), Zhang *et al*. (2006a) and Donat and Mulet (2010) is based on a fixed coordinate system. We take a different approach by applying a moving coordinate system. Furthermore, we extend the results of Zhang *et al*. and Benzoni-Gavage and Colombo to models with state-dependent pce-values and focus on weak hyperbolicity and anisotropy. Therefore, we first introduce a framework based on the Lagrangian (moving) coordinate system. The Lagrangian coordinate system for the kinematic wave model was introduced in Daganzo (2006), Leclercq *et al*. (2007, 2008) and van Wageningen-Kessels *et al*. (2009b). A multi-class version of the kinematic wave model was formulated in Lagrangian coordinates in van Wageningen-Kessels *et al*. (2009a, c). In this formulation the moving coordinates have the same velocity as the fastest vehicle class. Hyperbolicity and anisotropy were proven for the two-class model (van Wageningen-Kessels *et al*. 2009c). However, the method introduced there can only be applied to models with two classes because it is based on an eigenvalue analysis which can only be applied to 2×2-systems. The framework introduced in this article can also be applied to analyse hyperbolicity and anisotropy of larger systems and of other one dimensional flow models.

The main objective of this article is to analyse anisotropy of a generic multi-class kinematic traffic model. This means that, in contrast to previous similar analyses, that the pce-value is not assumed to be constant. Therefore, we discuss several previously developed multi-class models and introduce a generic form of these models (Section 2). In Section 3 we introduce the Lagrangian formulation of the generic model. We analyse hyperbolicity and anisotropy of the model in Section 4, which is the main contribution of this article. In Section 5 we show that under mild conditions anisotropy is represented by the model and we discuss the implications of these conditions. We conclude with a summary and outlook in Section 6.

2. Multi-class kinematic wave modelling

In multi-class traffic flow models differences in properties of vehicles are taken into account. Vehicles are divided into classes, based on their origins and destinations and/or on vehicle properties such as length and maximum velocity. In the latter case the model usually makes a distinction between passenger cars and trucks; sometimes these classes are subdivided further into, for example, light and heavy trucks. In this study we focus on classes based on vehicle properties. This is more generic than classes based on origin or destination, which can be analysed using the same approach.

We study the following generic multi-class kinematic wave model with U user classes:

$$\frac{\partial \rho_u}{\partial t} + \frac{\partial q_u}{\partial x} = 0, \quad \forall u, \text{ class specific conservation equation,} \tag{1}$$

$$q_u = Q_u(\rho_1, \ldots, \rho_U), \quad \forall u, \text{ class specific fundamental relation,} \tag{2}$$

where ρ_u denotes average class specific density: average number of vehicles of class u per metre, $q_u = \rho_u\, v_u$ is average flow of class u in vehicles per second and v_u is average velocity of class u in metre per second. The flow function $Q_u(\rho_1, \ldots, \rho_U)$ describes the class specific flow as a function of all class specific densities. In most multi-class models the flow q_u is a function of the effective (or total) density ρ in pce vehicles per metre. In turn, the effective density ρ is a function (usually a weighted sum, with the pce-values as the weights) of the class specific densities ρ_u. This implies that the class specific fundamental relation (2) is replaced by

$$q_u = Q_u(\rho), \tag{3}$$

$$\rho = \rho(\rho_1, \ldots, \rho_U). \tag{4}$$

The class specific flow function q_u (whether defined as (2) or as (3)) increases until a certain threshold, the critical density ρ_{crit}, and it decreases for values above the critical density. The critical density can be class specific, resulting in semi-congested traffic states where for example trucks behave as if they are in a congested regime while cars still are in a free flow regime. See Figure 1 for some examples of fundamental relations.

Using the definition $q_u = v_u \rho_u$, the fundamental relation (2) is sometimes reformulated as the relation between density and velocity:

$$v_u = V_u(\rho_1, \ldots, \rho_U), \quad \forall u, \text{ class specific fundamental relation.} \tag{5}$$

Below we describe some multi-class models based on the first-order kinematic wave model. All models are based on the conservation of vehicles Equation (1). The fundamental relation (2) or (5) differs from model to model. In most, but not all, models an effective density (4) combined with a fundamental relation as in (3) is used.

2.1. *Basic multi-class models*

Wong and Wong (2002b), Benzoni-Gavage and Colombo (2003), Chanut and Buisson (2003) and Zhang *et al.* (2006a) have introduced and analysed basic multi-class models. Zhang *et al.* and Wong and Wong use an effective density that is an unweighted summation of the class-specific densities:

$$\rho = \sum_u \rho_u. \tag{6}$$

Benzoni-Gavage and Colombo use a weighted summation based on the vehicle lengths L_u:

$$\rho = \sum_u L_u \rho_u, \tag{7}$$

with L_u the average vehicle length of class u. Chanut and Buisson only consider two user classes and the weighted summation is based on the ratio between the vehicle lengths L_u:

$$\rho = \rho_1 + \frac{L_2}{L_1}\rho_2. \tag{8}$$

Note that the formulation used in the original paper by Chanut and Buisson is somewhat different. Rewriting the equations into the form used here (Equations (8) and (11)) is tedious but straightforward.

Most basic multi-class models use simple fundamental relations, either Greenshields fundamental relation (Benzoni-Gavage and Colombo 2003):

$$v_u(\rho) = v_{u,\max}\left(1 - \frac{\rho}{\rho_{\max}}\right), \tag{9}$$

with $v_{u,\max}$ the class specific maximum velocity, ρ_{crit} the critical density and $\rho_{\max}$ the maximum density, or Drake's fundamental relation (Wong and Wong 2002a, Benzoni-Gavage and Colombo 2003):

$$v_u(\rho) = v_{u,\max} e^{-\frac{1}{2}(\rho/\rho_{\text{crit}})^2}. \tag{10}$$

These fundamental relations are rarely used by traffic engineers because they are regarded as unrealistic. Analysis of traffic flow data shows that Greenshields' assumption that the relation between density ρ and velocity v_u is linear is unrealistic. Drake's fundamental relation is regarded as unrealistic because there is no maximum density for which the velocities are zero. Moreover, both in (9) and (10) velocities are class dependent not only in free flow, but also in congestion. However, in congestion there is no room for overtaking and vehicles of all classes drive at the same speed (Kerner 2004a). Still, this class-independent speed might vary over time due to, for example, weather conditions. The authors of the above-mentioned papers do not address the latter issue, but do claim that their results can be generalised to other (more realistic) fundamental relations.

Chanut and Buisson use the Smulders fundamental diagram (Smulders 1990):

$$v_u(\rho) = \begin{cases} v_{u,\max} - \dfrac{v_{u,\max} - v_{\text{crit}}}{\rho_{\text{crit}}}\rho & \text{if } \rho < \rho_{\text{crit}}(\rho_1, \ldots, \rho_U), \\ \dfrac{\rho_{\text{crit}} v_{\text{crit}}}{\rho_{\max} - \rho_{\text{crit}}}\left(\dfrac{\rho_{\max}}{\rho} - 1\right) & \text{if } \rho \geq \rho_{\text{crit}}(\rho_1, \ldots, \rho_U), \end{cases} \tag{11}$$

with $v_{u,\max}$ the class specific maximum velocity, v_{crit} the critical velocity (velocity at critical density), ρ_{crit} the critical density and $\rho_{\max}$ the maximum density.

