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Executive summary

The accuracy of temporal basis functions used in the Time Domain Integral Equation method

Problem area
Improving stealth technology is an important field of research that is extensively used for both aircraft design and operations. Next to platform alignment, further reduction of the Radar Cross Section can be achieved by the use of advanced radar absorbing materials. In particular ferromagnetic materials can be utilised as a coating or as particles in a composite structure. To complement physical experiments, numerical methods provide a viable tool to analyse the backscattered radar field of a platform.

Description of work
In this report a promising method to numerically predict the Radar Cross Section will be presented. This method is designed for future applications such as predicting the radar signature of aircraft with advanced radar absorbing materials. Since nonlinear responses are expected, the method is fully formulated in the time domain, as opposed to standard approaches formulated in the frequency domain. A basic choice in this time domain method is the definition of temporal basis functions. The design of these functions will be motivated by a thorough accuracy analysis.

Results and conclusions
The widespread use of the shifted Lagrange basis functions are inspired by its optimal accuracy for the given class of functions that provide an efficient algorithm. The additional requirement of smooth basis functions can be achieved without impinging on the global accuracy. These novel spline basis functions result in both an accurate interpolation scheme and smooth electromagnetic fields. The presented analysis of temporal basis functions gives a careful foundation for high-order accurate methods in time. Numerical results obtained with the proposed method show electric surface currents that satisfy the accuracy and smoothness claimed by the mathematical analysis.

Applicability
Time Domain Integral Equation methods are a prerequisite to numerically model the nonlinear response of ferromagnetic radar absorbing materials. Large scale computations are expected to be feasible because use is made of a boundary integral formulation for which only the surface of the aircraft needs to be modelled.

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The accuracy of temporal basis functions used in the Time Domain Integral Equation method

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Abstract

A key parameter in the design of integral equation methods for transient electromagnetic scattering is the choice of temporal basis functions. Newly constructed basis functions have to meet requirements on accuracy, smoothness and efficiency, while the requirement of bandlimitedness is dropped for the nonlinear case. An analysis of the interpolation accuracy will justify the widespread use of the shifted Lagrange basis functions, because these have optimal accuracy, but introduce nonsmoothness in the calculated fields. Alternatively, a novel spline basis function is proposed that has optimal accuracy under an additional smoothness constraint. Computational results confirm the expected smoothness and accuracy.

Contents

1 Introduction

Time Domain Integral Equation (TDIE) methods are used to model transient electromagnetic scattering from complex structures. When, for example, the scatterer is coated with ferromagnetic or chiral radar absorbing materials, its response will be nonlinear, and standard methods in the frequency domain are not applicable. The boundary integral formulation has the added advantage of being efficient because the number of degrees of freedom scales squared instead of cubed with the electrical size of the object. For realistic problems, efficiency can be improved with plane-wave techniques (Refs. 6, 1, 5).

Historically, many choices for the definition of TDIE methods have been inspired by the Method of Moments (MoM), its frequency domain analogue. However, temporal basis functions are not encountered in the MoM. Linear functions have first been used as building block of the temporal basis functions (Ref. 4), followed by the introduction of basis functions with quadratic Lagrange polynomials (Ref. 3). The associated family of *shifted Lagrange basis functions* is still the most popular choice as temporal basis function. Based on different design criteria many other temporal basis functions have been proposed, but with varying success (Ref. 2).

To the best of the authors' knowledge, no comprehensive motivation for the choice of shifted Lagrange basis functions has been found in literature. This paper will show the unique advantage of the Lagrange polynomials: its optimal accuracy for the given class of temporal basis functions. Although optimal accurate, the piecewise smooth character of the shifted Lagrange basis functions can result in nonsmooth solutions. Smoothness can only be enforced when alleviating accuracy requirements. As will be shown, a clever choice of requirements can maintain the global accuracy while introducing smoothness. Aware of possible instabilities, this paper only concentrates on the accuracy of the temporal basis functions. All computational results presented are stable.

This paper starts in Section 2 by formulating the governing integral equations for transient electromagnetic scattering. How to construct temporal basis function for different criteria is explained in Section 3. The accuracy analysis of the temporal basis functions is presented in Section 4. For different accuracy conditions the analysis not only results in the known Lagrange polynomials and quadratic splines, but also gives a novel temporal basis function based on cubic splines. In Section 5 computational results are shown for the TDIE method with different temporal basis functions.

2 Formulation

For a closed and perfect electric conductor (PEC) with its surface denoted by S, integral equations give a relation between the incident and the scattered electromagnetic field. The scatterer is surrounded by free space and the incident fields excite an electric surface current $\mathbf{J}(\mathbf{r}, t)$ on S. This electric surface current induces a scattered electric field $\mathbf{E}^s(\mathbf{r}, t)$ and scattered magnetic field $\mathbf{H}^s(\mathbf{r}, t)$. The differentiated scattered fields are given by

$$\dot{\mathbf{E}}^s(\mathbf{r}, t) = \iint_S \left(\mu \frac{\ddot{\mathbf{J}}(\mathbf{r}', \tau)}{4\pi R} - \frac{1}{\epsilon} \nabla \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}', \tau)}{4\pi R} \right) d\mathbf{r}', \tag{1}$$

$$\dot{\mathbf{H}}^s(\mathbf{r}, t) = -\iint_S \left(\nabla \times \frac{\dot{\mathbf{J}}(\mathbf{r}', \tau)}{4\pi R} \right) d\mathbf{r}', \tag{2}$$

for $R = |\mathbf{r} - \mathbf{r}'|$, $\tau = t - \frac{R}{c}$ denoting the retarded time, and ∇ and ∇' denoting the nabla operator with respect to $\mathbf{r}$ and $\mathbf{r}'$, respectively. The surface conditions on the PEC scatterer are

$$-\mathbf{n} \times \mathbf{n} \times \left(\dot{\mathbf{E}}^i(\mathbf{r}, t) + \dot{\mathbf{E}}^s(\mathbf{r}, t) \right) = \mathbf{0}, \tag{3}$$

$$\mathbf{n} \times \left(\dot{\mathbf{H}}^i(\mathbf{r}, t) + \dot{\mathbf{H}}^s(\mathbf{r}, t) \right) = \dot{\mathbf{J}}(\mathbf{r}, t), \tag{4}$$

with $\mathbf{n}$ denoting the outward pointing unit normal on S. Substitution of the scattered fields (1) and (2) into the surface conditions (3) and (4) results in the EFIE and MFIE, respectively. After discretization, the CFIE is given by a linear combination of the EFIE and MFIE, that is,

$$(1 - \alpha)\dot{\mathbf{J}} - \mathbf{n} \times \left(\frac{\alpha}{\eta} \mathbf{n} \times \dot{\mathbf{E}}^s + (1 - \alpha)\dot{\mathbf{H}}^s \right) = \mathbf{n} \times \left(\frac{\alpha}{\eta} \mathbf{n} \times \dot{\mathbf{E}}^i + (1 - \alpha)\dot{\mathbf{H}}^i \right), \tag{5}$$

for $0 \leq \alpha \leq 1$ and $\eta = \sqrt{\mu/\epsilon}$ the impedance.

To solve the CFIE, the surface current $\mathbf{J}(\mathbf{r}, t)$ is expanded in terms of N_S spatial and N_t temporal basis functions as

$$\mathbf{J}(\mathbf{r}, t) = \sum_{n=1}^{N_S} \sum_{j=1}^{N_t} J_{n,j} \mathbf{f}_n(\mathbf{r}) T_j(t). \tag{6}$$

Spatially, a Galerkin testing procedure is applied. As spatial test and basis functions the RWG function is used. The time span is divided into uniform intervals with time steps $t_i = i\Delta t$ for $i = 0, 1, 2, \ldots$. Temporal point matching in subsequent time levels t_k results in the Marching on in Time (MOT) algorithm (Ref. 7).

3 Construction of temporal basis functions

Temporal basis functions are constructed according to user defined design criteria. Important are accuracy, smoothness, efficiency, and bandlimitedness, which can conflict with each other. How much emphasis is put on which criterium can be inspired by the application of the TDIE method. In this section it will be explained how these design criteria are used to construct temporal basis functions.

Efficiency of the TDIE method is based on a trade-off between the amount of work and accuracy. Generally speaking, basis functions with a small support result in an inexpensive TDIE method, while a large support results in an accurate method. In this paper, only temporal basis functions of a small support will be used: no entire-domain functions are taken into account. Moreover, TDIE methods are solved as a MOT algorithm, for which the unknown surface current is solved efficiently from solutions at previous time levels. Causal temporal basis functions naturally satisfy the marching criterion. Therefore, the analysis is restricted to causal basis functions only.

Bandlimitedness has been imposed on temporal basis functions to exclude undesirable frequency content from the TDIE method (Ref. 7). Since our main interest of application is nonlinear scattering, bandlimitedness of the backscattered field can not be assumed. Therefore, no limits will be imposed on the frequency band of the temporal basis function.

Smoothness is desired because smooth surface currents are expected from physical principles. Furthermore, smooth functions will result in more accurate quadrature of the spatial integrals.