Zhang *et al.* (2006a) do not specify the fundamental diagram, but use a generic velocity–density relationship.

2.2. *Multi-class model with 3 states*

Logghe and Immers (2008) introduced a two-class model with 3 states. This model is characterised by the existence, next to the free flow and the congested state, of a third state: semi-congestion. The critical density is class specific. In free flow both vehicle classes are in the free flow branch of the fundamental diagram (i.e. below the class specific critical density). In semi-congestion vehicles of class 1 are in the congested branch of the fundamental diagram (i.e. above class specific critical density) and vehicles of class 2 are in the free flow branch of the fundamental diagram (i.e. below class specific critical density). In congestion both vehicle classes are in the congested branch of the fundamental diagram (i.e. above class specific critical density). The basic idea behind this model is that a certain fraction of the road is assigned to each class. Then the fundamental diagram is scaled according to this fraction and the velocity is determined using this scaled fundamental diagram. The effective density ρ is an unweighted summation of the class specific densities (6). Logghe and Immers use the Daganzo fundamental relation:

$$v_u = \begin{cases} v_{u,\max} & \text{if } \rho_u < \rho_{u,\mathrm{crit}}(\rho_1, \rho_2), \\ \dfrac{\rho_{u,\mathrm{crit}} v_{u,\max}}{\rho_{u,\max} - \rho_{u,\mathrm{crit}}} \left(\dfrac{\rho_{u,\max}}{\rho} - 1 \right) & \text{if } \rho_u \geq \rho_{u,\mathrm{crit}}(\rho_1, \rho_2), \end{cases} \tag{12}$$

with $v_{u,\max}$ the class specific maximum velocity, $v_{u,\mathrm{crit}}$ the class specific critical velocity and with the class specific critical and maximum densities:

$$\rho_{u,\mathrm{crit}}(\rho_1, \rho_2) = \alpha_u \rho_{u,\mathrm{crit}}(\rho_1, 0), \quad \rho_{u,\max}(\rho_1, \rho_2) = \alpha_u \rho_{u,\max}(\rho_i, 0), \tag{13}$$

and the scaling fraction:

$$\alpha_u = \frac{\rho_u}{\rho}. \tag{14}$$

We note that the velocity v_u is class specific both in free flow and in congestion. As was argued before, this is not realistic in congestion.

2.3. *Multi-class models with generic pce's*

Multi-class models with generic pce-functions are introduced and analysed by Ngoduy and Liu (2007) and van Lint *et al.* (2008). In these models the pce-value depends on the current traffic state. The Fastlane model by van Lint *et al.* uses a pce-value η_u which is an implicit function of the effective density ρ:

$$\rho = \sum \eta_u(\rho) \rho_u, \tag{15}$$

$$\eta_u(\rho) = \frac{L_u + T_u v_u(\rho)}{L_1 + T_1 v_1(\rho)}, \tag{16}$$

where L_u denotes the class specific vehicle gross length (i.e. the vehicle length plus the minimum distance between two vehicles, normally about 1 m), $T_u v_u(\rho)$ denotes the class specific minimum space headway: the minimum distance the front of a vehicle keeps to the tail of its predecessor minus the minimum distance between two vehicles which was already incorporated in L_u. This pce-function is based on a safe-distance car-following rule: at higher speeds (i.e. lower densities) the safe distance of a vehicle to its leader is larger. As a result, the pce-value of class u is the ratio of the space a vehicle of that class u takes and the space a vehicle of the reference class takes. As we noted before, the pce-function is an implicit function of the effective density. Combining Equations (15) and (16) with the velocity function (11) results in a bi-valued effective density function. However, if we furthermore demand that vehicle velocities v_u are nonnegative and the effective density ρ is a continuous function of the class specific density of class 1 ρ_1, the effective density ρ can be written as a single-valued function of the class specific densities ρ_u.

Ngoduy and Liu use an unweighted summation for the effective density (6). They use the pce-value and the current state to determine the critical and the maximum density:

$$\rho_{\text{crit}} = \rho_{\text{ref, crit}} \sum_u \frac{\alpha_u}{\beta_u}, \quad \rho_{\max} = \rho_{\text{ref, max}} \sum_u \frac{\alpha_u}{\beta_u}, \tag{17}$$

with α_u the share (14) and β_u the pce-value. Ngoduy and Liu refer to the Highway Capacity Manual (HCM 2000) to look up pce-values, which can depend on the vehicle type, vehicle length, slope of the road and fraction of heavy vehicles. Since the fraction of heavy vehicles depends on the current traffic state, the pce-value depends on the traffic state: $\beta_u = \beta_u(\rho_1, \ldots, \rho_U)$.

Both the model by Ngoduy and Liu and the Fastlane model use the Smulders fundamental relation (11).

3. Lagrangian formulation of the multi-class model

The generic multi-class model in Eulerian coordinates (1) and (2) (or (3) and (4)) can be reformulated in Lagrangian coordinates by using definitions

$$q_u = \rho_u v_u, \quad s_u = \frac{1}{\rho_u} \quad \text{and} \quad s = \frac{1}{\rho}, \tag{18}$$

and the Lagrangian time derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}, \tag{19}$$

with $s_u = 1/\rho_u$ the average vehicle spacing of class u in metre per vehicle and $s = 1/\rho$ the effective vehicle spacing in metre per pce-vehicle. n denotes the vehicle 'number'. Since the flow is regarded as a continuum, n is not integer but can take any real value. Vehicles of user class 1 are numbered in opposite driving direction. Other user classes are not numbered. Combining (18) and (19) and using that $\partial n/\partial x = -\rho_1$ yields:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial n}\frac{\partial n}{\partial x} = -\rho_1 \frac{\partial}{\partial n} = -\frac{1}{s_1}\frac{\partial}{\partial n}, \tag{20}$$

$$\frac{\partial}{\partial t} = \frac{\mathrm{D}}{\mathrm{D}t} - v_1 \frac{\partial}{\partial x} = \frac{\mathrm{D}}{\mathrm{D}t} + \frac{v_1}{s_1}\frac{\partial}{\partial n}. \tag{21}$$

Substituting (18), (20) and (21) into the multi-class model equations (1)–(2) and rewriting the resulting equations gives the Lagrangian multi-class kinematic wave model:

$$\frac{\mathrm{D}s_1}{\mathrm{D}t} + \frac{\partial v_1}{\partial n} = 0, \quad \text{conservation class 1}, \tag{22}$$

$$\frac{\mathrm{D}s_u}{\mathrm{D}t} + \frac{v_1 - v_u}{s_1}\frac{\partial s_u}{\partial n} + \frac{s_u}{s_1}\frac{\partial v_u}{\partial n} = 0, \quad \forall u \neq 1, \text{ conservation other classes}, \tag{23}$$

$$v_u = V_u(s_1, s_2, \ldots, s_U), \quad \forall u, \text{class specific fundamental relation}, \tag{24}$$