Accuracy will be the most important issue for the construction of temporal basis functions. The accuracy analysis restricts to the error stemming from the time discretization only, which is one part of the truncation error of the TDIE method. With the aid of the accuracy analysis not only existing basis functions can be justified, it also provides a structured procedure to develop new basis functions. Abandoning optimal accuracy gives way to incorporating smoothness into the interpolation. Novel temporal basis functions can then be constructed with a user-defined trade-off between accuracy and smoothness.

Classic EFIE formulations use an integral with respect to time, acting on the surface current. Because numerical procedures for evaluating these integrals are tedious, a differentiated formulation is often used. This liberates the equations from time integrals, but is at the expense of a second order time derivative. The time derivatives of the unkown functions are represented by the derivatives of the interpolator. Because differentiation reduces the interpolation accuracy by one order, also the accuracy of the time differentiated functions is reduced by one order.

4 Accuracy of temporal basis functions

A key factor for the success of the MOT algorithm is the proper choice of temporal basis functions $T_j(t)$ for $j = 1, 2, \ldots, N_t$. Therefore a thorough analysis of the interpolation accuracy will be presented in this section. The basis functions are chosen as $T_j(t) = T(t - j\Delta t)$ for a predefined function $T(t)$. The function T is called the interpolant, since it interpolates functions at a retarded time level between several discrete time steps. The interpolant has a bounded support around zero, given by a fixed multiple of Δt.

The temporal accuracy is measured as the order with respect to the time step size of the pointwise interpolation error. The analysis starts with a general representation of temporal basis functions. Imposing the orders of accuracy results in conditions on the interpolation functions. The unique interpolator satisfying the proposed conditions gives a representation of the temporal basis function.

4.1 Optimal order accurate three-point interpolant

First, three-point interpolants will be analyzed. The corresponding temporal basis function can be represented by a set of three independent functions. That is,

$$T(t) = \begin{cases} F_0(t), & -\Delta t < t \leq 0, \\ F_1(t), & 0 < t \leq \Delta t, \\ F_2(t), & \Delta t < t \leq 2\Delta t, \\ 0, & \text{else}, \end{cases} \tag{7}$$

for arbitrary functions F_0, F_1, F_2 that are twice continuously differentiable inside their respective time intervals.

Consider an arbitrary retarded time instant τ_k, that is situated in the time interval $t_{\ell-1} < \tau_k \leq t_\ell$. Substitution of the interpolant (7) into the series expansion (6) of the surface current results in

$$\mathbf{J}(\mathbf{r}, \tau_k) = \sum_{n=1}^{N_S} \mathbf{f}_n(\mathbf{r}) \big(J_{n,\ell} F_0(\tau_k - t_\ell) + J_{n,\ell-1} F_1(\tau_k - t_{\ell-1}) + J_{n,\ell-2} F_2(\tau_k - t_{\ell-2}) \big). \tag{8}$$

This interpolation has to represent the surface current accurate for arbitrary τ_k. For three-point interpolators, third order accuracy can be imposed. Because time differentiation drops the accuracy by one order, the first time derivative will thus be interpolated second order accurate, and the second time derivative will have first order accuracy. In total this results in a first order accurate TDIE method. These accuracy conditions are represented by

$$F_0 + F_1 + F_2 = 1, \tag{9}$$

$$F_1 + 2F_2 = \frac{t_\ell - \tau_k}{\Delta t}, \tag{10}$$

$$F_1 + 4F_2 = \left(\frac{t_\ell - \tau_k}{\Delta t}\right)^2, \tag{11}$$

as derived in Appendix A. The first equation is the well-known unit sum condition for interpolants. These three accuracy conditions uniquely determine the three-point interpolant as

$$T(t) = \begin{cases} \frac{1}{2}\tilde{t}^2 + \frac{3}{2}\tilde{t} + 1, & -1 < \tilde{t} \le 0, \\ -\tilde{t}^2 + 1, & 0 < \tilde{t} \le 1, \\ \frac{1}{2}\tilde{t}^2 - \frac{3}{2}\tilde{t} + 1, & 1 < \tilde{t} \le 2, \\ 0, & \text{else}, \end{cases} \tag{12}$$

for the scaled time $\tilde{t} = t/\Delta t$. This temporal basis function can be recognized as the quadratic Lagrange basis function (Ref. 3).

4.2 Smooth and accurate three-point interpolant

Given optimal accuracy for three-point interpolation, the analysis results in a unique representation of the temporal basis function. The resulting quadratic Lagrange basis function is continuous, but nonsmooth in the discrete time levels. Because the shifted Lagrange basis function is the unique solution, the accuracy conditions have to be relaxed to give some freedom for other conditions, like smoothness.

Quadratic Lagrange interpolants have a global accuracy of order one, but the function itself and its first derivative are third and second order accurate, respectively. A natural choice of alleviating accuracy is to impose first order accuracy on the basis function, as well as its first and second time derivative. The global accuracy of the TDIE method is thus order one. Additionally, a continuous derivative of the basis functions is required. For these conditions, the interpolant is uniquely defined as

$$T(t) = \begin{cases} \frac{1}{2}\tilde{t}^2 + \tilde{t} + \frac{1}{2} & -1 < \tilde{t} \le 0, \\ -\tilde{t}^2 + \tilde{t} + \frac{1}{2} & 0 < \tilde{t} \le 1, \\ \frac{1}{2}\tilde{t}^2 - 2\tilde{t} + 2 & 1 < \tilde{t} \le 2, \\ 0 & \text{else}. \end{cases} \tag{13}$$

This gives the temporal basis function based on quadratic splines (Ref. 2).

4.3 Optimal order accurate four-point interpolant

The quadratic Lagrange polynomials are the only three-point interpolants that approximate functions third order accurate. Improvement in accuracy can thus only be achieved by increasing the number of support points. Four-point interpolants are represented by

$$T(t) = \begin{cases} F_0(t), & -\Delta t < t \leq 0, \\ F_1(t), & 0 < t \leq \Delta t, \\ F_2(t), & \Delta t < t \leq 2\Delta t, \\ F_3(t), & 2\Delta t < t \leq 3\Delta t, \\ 0, & \text{else.} \end{cases} \tag{14}$$

Imposing the conditions

$$F_0 + F_1 + F_2 + F_3 = 1, \tag{15}$$

$$F_1 + 2F_2 + 3F_3 = \frac{t_\ell - \tau_k}{\Delta t}, \tag{16}$$

$$F_1 + 4F_2 + 9F_3 = \left(\frac{t_\ell - \tau_k}{\Delta t}\right)^2, \tag{17}$$

$$F_1 + 8F_2 + 27F_3 = \left(\frac{t_\ell - \tau_k}{\Delta t}\right)^3, \tag{18}$$

results in interpolants that approximate functions fourth-order accurate. The first and second time derivative is then interpolated third and second order accurate, respectively, resulting in a second order accurate TDIE method. The unique interpolant satisfying these accuracy conditions is given by

$$T(t) = \begin{cases} \frac{1}{6}\tilde{t}^3 + \tilde{t}^2 + \frac{11}{6}\tilde{t} + 1, & -1 < \tilde{t} \leq 0, \\ -\frac{1}{2}\tilde{t}^3 - \tilde{t}^2 + \frac{1}{2}\tilde{t} + 1, & 0 < \tilde{t} \leq 1, \\ \frac{1}{2}\tilde{t}^3 - \tilde{t}^2 - \frac{1}{2}\tilde{t} + 1, & 1 < \tilde{t} \leq 2, \\ -\frac{1}{6}\tilde{t}^3 + \tilde{t}^2 - \frac{11}{6}\tilde{t} + 1, & 2 < \tilde{t} \leq 3, \\ 0, & \text{else,} \end{cases} \tag{19}$$

which is the cubic Lagrange interpolant (Ref. 2).

4.4 Smooth and accurate four-point interpolant

Optimal accuracy is uniquely obtained by cubic Lagrange basis functions, which are continuous but have a discontinuous time derivative in the discrete time levels. To obtain smooth interpolants, the accuracy conditions have to be softened. However, the same global accuracy can be achieved by alleviating the accuracy conditions on the interpolant and its first derivative only.

Continuity of the derivative can be imposed additionally. The unique solution is defined by the cubic spline interpolant, given by

$$T(t) = \begin{cases} \frac{1}{6}\tilde{t}^3 + \tilde{t}^2 + \frac{3}{2}\tilde{t} + \frac{2}{3}, & -1 < \tilde{t} \leq 0, \\ -\frac{1}{2}\tilde{t}^3 - \tilde{t}^2 + \frac{3}{2}\tilde{t} + \frac{2}{3}, & 0 < \tilde{t} \leq 1, \\ \frac{1}{2}\tilde{t}^3 - \tilde{t}^2 - \frac{3}{2}\tilde{t} + \frac{8}{3}, & 1 < \tilde{t} \leq 2, \\ -\frac{1}{6}\tilde{t}^3 + \tilde{t}^2 - \frac{3}{2}\tilde{t}, & 2 < \tilde{t} \leq 3, \\ 0, & \text{else.} \end{cases} \tag{20}$$

In (Ref. 2) also a temporal basis function based on cubic splines has been used. Although both named cubic spline interpolators, the definition of the basis function in (Ref. 2) differs from definition (20). The novel temporal basis function based on the cubic spline interpolator (20) has the special property of being both smooth and second order accurate. Concluding, cubic spline basis functions have the same accuracy as cubic Lagrange basis functions, and have the added advantage of a continuous derivative. In Figure 1 the considered temporal basis functions are shown and their characteristics are listed in Table 1.