If the class specific fundamental relation is based on the effective density (3) and (4) then in Lagrangian formulation the class-specific fundamental relation is based on the effective spacing:

$$v_u = V_u(s), \quad \text{with } s = s(s_1, s_2, \ldots, s_U), \tag{25}$$

Equations (22) and (23) are rewritten as a system of equations:

$$\frac{\partial \vec{s}}{\partial t} + J(\vec{s})\frac{\partial \vec{s}}{\partial n} = \vec{0}, \tag{26}$$

$$\text{with} \quad \vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_U \end{pmatrix}, \quad J(\vec{s}) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,U} \\ \vdots & \ddots & \vdots \\ a_{U,1} & \cdots & a_{U,U} \end{pmatrix}, \tag{27}$$

$$\text{and} \quad \begin{cases} a_{i,i} = \dfrac{s_i}{s_1}\dfrac{\partial v_i}{\partial s_i} + \dfrac{v_1 - v_i}{s_1} & \text{on the diagonal,} \\ a_{i,j} = \dfrac{s_i}{s_1}\dfrac{\partial v_i}{\partial s_j} & \text{for } i \neq j, \text{ off diagonal.} \end{cases} \tag{28}$$

In (28) we use

$$\frac{\partial v_u}{\partial n} = \frac{\partial v_u}{\partial s_1}\frac{\partial s_1}{\partial n} + \frac{\partial v_u}{\partial s_2}\frac{\partial s_2}{\partial n} + \cdots + \frac{\partial v_u}{\partial s_U}\frac{\partial s_U}{\partial n}, \tag{29}$$

to find the elements $a_{i,j}$ of the Jacobian matrix $J(\vec{s})$.

The LWR-model in Lagrangian coordinates can also be derived in a less formal (but more intuitive) way by using graphical techniques (van Wageningen-Kessels *et al*. 2009b). In van Wageningen-Kessels *et al*. (2010) this graphical derivation is extended to the multi-class model in Lagrangian coordinates (22)–(25).

In the next section we show that traffic anisotropy is represented by the multi-class model (22)–(25) if the model has certain properties. We use the Lagrangian formulation of the model. Obviously, if anisotropy is represented by the model in Lagrangian formulation, it is also represented in other formulations such as the Eulerian formulation.

4. Anisotropy

In this section we first show that the generic macroscopic multi-class traffic flow model (1), (3), (4) or equivalently (22)–(24), under certain conditions, is hyperbolic, implying that perturbations only propagate at finite velocity. Second, we show that the model is furthermore anisotropic under certain more strict conditions. Therefore, we show that perturbations do not travel faster than vehicles of the fastest class. We recall that characteristics are curves (or waves) at which information (disturbances or perturbations) propagate. We note that in the solution of the multi-class kinematic wave model other waves such as shocks and rarefaction waves occur. However, the velocity of these waves are equal to the velocities of a characteristic, or between the velocities of two characteristics and consequently their velocity is always lower than or equal to that of the fastest characteristic.

4.1. *Preliminaries*

Before we introduce and prove the conditions under which the multi-class model is hyperbolic and anisotropic, we need some definitions and preliminaries. First of all, we need some results from differential equations theory and linear algebra (see, e.g. LeVeque 2002, Section 2.9):

Definition 1 ((Weak) hyperbolicity)**:** The system of equations $\partial\vec{s}/\partial t + J(\vec{s})\partial\vec{s}/\partial n = 0$ is weakly hyperbolic for those values of $\vec{s}$ for which the eigenvalues of $J(\vec{s})$ are real.

Preliminary 1: *The characteristic velocities with respect to the coordinate velocity of a weakly hyperbolic system of equations* $\partial\vec{s}/\partial t + J(\vec{s})\partial\vec{s}/\partial n = 0$ *are equal to the eigenvalues of* $J(\vec{s})$.

In the rest of this article we also use 'hyperbolic' where we mean 'weakly hyperbolic'. Furthermore, we need some results from basic linear algebra (see, e.g. Strang 1988):

Preliminary 2: *The matrices* A *and* DAD^{-1} *have the same eigenvalues.*

Preliminary 3: *The eigenvalues of a real and symmetric matrix are real.*

Preliminary 4: *Suppose* S *is a real and symmetric matrix.* S *has nonnegative eigenvalues if and only if* S *has nonnegative pivots after applying Gaussian elimination.*

The conditions described below must hold for all relevant traffic states. Therefore, we only have to consider relevant densities. Negative densities are physically impossible, as well as effective densities higher than the maximum effective density. Therefore, from now on, we only consider states with class specific densities $\rho_1, \ldots, \rho_U \geq 0$ and effective density $\rho = \rho(\rho_1, \ldots, \rho_U) \in [0, \rho_{\max}]$, or equivalently traffic states with effective vehicle spacing $s = s(s_1, \ldots, s_U) \in [s_{\min}, \infty)$.

4.2. *Anisotropy of the generic multi-class model*

We now define anisotropy and present the conditions for anisotropy. The conditions will be proven to be sufficient conditions in the remainder of this section.

Definition 2 (Anisotropy)**:** A multi-class macroscopic traffic flow model is anisotropic if for any relevant traffic state characteristics travel at velocities smaller than or equal to vehicle velocities of the fastest class in this traffic state.

Condition 1 (Hyperbolicity)**:** *For all relevant spacings the fundamental relation in Lagrangian formulation* (24) *is nondecreasing*:

$$\frac{\partial v_u(\vec{s})}{\partial s_w} \geq 0, \tag{30}$$

for all user classes u *and* w.

Condition 2 (Anisotropy): *There is a 'fastest' class u^* such that for all relevant spacings*

$$v_{u^*}(\vec{s}) \geq v_u(\vec{s}). \tag{31}$$

We note that these conditions 1 and 2 are a generalisation of the conditions for the two-class model as they were introduced in van Wageningen-Kessels *et al.* (2009c).

Theorem 1: *The nonlinear system of partial differential equations* (26) *is hyperbolic for all relevant values of $\vec{s}$ if Condition* 1 *is satisfied.*

Proof: We use Definition 1 and show that the model is hyperbolic by showing that the eigenvalues of $J(\vec{s})$ are real. We show that the eigenvalues are real by applying Preliminary 2 and 3.

Therefore, we first define matrix $M(\vec{s})$ as follows:

$$M(\vec{s}) = D(\vec{s})J(\vec{s})\big(D(\vec{s})\big)^{-1}, \tag{32}$$

with diagonal matrix:

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_U \end{pmatrix}, \quad \text{with } d_i = \sqrt{\frac{\frac{\partial s}{\partial s_i}}{s_i \frac{\partial v_i}{\partial s}}}. \tag{33}$$

Later we will also need

$$\frac{d_i}{d_j} = \sqrt{\frac{s_j \frac{\partial v_j}{\partial s}\frac{\partial s}{\partial s_i}}{s_i \frac{\partial v_i}{\partial s}\frac{\partial s}{\partial s_j}}} = \sqrt{\frac{s_j}{s_i}}\sqrt{\frac{\partial v_j/\partial s_i}{\partial v_i/\partial s_j}}, \tag{34}$$

in the last equality we use that $(\partial v_j/\partial s)(\partial s/\partial s_i) = \partial v_j/\partial s_i$.

The elements of matrix $M(\vec{s})$ are

$$m_{i,j} = \frac{d_i}{d_j} a_{i,j}. \tag{35}$$

The main diagonal of $M(\vec{s})$ is equal to the main diagonal of matrix $J(\vec{s})$. Therefore, the elements at the main diagonal of $M(\vec{s})$ are real. For $i \neq j$ the elements of matrix $M(\vec{s})$ are

$$m_{i,j} = \frac{d_i}{d_j} a_{i,j} = \sqrt{\frac{s_j}{s_i}}\sqrt{\frac{\partial v_j/\partial s_i}{\partial v_i/\partial s_j}}\frac{s_i}{s_1}\frac{\partial v_i}{\partial s_j} = \frac{\sqrt{s_i s_j}}{s_1}\sqrt{\frac{\partial v_j}{\partial s_i}\frac{\partial v_i}{\partial s_j}}. \tag{36}$$

If Condition 1 holds then the terms under the square root signs are positive and consequently matrix $M(\vec{s})$ is real. We verify that $M(\vec{s})$ is symmetric:

$$m_{j,i} = \frac{d_j}{d_i} a_{j,i} = \frac{\sqrt{s_j s_i}}{s_1}\sqrt{\frac{\partial v_j}{\partial s_i}\frac{\partial v_i}{\partial s_j}} = m_{i,j}. \tag{37}$$

Under Condition 1 we can conclude that matrix $M(\vec{s})$ is real and symmetric which implies by Preliminary 3 that its eigenvalues are real. From Preliminary 2 we conclude that the eigenvalues of $M(\vec{s})$ and $J(\vec{s})$ are equal and consequently the eigenvalues of $J(\vec{s})$ are also real. Therefore, the system (26) is hyperbolic if Condition 1 holds. □

We note that Condition 1 is sufficient but not necessary for hyperbolicity. For example, even if $M(\vec{s})$ has imaginary entries the eigenvalues of the matrix $M(\vec{s})$ can be real, and thus the model can be hyperbolic.

Lemma 1: *A multi-class macroscopic traffic flow model in Lagrangian formulation with coordinate velocity equal to the vehicle velocity of the fastest class is anisotropic if it can be described by a hyperbolic system of equations* $\partial\vec{s}/\partial t + J(\vec{s})\partial\vec{s}/\partial n = 0$ *with eigenvalues of the Jacobian matrix* $J(\vec{s})$ *all nonnegative.*

Proof: This lemma follows readily from Definitions 1 and 2 and Preliminary 1. □

Theorem 2: *The model represented by the nonlinear system of partial differential equations* (26) *is anisotropic if Conditions* 1 *and* 2 *are satisfied.*

Proof: Since Condition 1 holds, it follows from Theorem 1 that the system is hyperbolic. Without loss of generality in the following we will assume that the fastest class u^* is class 1 and that in the Lagrangian formulation the coordinates have the same velocity as the vehicles of class 1. According to Lemma 1, the model is anisotropic if the eigenvalues of the Jacobian matrix $J(\vec{s})$ are all nonnegative. We will show that the eigenvalues are indeed nonnegative by making use of matrix $M(\vec{s})$ as it was defined in (32) and applying Preliminary 4.

We apply Gaussian elimination to matrix $M(\vec{s})$ to find the pivots, i.e. we subtract $m_{i,1}/m_{1,1}$ times row 1 from row i which gives zeros on column 1. Substituting (28), (33) and (35) gives matrix $\tilde{M}(\vec{s})$ with:

$$\text{on row 1:}$$
$$\tilde{m}_{1,j} = m_{1,j} = \frac{d_1}{d_j}a_{j,1} = \sqrt{\frac{s_j}{s_1}}\sqrt{\frac{\partial v_j/\partial s_1}{\partial v_1/\partial s_j}\frac{s_1}{s_1}\frac{\partial v_1}{\partial s_j}} = \sqrt{\frac{s_j}{s_1}}\sqrt{\frac{\partial v_j}{\partial s_1}\frac{\partial v_1}{\partial s_j}} = \sqrt{\frac{s_j}{s_1}\frac{\partial v_1}{\partial s_1}\frac{\partial v_j}{\partial s_j}},$$

$$\text{on column 1, except for row 1 } (i \neq 1):$$
$$\tilde{m}_{i,1} = 0,$$

$$\text{on the diagonal, except for row 1 } (i \neq 1):$$
$$\begin{aligned}\tilde{m}_{i,i} &= m_{i,i} - \frac{m_{i,1}}{m_{1,1}}m_{1,i} = m_{i,i} - \frac{m_{1,i}^2}{m_{1,1}} = a_{i,i} - \left(\frac{d_1}{d_i}a_{1,i}\right)^2\frac{1}{a_{1,1}}\\ &= \frac{s_i}{s_1}\frac{\partial v_i}{\partial s_i} + \frac{v_1 - v_i}{s_1} - \frac{s_i}{s_1}\frac{\partial v_i/\partial s_1}{\partial v_1/\partial s_i}\left(\frac{\partial v_1}{\partial s_i}\right)^2\frac{1}{\partial v_1/\partial s_1}\\ &= \frac{s_i}{s_1}\frac{\partial v_i}{\partial s_i} - \frac{s_i}{s_1}\frac{\partial v_i}{\partial s_i} + \frac{v_1 - v_i}{s_1} = \frac{v_1 - v_i}{s_1},\end{aligned}$$

$$\text{off the diagonal, except for row 1 } (i \neq 1, i \neq j):$$
$$\begin{aligned}\tilde{m}_{i,j} &= m_{i,j} - \frac{m_{i,1}}{m_{1,1}}m_{1,j} = \frac{d_i}{d_j}a_{i,j} - \frac{\frac{d_i}{d_1}a_{i,1}}{a_{1,1}}\frac{d_1}{d_j}a_{1,j} = \frac{d_i}{d_j}\left(a_{i,j} - \frac{a_{i,1}}{a_{1,1}}a_{1,j}\right)\\ &= \frac{d_i}{d_j}\left(\frac{s_i}{s_1}\frac{\partial v_i}{\partial s_j} - \frac{\frac{s_i}{s_1}\frac{\partial v_i}{\partial s_1}}{\frac{s_1}{s_1}\frac{\partial v_1}{\partial s_1}}\frac{s_1}{s_1}\frac{\partial v_1}{\partial s_j}\right) = \frac{d_i}{d_j}\frac{s_i}{s_1}\left(\frac{\partial v_i}{\partial s_j} - \frac{\partial v_i}{\partial s_j}\right) = 0.\end{aligned}$$

Or in matrix form:

$$\tilde{M} = \begin{pmatrix} \frac{\partial v_1}{\partial s_1} & \sqrt{\frac{s_2}{s_1}\frac{\partial v_1}{\partial s_1}\frac{\partial v_2}{\partial s_2}} & \cdots & \sqrt{\frac{s_U}{s_1}\frac{\partial v_1}{\partial s_1}\frac{\partial v_U}{\partial s_U}} \\ & \frac{v_1 - v_2}{s_1} & & \emptyset \\ & & \ddots & \\ \emptyset & & & \frac{v_1 - v_U}{s_1} \end{pmatrix}. \tag{38}$$

Only one step of Gaussian elimination is enough: all entries of $\tilde{M}$ below the main diagonal are zero. The pivots are the elements on the diagonal: $\tilde{m}_{1,1} = \partial v_1/\partial s_1$ and $\tilde{m}_{i,i} = (v_1 - v_i)/s_1$, $\forall i > 1$. The pivots, and thus the eigenvalues and the characteristic velocities, are nonnegative if both Conditions 1 and 2 hold. We conclude that the multi-class model (22)–(25) represents anisotropy if both conditions are satisfied. □

5. Implications of the conditions

We have shown above that under certain conditions (Condition 2) anisotropy is represented by the generic multi-class kinematic wave model (22)–(25), and that under somewhat weaker conditions (Condition 1) the model is hyperbolic. We discuss the practical implications of these conditions and verify whether they hold for the more specific models that were discussed in Sections 2.1–2.3.