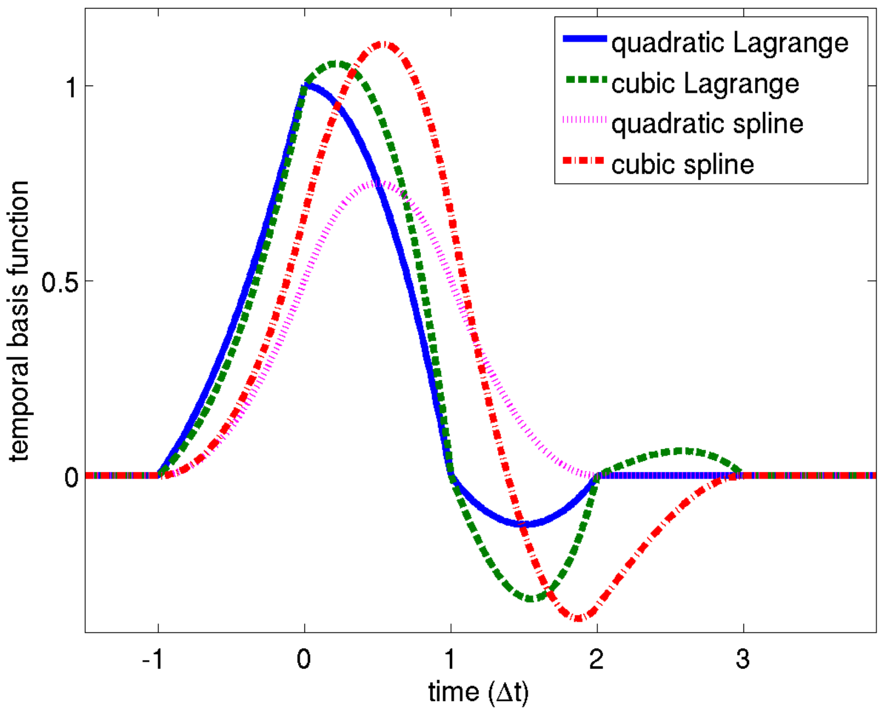

Fig. 1 Temporal basis functions.

basis function	acc. T	acc. T'	acc. T''	total acc.	smoothness
quadratic Lagrange	$\mathcal{O}(\Delta t^3)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$	$\mathcal{C}^0(\mathbb{R})$
quadratic spline	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$	$\mathcal{C}^1(\mathbb{R})$
cubic Lagrange	$\mathcal{O}(\Delta t^4)$	$\mathcal{O}(\Delta t^3)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{C}^0(\mathbb{R})$
cubic spline	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{C}^1(\mathbb{R})$

Table 1 Accuracy for the three temporal terms, the total accuracy in time of the CFIE and the smoothness of the various temporal basis functions.

5 Numerical results

The TDIE method with the temporal basis functions described in previous sections and shown in Figure 1 is applied to a PEC cube with edges of 1 m. A Gaussian plane wave is used as incident field, that is, $\mathbf{E}^i(\mathbf{r},t) = 120\pi\mathbf{p}\frac{4}{\sqrt{\pi}T}e^{-(4(c(t-t_0)-\mathbf{r}\cdot\mathbf{k})/T)^2}$. The parameters are given by: polarization $\mathbf{p} = \hat{\mathbf{x}}$; propagation $\mathbf{k} = -\hat{\mathbf{z}}$; width $T = 6$ lm; and delay $t_0 = 4$ lm. The CFIE-0.5 is used with an implicit time step size of 0.71 lm.