In the following discussion we assume that the fundamental diagram describing the relation between density ρ and flow q_u, or equivalently the relation between density ρ or spacing s and velocity v_u is differentiable for all relevant traffic states. In practice many fundamental diagrams are not continuously differentiable, and sometimes even not continuous at the critical density. However, most fundamental diagrams such as the Daganzo fundamental diagram and the Smulders fundamental diagram (Smulders 1990) are piecewise differentiable. Moreover, the Daganzo fundamental diagram can be approximated by a continuously differentiable function (del Castillo 2010). In fact, any other continuous fundamental diagram can be approximated by a generalisation of the Taylor series due to Hille and Phillips (1957) which converges to the original fundamental diagram. The resulting series is continuously differentiable.

We now first rewrite the conditions in the traditional Eulerian formulation using the definitions (18).

5.1. *Hyperbolicity condition*

Condition 1 guarantees that the multi-class model is hyperbolic. It can be rewritten as

$$\frac{\partial v_u}{\partial \rho_w} \leq 0, \quad \forall u, w, \forall \rho \in [0, \rho_{\max}]. \tag{39}$$

This condition implies that if the class specific density ρ_u increases, the velocity of any class v_w does not increase: it either decreases or remains the same. This is a very intuitive condition: if there are more vehicles on the road, they drive slower. However, depending on the formulation of the model, this is not always satisfied, as we will see below.

5.1.1. *Hyperbolicity of basic multi-class models*

For the analysis of the basic multi-class models we rewrite the velocity derivative as

$$\frac{\partial v_u}{\partial \rho_w} = \frac{\partial v_u}{\partial \rho} \frac{\partial \rho}{\partial \rho_w}. \tag{40}$$

By substituting the fundamental relation (9), (10), (11) or (12) in (40) it can easily be verified that the velocity derivative to the effective density is negative or zero: $\partial v_u / \partial \rho \leq 0$. Furthermore, the derivative of the effective density to the class specific density is positive: $\partial \rho / \partial \rho_w > 0$, independent of whether an unweighted summation as in (6) or a weighted summation as in (7) or (8) is used. We conclude that the velocity derivative condition is satisfied for the basic multi-class models that were discussed in Section 2.1.

5.1.2. *Hyperbolicity of the 3-state model*

For the analysis of the multi-class model with 3 states we can apply a simple eigenvalue analysis as proposed in van Wageningen-Kessels *et al*. (2009c) because there are only 2 classes. We do not need all of the framework developed before. In fact, we cannot use parts of it because the matrix $M(\vec{s})$ is not real if $s_1 \neq s_2$.

However, we do need the derivatives of the fundamental relation (12). In free flow all derivatives are zero. In congestion they are

$$\begin{aligned} \frac{\partial v_1}{\partial s_1} &= C_1(s_1+s_2)\frac{s_2-s_1}{s_1^2}, \quad &\frac{\partial v_1}{\partial s_2} = 2C_1\frac{s_1+s_2}{s_1}, \\ \frac{\partial v_2}{\partial s_2} &= -C_2(s_1+s_2)\frac{s_2-s_1}{s_2^2}, \quad &\frac{\partial v_2}{\partial s_1} = 2C_2\frac{s_1+s_2}{s_2}, \end{aligned} \tag{41}$$

with the parameters of the fundamental relation:

$$C_1 = \frac{\rho_{1,\mathrm{crit}}(\rho_1,0)\rho_{1,\max}(\rho_1,0)v_{1,\max}}{\rho_{1,\max}(\rho_1,0)-\rho_{1,\mathrm{crit}}(\rho_1,0)}, \quad C_2 = \frac{\rho_{2,\mathrm{crit}}(0,\rho_2)\rho_{2,\max}(0,\rho_2)v_{2,\max}}{\rho_{2,\max}(0,\rho_2)-\rho_{2,\mathrm{crit}}(0,\rho_2)}. \tag{42}$$

The eigenvalues of the model are

$$\lambda_{1,2} = \frac{1}{2}\left(a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right), \tag{43}$$

$$\text{with} \quad a = \frac{\partial v_1}{\partial s_1}, \quad b = \frac{\partial v_1}{\partial s_2}, \quad c = \frac{s_2}{s_1}\frac{\partial v_2}{\partial s_1}, \quad d = \frac{s_2}{s_1}\frac{\partial v_2}{\partial s_2} + \frac{v_1 - v_2}{s_1}. \tag{44}$$

The eigenvalues are real if the term under the square root sign in (43) is nonnegative. From (41) we can conclude that both $\partial v_1/\partial s_2$ and $\partial v_2/\partial s_1$ are nonnegative. Therefore, the term under the square root sign in (43) is nonnegative:

$$(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc = (a-d)^2 + 4\frac{s_2}{s_1}\frac{\partial v_1}{\partial s_2}\frac{\partial v_2}{\partial s_1} \geq 0. \tag{45}$$

Since the eigenvalues are real, we can conclude from Preliminary 1 that the model is hyperbolic.

5.1.3. *Hyperbolicity of the model by Ngoduy and Liu*

Both multi-class models with generic pce's are analysed separately. In free flow in the model by Ngoduy and Liu the velocity derivative is

$$\frac{\partial v_u}{\partial \rho_w} = -(v_{\max} - v_{\text{crit}})\frac{\partial}{\partial \rho_w}\left(\frac{\rho}{\rho_{\text{crit}}}\right) = -(v_{\max} - v_{\text{crit}})\frac{\rho_{\text{crit}}\frac{\partial \rho}{\partial \rho_w} - \rho\frac{\partial \rho_{\text{crit}}}{\partial \rho_w}}{(\rho_{\text{crit}})^2}. \tag{46}$$

The hyperbolicity condition (39) or equivalently Condition 1 holds if the nominator is nonnegative. Therefore, we first calculate

$$\frac{\partial \rho}{\partial \rho_w} = 1, \text{ and} \tag{47}$$

$$\frac{\partial \rho_{\text{crit}}}{\partial \rho_w} = \frac{\rho_{\text{ref, crit}}}{\rho}\frac{\partial}{\partial \rho_w}\left(\sum_i \frac{\rho_i}{\beta_i}\right) = \frac{\rho_{\text{ref, crit}}}{\rho}\left(\left(\sum_i \frac{1}{(\beta_i)^2}\frac{\partial \beta_i}{\partial \rho_i}\right) + \frac{1}{\beta_w}\right). \tag{48}$$