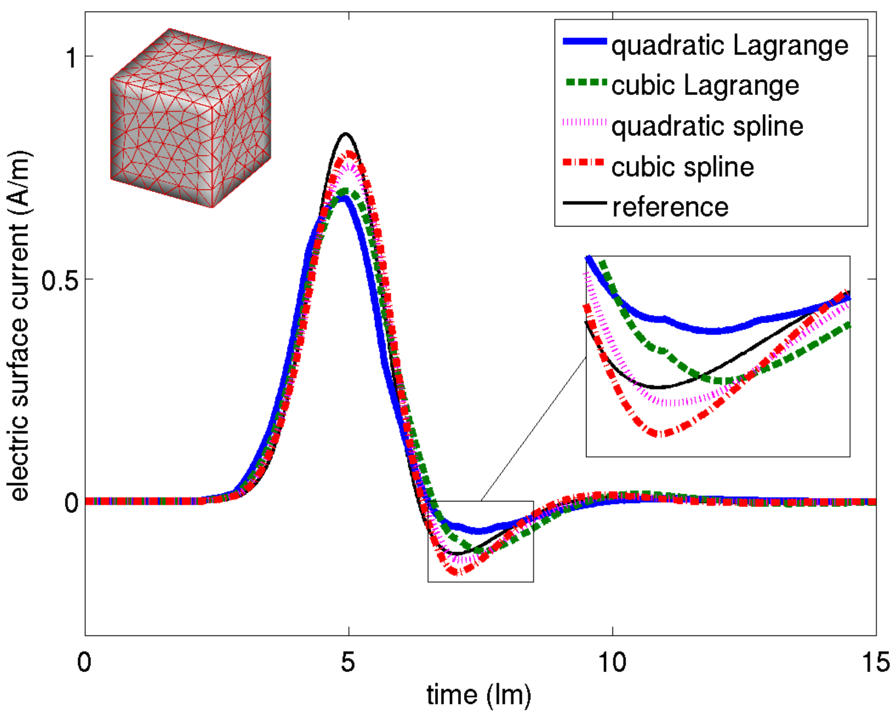

Fig. 2 Electric surface current at the top face of a cube.

In Figure 2 the induced electric surface currents are shown. The reference solution is obtained with a small time step size of $0.012\,\mathrm{lm}$. The surface currents calculated with the TDIE method using shifted Lagrange basis functions show discontinuities in the gradient. These nonsmooth solutions in discrete time levels correspond with the piecewise smooth character of the shifted Lagrange basis function. On the contrary, the surface currents from the spline basis functions are smooth, as expected from the its construction.

5.1 Richardsons method

Richardsons extrapolation algorithm can be used to approximate the accuracy of numerical methods. Considering the same test problem with time step sizes of 0.012, 0.023, 0.047, 0.093, and $0.186\,\mathrm{lm}$, the order of accuracy can be predicted three times. The accuracy according to Richardsons method, shown in Table 2, is very close to the expected order of one for the quadratic and two for the cubic functions, confirming the equal global order of accuracy of spline and Lagrange interpolants.

basis function	$0.012\,\mathrm{lm}$	$0.023\,\mathrm{lm}$	$0.047\,\mathrm{lm}$
quadratic Lagrange	1.096	1.099	1.185
quadratic spline	0.981	0.963	0.922
cubic Lagrange	2.064	2.086	2.344
cubic spline	1.970	1.929	1.834

Table 2 Order of accuracy, with the smallest time step size listed.

6 Conclusions

In this paper, a thorough analysis of the accuracy of the temporal basis functions used in the TDIE method is presented. For a causal interpolator with a fixed bounded support, only the Lagrange polynomials give an optimal order of accuracy. The shifted Lagrange basis functions are piecewise smooth. Global smoothness can only be obtained when alleviating the accuracy conditions. However, by alleviating the accuracy condition cleverly, the total accuracy of the TDIE method can remain the same while contuinity of the derivative is required additionally. This results in temporal basis functions based on splines. For example, the novel cubic spline basis function has a continuous derivative and is second order accurate, which is the same global accuracy as the cubic Lagrange basis functions. Numerical results confirm the expected accuracy and smoothness of the TDIE method.

Appendix A Derivation of accuracy conditions

Temporal basis functions in the MOT scheme are denoted by $T_j(t)$ and formulated as $T_j(t) = T(t - j\Delta T)$. The definition of T will then give a representation of all temporal basis functions. The class of causal basis functions with a small support can be written as

$$T(t) = \begin{cases} F_0(t), & -\Delta t < t \leq 0, \\ F_1(t), & 0 < t \leq \Delta t, \\ F_2(t), & \Delta t < t \leq 2\Delta t, \\ 0, & \text{else}, \end{cases} \tag{21}$$

for F_0, F_1 and F_2 denoting quadratic polynomials and Δt the time step size. Suppose the model equations are evaluated in an arbitrary retarded time level τ_k, one can find a discrete time level $t_\ell = \ell\Delta t$ such that $t_{\ell-1} < \tau_k \leq t_\ell$. With σ defined as $\sigma = t_\ell - \tau_k$, the analysis can be restricted to an arbitrary $\sigma \in [0, \Delta t)$.