For the conditions to hold, the nominator of (46) should be nonnegative:

$$\rho_{\text{crit}}\frac{\partial \rho}{\partial \rho_w} - \rho\frac{\partial \rho_{\text{crit}}}{\partial \rho_w} = \rho_{\text{crit}} - \rho_{\text{ref, crit}}\left(\left(\sum_i \frac{1}{(\beta_i)^2}\frac{\partial \beta_i}{\partial \rho_i}\right) + \frac{1}{\beta_w}\right) \geq 0. \tag{49}$$

Because no pce-function was suggested by Ngoduy and Liu, we assume that the pce-values β_u are constant. In that case we find:

$$\rho_{\text{crit}}\frac{\partial \rho}{\partial \rho_w} - \rho\frac{\partial \rho_{\text{crit}}}{\partial \rho_w} = \rho_{\text{crit}} - \frac{\rho_{\text{ref, crit}}}{\beta_w}. \tag{50}$$

It now depends on the choice of the reference class and the pce-values whether (50) is indeed nonpositive. If it is nonpositive, then the hyperbolicity condition (39) or equivalently Condition 1 holds and consequently the model is hyperbolic in free flow. The conditions are for example satisfied if the class with the smallest vehicle length and headway and consequently the largest critical density is chosen as the reference class and the pce-value is chosen such that all other classes have pce-values larger than or equal to this class.

In congestion in the model by Ngoduy and Liu the velocity derivative is

$$\frac{\partial v_u}{\partial \rho_w} = \frac{\rho_{\text{ref, crit}} v_{\text{crit}}}{\rho_{\text{ref, max}} - \rho_{\text{ref, crit}}}\frac{\rho\frac{\partial \rho_{\max}}{\partial \rho_w} - \rho_{\max}\frac{\partial \rho}{\partial \rho_w}}{\rho^2}. \tag{51}$$

The hyperbolicity condition (39) or equivalently Condition 1 holds if the nominator is nonnegative. Similarly, to the free flow case we find that the following holds if:

$$\frac{\rho_{\text{ref, max}}}{\beta_w} - \rho_{\max} \leq 0 \tag{52}$$

Similarly to the free flow case it now depends on the choice of the reference class and the pce-values whether (52) holds and consequently whether the model is hyperbolic in congestion. Again the conditions are for example satisfied by taking the class with the smallest vehicle length and headway as the reference class.

5.1.4. *Hyperbolicity of the Fastlane model*

For the Fastlane multi-class model we again rewrite the velocity derivative as in (40). Substituting the fundamental relation (11), it can easily be verified that the velocity derivative to the effective density is negative or zero: $\partial v_u/\partial \rho \leq 0$. Now we have to check whether the effective density increases if the class-specific density increases:

$$\frac{\partial \rho}{\partial \rho_u} \geq 0. \tag{53}$$

This would be reasonable to assume, however it appears that this condition is not satisfied for certain pce-functions.

Recall that the effective density is a (weighted) summation over all user classes (15). If the pce-value depends on effective density, the sign of the derivative depends on the actual pce-function $\eta_u(\rho)$ and its parameters. In that case we will have to check whether (53) holds for all user classes u and all relevant densities.

Theorem 3: *Suppose that the effective density ρ is a weighted summation of all user class specific densities ρ_u as in* (15) *and weights are larger than or equal to* 1: $\eta_u(\rho) \geq 1$, $\forall u$, $\forall \rho \in [0, \rho_{\max}]$. *If and only if for all relevant values of effective density $\rho = \rho^1$ and $\rho = \rho^2$ with $\rho^1 < \rho^2$ the pce-functions $\eta_u(\rho)$ of all user classes u satisfy*:

$$\frac{\eta_u(\rho^2)}{\rho^2} < \frac{\eta_u(\rho^1)}{\rho^1}, \tag{54}$$

then condition (53) *holds for all relevant spacings.*

Proof: We first note that (54) is equivalent to

$$\frac{\mathrm{d}\eta_u(\rho)}{\mathrm{d}\rho} < \frac{\eta_u(\rho)}{\rho}. \tag{55}$$

This can be shown by dividing both sides of (55) by ρ, integrating over ρ from $\rho = \rho^1$ to $\rho = \rho^2$, solving this integral and rewriting the result. In the following we will show that the hyperbolicity condition (53) is equivalent to condition (55).

We reformulate (53):

$$\begin{aligned}\frac{\partial \rho}{\partial \rho_u} = \sum_{i=1}^{U} \rho_i \frac{\partial \eta_i}{\partial \rho_u} + \sum_{i=1}^{U} \eta_i \frac{\partial \rho_i}{\partial \rho_u} = \eta_u + \frac{\partial \rho}{\partial \rho_u} \sum_{i=1}^{U} \rho_i \frac{\mathrm{d}\eta_i}{\mathrm{d}\rho} > 0, \ \Leftrightarrow \\ \frac{\partial \rho}{\partial \rho_u} = \frac{\eta_u}{1 - \sum_{i=1}^{U} \rho_i \frac{\mathrm{d}\eta_i}{\mathrm{d}\rho}} > 0, \ \Leftrightarrow \\ \sum_{i=1}^{U} \rho_i \frac{\mathrm{d}\eta_i}{\mathrm{d}\rho} < 1.\end{aligned} \tag{56}$$

This shows that (56) is equivalent to (53) for all relevant spacings.

We will now show that if (55) is satisfied for all relevant densities, than (56) is also satisfied for all relevant densities. We note that the sum in (56) can also be written as a weighted average:

$$\sum_{u=1}^{U} \rho_u \frac{\mathrm{d}\eta_u}{\mathrm{d}\rho} = \sum_{u=1}^{U} \frac{\eta_u \rho_u}{\rho} \frac{\rho}{\eta_u} \frac{\mathrm{d}\eta_u}{\mathrm{d}\rho}. \tag{57}$$

(All weights $\eta_u \rho_u / \rho$ are nonnegative and they sum up to one: $\sum_{u=1}^{U} \eta_u \rho_u / \rho = 1$.) Inequality (55) guarantees that all elements $(\rho/\eta_u)\mathrm{d}\eta_u/\mathrm{d}\rho$ in the weighted average (57) are smaller than 1. Therefore, also the weighted average is smaller than 1, i.e. (56) holds.

We will now show that if (56) is satisfied for all relevant densities, then (55) is also satisfied for all relevant densities. We will do this by showing that if there is a relevant state that does not satisfy (55), then (56) is also not satisfied for this state. Let us call the effective density of this state $\tilde{\rho}$ and suppose that only vehicles of class $\tilde{u}$ are present in this state. This implies that the class specific density of this class is $\rho_{\tilde{u}} = \tilde{\rho}/\eta_{\tilde{u}}(\tilde{\rho})$ and all other class specific densities are zero. Furthermore, suppose that for this user class $\tilde{u}$ (55) does not hold at the effective density $\tilde{\rho}$, i.e.

$$\frac{\mathrm{d}\eta_{\tilde{u}}(\tilde{\rho})}{\mathrm{d}\rho} \geq \frac{\eta_{\tilde{u}}(\tilde{\rho})}{\tilde{\rho}}. \tag{58}$$

We substitute (58) in the left-hand side of (56) and find:

$$\sum_{u=1}^{U} \rho_u \frac{\mathrm{d}\eta_u}{\mathrm{d}\rho} = \rho_{\tilde{u}} \frac{\mathrm{d}\eta_{\tilde{u}}}{\mathrm{d}\rho} = \frac{\tilde{\rho}}{\eta_{\tilde{u}}(\tilde{\rho})} \frac{\mathrm{d}\eta_{\tilde{u}}}{\mathrm{d}\rho} \geq 1. \tag{59}$$

This implies that (56) does not hold.