In the TDIE method one obtains an approximation of the unknown electric surface current. In the present analysis, the unknown function is denoted by $u(t)$. The MOT scheme provides approximations u_n of the unknown at discrete time levels t_n. Since only the interpolation error will be analyzed, one can assume this to be the unknown at this time level, so $\tilde{u}_n = u(t_n)$. The interpolation procedure results in an function $\hat{u}(t)$ that approximates $u(t)$. In an arbitrary retarded time level, this is given by

$$\hat{u}(\tau_k) = \sum_{n=0}^{k} T_n(\tau_k)\tilde{u}_n, \tag{22}$$

and similar for its time derivatives, i.e.,

$$\hat{u}'(\tau_k) = \sum_{n=0}^{k} T_n'(\tau_k)\tilde{u}_n \tag{23}$$

and

$$\hat{u}''(\tau_k) = \sum_{n=0}^{k} T_n''(\tau_k)\tilde{u}_n. \tag{24}$$

For quadratic polynomial basis functions (21) the interpolation rule gives

$$\hat{u}(\tau_k) = F_0(-\sigma)\tilde{u}_\ell + F_1(\Delta t - \sigma)\tilde{u}_{\ell-1} + F_2(2\Delta t - \sigma)\tilde{u}_{\ell-2}, \tag{25}$$

$$\hat{u}'(\tau_k) = F_0'(-\sigma)\tilde{u}_\ell + F_1'(\Delta t - \sigma)\tilde{u}_{\ell-1} + F_2'(2\Delta t - \sigma)\tilde{u}_{\ell-2}, \tag{26}$$

$$\hat{u}''(\tau_k) = F_0''(-\sigma)\tilde{u}_\ell + F_1''(\Delta t - \sigma)\tilde{u}_{\ell-1} + F_2''(2\Delta t - \sigma)\tilde{u}_{\ell-2}. \tag{27}$$

The interpolation error of $\hat{u}$, $\hat{u}'$, and $\hat{u}''$ at an arbitrary retarded time level is given by $|\hat{u}(\tau_k)-u_\sigma|$, $|\hat{u}'(\tau_k)-u'_\sigma|$, and $|\hat{u}''(\tau_k)-u''_\sigma|$, respectively, for $u_\sigma = u(\tau_k)$, $u'_\sigma = u'(\tau_k)$, and $u''_\sigma = u''(\tau_k)$. To obtain the order of accuracy, the functions in the nodal points are written as a Taylor series in the unknown functions in the retarded time level. That is,

$$\tilde{u}_\ell = u_\ell = u_\sigma + \sigma u'_\sigma + \frac{\sigma^2}{2}u''_\sigma + \mathcal{O}(\sigma^3). \tag{28}$$

$$\tilde{u}_{\ell-1} = u_{\ell-1} = u_\sigma + (\sigma-\Delta t)u'_\sigma + \frac{(\sigma-\Delta t)^2}{2}u''_\sigma + \mathcal{O}(\sigma^3), \tag{29}$$

$$\tilde{u}_{\ell-2} = u_{\ell-2} = u_\sigma + (\sigma-2\Delta t)u'_\sigma + \frac{(\sigma-2\Delta t)^2}{2}u''_\sigma + \mathcal{O}(\sigma^3), \tag{30}$$

Substitution of the Taylor series (28)-(30) into the approximations (25)-(27) and using $\mathcal{O}(\sigma) = \mathcal{O}(\Delta t)$ results in

$$\hat{u}(\tau_k) = \mathcal{A}_{11}u_\sigma + \mathcal{A}_{12}u'_\sigma + \mathcal{A}_{13}u''_\sigma + \mathcal{O}(\Delta t^3), \tag{31}$$

$$\hat{u}'(\tau_k) = \mathcal{A}_{21}u_\sigma + \mathcal{A}_{22}u'_\sigma + \mathcal{A}_{23}u''_\sigma + \mathcal{O}(\Delta t^2), \tag{32}$$

$$\hat{u}''(\tau_k) = \mathcal{A}_{31}u_\sigma + \mathcal{A}_{32}u'_\sigma + \mathcal{A}_{33}u''_\sigma + \mathcal{O}(\Delta t), \tag{33}$$

with $\mathcal{A}_{ij}$ given by

$$\mathcal{A}_{11} = F_{0,\sigma} + F_{1,\sigma} + F_{2,\sigma}, \tag{34}$$

$$\mathcal{A}_{12} = \sigma F_{0,\sigma} + (\sigma-\Delta t)F_{1,\sigma} + (\sigma-2\Delta t)F_{2,\sigma}, \tag{35}$$