We conclude that if and only if (54) holds then the condition (53) holds and the effective density increases if one class-specific density increases. This is a necessary but not sufficient condition for anisotropy. □

Note that condition (55), or equivalently condition (54), can easily be checked graphically: that is, by drawing the pce-values η_u as a function of density ρ (Figure 2). If one can now draw a straight line through the origin that intersects the pce-function twice or more between zero and maximum density, the condition is not satisfied. For example, any positive constant, any positive linear or any positive and concave pce-function $\eta(\rho)$ satisfies (55). Any positive and piecewise constant pce-function $\eta(\rho)$ with at least one

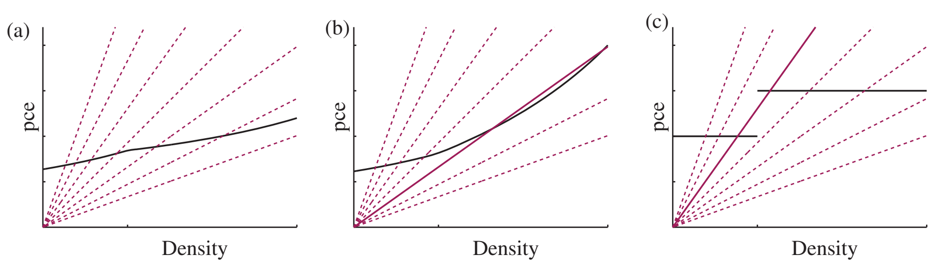

Figure 2. Pce-functions (black solid lines) are checked graphically for hyperbolicity. If any straight line (thin red broken lines) that goes through the origin intersects the pce-function twice or more, the pce-function is invalid (thick red broken lines). (a) Fastlane pce with valid parameters; (b) fastlane pce with invalid parameters; (c) piecewise constant pce with invalid parameters.

discontinuity with the pce-value low before and high after the discontinuity does not satisfy (55). We note that the latter kind of pce-functions are regularly used. For example in models where passenger cars have pce-value 1 and trucks have a low constant pce-value in free flow conditions and the pce-function shows a discontinuity to a high constant pce-value in congested conditions.

The difficulty with this condition (54) arises because in the Fastlane model the pce-value η is an implicit function of density ρ. So, if the density of a certain class increases, the effective density must increase, this would result in an increase of the pce-value of that class, and the effective density would increase even more, etc. To avoid this getting out of hand, the pce-function should not be too steep, in order to 'damp out' this effect. That is exactly what the condition formulated as (54) tells us.

5.2. *Anisotropy condition*

Conditions 1 and 2 guarantee that the multi-class model is anisotropic. Condition 2 can be rewritten as

$$\exists u^* \text{ such that } v_{u^*}(\vec{\rho}) \geq v_u(\vec{\rho}), \quad \forall u, \forall \rho \in [0, \rho_{\max}]. \tag{60}$$

The condition guarantees that no user class is faster than user class u^*.

In the Lagrangian formulation one should choose user class u^* the reference class. Therefore, the coordinate system moves with the same velocity as class u^* and no other class will move faster than the coordinate system. So, if the model is anisotropic, information will not travel faster than the coordinates. This gives some important advantages for the numerical methods to be applied in a simulation.

In most practical cases passenger cars are faster than (or just as fast as) all other vehicles for all traffic states. In that case, passenger cars should be taken as the reference class u^*.

5.2.1. *Anisotropy of the multi-class model with 3 states*

Even though the multi-class model with 3 states is hyperbolic, it does not satisfy Condition 1 for all relevant states (e.g. if $s_1 < s_2$ then $\partial v_2 / \partial s_2 < 0$). Therefore, we will analyse directly whether the eigenvalues are nonnegative. We already concluded in Section 5.1.2 that the eigenvalues are real. The smallest eigenvalue is nonnegative if

$$a + d - \sqrt{(a+d)^2 - 4(ad - bc)} > 0 \Leftrightarrow ad - bc > 0. \tag{61}$$

Substituting (44) and (41) we find in congestion:

$$ad - bc = -\frac{s_2}{s_1}\frac{(s_1 + s_2)^2}{s_1 s_2} C_1 C_2 \left(\frac{(s_2 - s_1)^2}{s_1 s_2} + 4\right) + C_1(s_1 + s_2)\frac{s_2 - s_1}{s_1^2}\frac{v_1 - v_2}{s_1}, \tag{62}$$

with C_1 and C_2 as in (41). It depends on the parameters of the fundamental diagram whether (62) is negative or not. However, for certain choices of these parameters (62) is negative and hence there is one negative eigenvalue and consequently the model is not anisotropic. As an example we take the parameters of the fundamental relations for both classes equal: $\rho_{1,\max}(\rho_1, 0) = \rho_{2,\max}(0, \rho_2)$, $\rho_{1,\text{crit}}(\rho_1, 0) = \rho_{2,\text{crit}}(0, \rho_2)$ and $v_{1,\max} = v_{2,\max}$.

Therefore, the velocities v_1 and v_2 are equal and in congestion the sign of (62) is equal to the sign of $-((s_2 - s_1)^2/(s_1 s_2) + 4)$. Therefore, (62) is negative and the model is not anisotropic.

6. Conclusion

Hyperbolicity and anisotropy are important features of a traffic flow model. If a traffic flow model is hyperbolic, information travels at finite velocity. If a traffic flow model is anisotropic, information travels at a velocity smaller than or equal to the vehicle velocity. It is noted that hyperbolicity is a necessary but not sufficient condition for a traffic flow model to be anisotropic.

There are two reasons why one would want to know whether a certain traffic flow model is hyperbolic and anisotropic: real traffic is hyperbolic and anisotropic and furthermore, if the model has these properties more efficient computational methods can be applied.

We developed a framework to analyse hyperbolicity and anisotropy for multi-class kinematic wave traffic flow models. This framework was applied to all models known from the literature.

It is concluded that all 'basic' models are hyperbolic and anisotropic. The 3-state model introduced by Logghe and Immers is hyperbolic but it depends on the choice of the parameters of the fundamental relation whether it is anisotropic. Other multi-class models have to satisfy some criteria to be both hyperbolic and anisotropic. These criteria are related to the pce-function and the way the different classes are treated. The criteria are further analysed future studies.

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References

Aw, A. and Rascle, M., 2000. Resurrection of 'second order models' of traffic flow? *SIAM Journal on Applied Mathematics*, 60 (3), 916–938.

Bagnerini, P. and Rascle, M., 2003. A multiclass homogenized hyperbolic model of traffic flow. *SIAM Journal on Mathematical Analysis*, 35 (4), 949–973.

Benzoni-Gavage, S. and Colombo, R.M., 2003. An *n*-populations model for traffic flow. *European Journal of Applied Mathematics*, 14 (05), 587–612.