$$\mathcal{A}_{13} = \frac{1}{2}\sigma^2 F_{0,\sigma} + \frac{(\sigma-\Delta t)^2}{2}F_{1,\sigma} + \frac{(\sigma-2\Delta t)^2}{2}F_{2,\sigma}, \tag{36}$$

$$\mathcal{A}_{21} = F'_{0,\sigma} + F'_{1,\sigma} + F'_{2,\sigma}, \tag{37}$$

$$\mathcal{A}_{22} = \sigma F'_{0,\sigma} + (\sigma-\Delta t)F'_{1,\sigma} + (\sigma-2\Delta t)F'_{2,\sigma}, \tag{38}$$

$$\mathcal{A}_{23} = \frac{1}{2}\sigma^2 F'_{0,\sigma} + \frac{(\sigma-\Delta t)^2}{2}F'_{1,\sigma} + \frac{(\sigma-2\Delta t)^2}{2}F'_{2,\sigma}, \tag{39}$$

$$\mathcal{A}_{31} = F''_{0,\sigma} + F''_{1,\sigma} + F''_{2,\sigma}, \tag{40}$$

$$\mathcal{A}_{32} = \sigma F''_{0,\sigma} + (\sigma-\Delta t)F''_{1,\sigma} + (\sigma-2\Delta t)F''_{2,\sigma}, \tag{41}$$

$$\mathcal{A}_{33} = \frac{1}{2}\sigma^2 F''_{0,\sigma} + \frac{(\sigma-\Delta t)^2}{2}F''_{1,\sigma} + \frac{(\sigma-2\Delta t)^2}{2}F''_{2,\sigma}, \tag{42}$$

with the abbreviations

$$F_{0,\sigma} = F_0(-\sigma), \tag{43}$$

$$F_{1,\sigma} = F_1(\Delta t - \sigma), \tag{44}$$

$$F_{2,\sigma} = F_2(2\Delta t - \sigma). \tag{45}$$

To obtain a first order accurate interpolation scheme, one needs $\hat{u}(\tau_k) = u_\sigma + \mathcal{O}(\Delta t)$, which is satisfied if $\mathcal{A}_{11} = 1$. So the accuracy condition for a first orde accurate interpolation of an unknown function value is given by

$$F_{0,\sigma} + F_{1,\sigma} + F_{2,\sigma} = 1. \tag{46}$$

A second order accurate interpolation is obtained if $\mathcal{A}_{11} = 1$ and $\mathcal{A}_{12} = 0$, which is equivalent to conditions (46) and

$$F_{1,\sigma} + 2F_{2,\sigma} = \frac{\sigma}{\Delta t}. \tag{47}$$

A third order accurate interpolation is obtained if $\mathcal{A}_{11} = 1$, $\mathcal{A}_{12} = 0$ and $\mathcal{A}_{13} = 0$, which is equivalent to conditions (46), (47) and

$$F_{1,\sigma} + 4F_{2,\sigma} = \big(\frac{\sigma}{\Delta t}\big)^2. \tag{48}$$

Notice that the three conditions (46), (47) and (48) imply $\mathcal{A}_{21} = 0$, $\mathcal{A}_{31} = 0$, $\mathcal{A}_{22} = 1$, $\mathcal{A}_{32} = 0$, $\mathcal{A}_{23} = 0$, $\mathcal{A}_{33} = 1$ for all $\sigma \in [0, \Delta t)$.

Concluding, conditions (46), (47) and (48) are sufficient to obtain optimal accuracy, i.e.,

$$|\hat{u}(\tau_k) - u_\sigma| = \mathcal{O}(\Delta t^3), \tag{49}$$

$$|\hat{u}'(\tau_k) - u'_\sigma| = \mathcal{O}(\Delta t^2), \tag{50}$$

$$|\hat{u}''(\tau_k) - u''_\sigma| = \mathcal{O}(\Delta t). \tag{51}$$

To obtain first order accuracy for all three terms, it is sufficient to require $\mathcal{A}_{11} = 1$, $\mathcal{A}_{22} = 1$, and $\mathcal{A}_{33} = 1$, which is equivalent with

$$F_{0,\sigma} + F_{1,\sigma} + F_{2,\sigma} = 1, \tag{52}$$

$$F'_{1,\sigma} + 2F'_{2,\sigma} = -\frac{1}{\Delta t}, \tag{53}$$

$$F''_{1,\sigma} + 4F''_{2,\sigma} = \frac{2}{\Delta t^2}, \tag{54}$$

for all $\sigma \in [0, \Delta t)$. Notice that conditions (52) and (53) imply $\mathcal{A}_{21} = 0$, $\mathcal{A}_{31} = 0$, and $\mathcal{A}_{32} = 0$, and thus first order accuracy is obtained.

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