Chanut, S. and Buisson, C., 2003. Macroscopic model and its numerical solution for two-flow mixed traffic with different speeds and lengths. *Transportation Research Record: Journal of the Transportation Research Board*, 1852, 209–219.

Daganzo, C.F., 1995. Requiem for second-order fluid approximations of traffic flow. *Transportation Research Part B: Methodological*, 29 (4), 277–286.

Daganzo, C.F. 2006. On the variational theory of traffic flow: well-posedness, duality and applications. *Networks and Heterogeneous Media*, 1 (4), 601–619.

del Castillo, J.M. 2010. Two new models for the flow-density relationship. *In*: *Transportation research board 89th Annual Meeting Compendium of Papers (DVD)*, Washington, DC.

Donat, R and Mulet, P., 2010. A secular equation for the Jacobian matrix of certain multispecies kinematic flow models. *Numerical Methods for Partial Differential Equations*, 26 (1), 159–175.

Garavello, M. and Piccoli, B., 2009. On fluido-dynamic models for urban traffic. *Networks and Heterogeneous Media*, 4 (1), 107–126.

Gupta, A.K. and Katiyar, V.K., 2007. A new multi-class continuum model for traffic flow. *Transportmetrica*, 3 (1), 73–85.

HCM 2000. *Highway capacity manual*. Washington, DC: Transportation Research Board.

Hille, E. and Phillips, R.S., 1957. Functional analysis and semi-groups. *American Mathematical Society*. Vol. 31, Providence, Rhode Island, 302–327, Chap. 10.

Hoogendoorn, S.P. and Bovy, P.H.L., 2000. Continuum modelling of multiclass traffic flow. *Transportation Research Part B: Methodological*, 34 (2), 123–146.

Hoogendoorn, S.P. and Bovy, P.H.L., 2001. State-of-the-art of vehicular traffic flow modeling. *Proceedings of the institution of mechanical engineers, Part I: Journal of Systems and Control Engineering*, 215, pp. 283–303.

Hoogendoorn, S.P., Bovy, P.H.L., and van Lint, H., 2002. Short-term prediction of traffic flow conditions in a multilane multiclass network. *In*: M.A.P. Taylor, ed. *Proceedings of the 15th International Symposium on Transportation and Traffic Theory*. Oxford, UK: Elsevier, 625–651.

Kerner, B.S., 2004a. The physics of traffic. empirical freeway pattern features, engineering applications, and theory. *Understanding Complex Systems*. Berlin: Springer.

Kerner, B.S., 2004b. Three-phase traffic theory and highway capacity. *Physica A: Statistical and Theoretical Physics*, 333, 379–440.

Kerner, B.S., 2009. *Introduction to modern traffic flow theory and control: the long road to three-phase traffic theory*. Heidelberg: Springer.

Leclercq, L., Laval, J., and Chevallier, E., 2007. The Lagrangian coordinates and what it means for first order traffic flow models. *In*: R.E. Allsop, M.G.H. Bell and B.G. Heydecker, eds. *Transportation and traffic theory 2007*. Oxford, UK: Elsevier, 735–753.

Leclercq, L., Laval, J., and Chevallier, E., 2008. The Lagrangian coordinates applied to the LWR model. *Hyperbolic Problems: Theory, Numerics, Applications*. Berlin Heidelberg: Springer, 671–678.

LeVeque, R.J., 2002. *Finite volume methods for hyperbolic problems*. *Cambridge texts in applied mathematics*. Cambridge: Cambridge University Press.

Lighthill, M.J. and Whitham, G.B., 1955. On Kinematic waves. II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London*. *Series A, Mathematical and Physical Sciences*, 229 (1178), 317–345.

Logghe, S. and Immers, L.H., 2008. Multi-class kinematic wave theory of traffic flow. *Transportation Research Part B: Methodological*, 42 (6), 523–541.

Ngoduy, D., 2010. Multiclass first-order modelling of traffic networks using discontinuous flow-density relationships. *Transportmetrica*, 6 (2), 121–141.

Ngoduy, D. and Liu, R., 2007. Multiclass first-order simulation model to explain non-linear traffic phenomena. *Physica A: Statistical Mechanics and its Applications*, 385 (2), 667–682.

Payne, H.J., 1971. Models of freeway traffic and control. *In*: G.A. Bekey, ed. *Mathematical Models of Public Systems*, 51–61.

Richards, Paul I., 1956. Shock waves on the highway. *Operations Research*, 4 (1), 42–51.

Smulders, S., 1990. Control of freeway traffic flow by variable speed signs. *Transportation Research Part B: Methodological*, 24 (2), 111–132.

Strang, G., 1988. *Linear algebra and its applications*. 3rd ed. San Diego: Harcourt Brace Jovanovich.

Treiber, M., Kesting, A., and Helbing, D., 2010. Three-phase traffic theory and twophase models with a fundamental diagram in the light of empirical stylized facts. *Transportation Research Part B: Methodological*, 44 (8–9), 983–1000.

van Lint, J.W.C., Hoogendoorn, S.P., and Schreuder, M., 2008. Fastlane: a new multi-class first order traffic flow model. *Transportation Research Record: Journal of the Transportation Research Board*, 2088, 177–187.

van Wageningen-Kessels, F.L.M., *et al.*, 2009a. Implicit and explicit numericalmethods formacroscopic traffic flow models. *In*: *Transportation research board 88th annual meeting compendium of papers (DVD)*, Washington, DC.

van Wageningen-Kessels, F.L.M., *et al.*, 2009b. Implicit time stepping schemes applied to the kinematic wave model in Lagrangian coordinates. *In*: *Proceedings of traffic and granular flow*, Shanghai.

van Wageningen-Kessels, F.L.M., *et al.*, 2009c. Multiple user classes in the kinematic wave model in Lagrangian coordinates. *In*: *Proceedings of traffic and granular flow*, Shanghai.

van Wageningen-Kessels, F.L.M., *et al.*, 2010. Lagrangian formulation of a multi-class kinematic wave model. *Transportation Research Record: Journal of the Transportation Research Board*, 2188, 29–36.

Wong, G.C.K. and Wong, S.C., 2002a. A multi-class traffic flow model – an extension of LWR model with heterogeneous drivers. *Transportation Research Part A: Policy and Practice*, 36 (9), 827–841.

Wong, S.C. and Wong, G.C.K., 2002b. An analytical shock-fitting algorithm for LWR kinematic wave model embedded with linear speed-density relationship. *Transportation Research Part B: Methodological*, 36 (8), 683–706.

Zhang, H. M., 2003. Anisotropic property revisited – does it hold in multi-lane traffic? *Transportation Research Part B: Methodological*, 37 (6), 561–577.

Zhang, P., *et al.*, 2006a. Hyperbolicity and kinematic waves of a class of multi-population partial differential equations. *European Journal of Applied Mathematics*, 17, 171–200.

Zhang, P., Wong, S.C., and Shu, C., 2006b. A weighted essentially non-oscillatory numerical scheme for a multi-class traffic flow model on an inhomogeneous highway. *Journal of computational physics*, 212, 739–756.

Zhang, P., Wong, S.C., and Xu, Z., 2008. A hybrid scheme for solving a multi-class traffic flow model with complex wave breaking. *ComputerMethods in Applied Mechanics and Engineering*, 197, 3816–3827.