Accurate Hessian computation using smooth finite elements and flux preserving meshes

Solving the shallow water equations in estuaries

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Background

An estuary is defined as the area where a river meets the sea. Examples in the Netherlands are the Scheldt and Ems estuary. Estuaries often have a great natural and economical value (Dijkstra, 2019). To preserve this value, it is essential to accurately simulate and understand the water motion in these systems. An example of an idealised Scheldt geometry and bathymetry is shown in Figure 1.

One of the best methods to increase the understanding of the estuarine dynamics, is using explanatory, or idealised, models (Murray, 2003). These models allow for, e.g., extensive parameter sensitivity studies. These models often make use of a perturbation method in a small parameter. The equations are ordered in this small parameter. The first-order equations are forced by the leading-order equations. For the computation of this forcing, the solution itself, its first-order derivatives and its second-order derivatives are required.

Most numerical techniques have no problem resolving the leading-order water motion, its gradient becomes more difficult and the Hessian is especially difficult to obtain accurately. That is why most idealised models have been restricted to highly simplified geometries. However, this restriction is purely numerical and resolving it is a crucial step in extending explanatory models to more complex geometries.

Current research efforts have been directed towards resolving these issues using more advanced numerical techniques. For example, the current approach consists of high-order, curved finite elements with adaptive h-refinements (see, e.g., Karniadakis and Sherwin, 1999). These and other classical elements are typically C^0-continuous. Meaning that the gradient and Hessian can be discontinuous at the cell boundaries. However, it is expected that the gradient and Hessian are generally smooth for the geometries considered. It is, therefore, interesting to investigate if higher-order inter-element continuity conditions yield faster converging gradients and Hessians. Another interesting method that could increase the accuracy of the solution is found by following the physics of the problem. By generating meshes that preserve the physical fluxes, it is anticipated that the transport between cells is captured more accurately. Which, in turn, could lead to faster convergence.

Problem description

The three-dimensional shallow water equations can be reduced using a scaling analysis, perturbation method and harmonic decomposition to a system of forced linear equations at each order of the small parameter. The vertical dimension can be solved analytically and the system of linear equations simplify to a single two-dimensional Helmholtz equation (Kumar, 2018). The leading-order complex Helmholtz equation solving for the complex free surface Z reads

$$\nabla \cdot [D(0)\nabla Z] + i\omega Z = 0.$$

Here, $D(0)$ is a complex diffusion tensor containing the vertical information, i the imaginary unit and ω the frequency of the semi-diurnal tide.

The goal of this master project is to investigate how we can better approximate the gradients and Hessians of the free surface using numerical methods. In particular, the study will focus on the utilization of smooth finite elements, called splines (Hughes *et al.*, 2005), that allow for new types of adaptive refinements in both

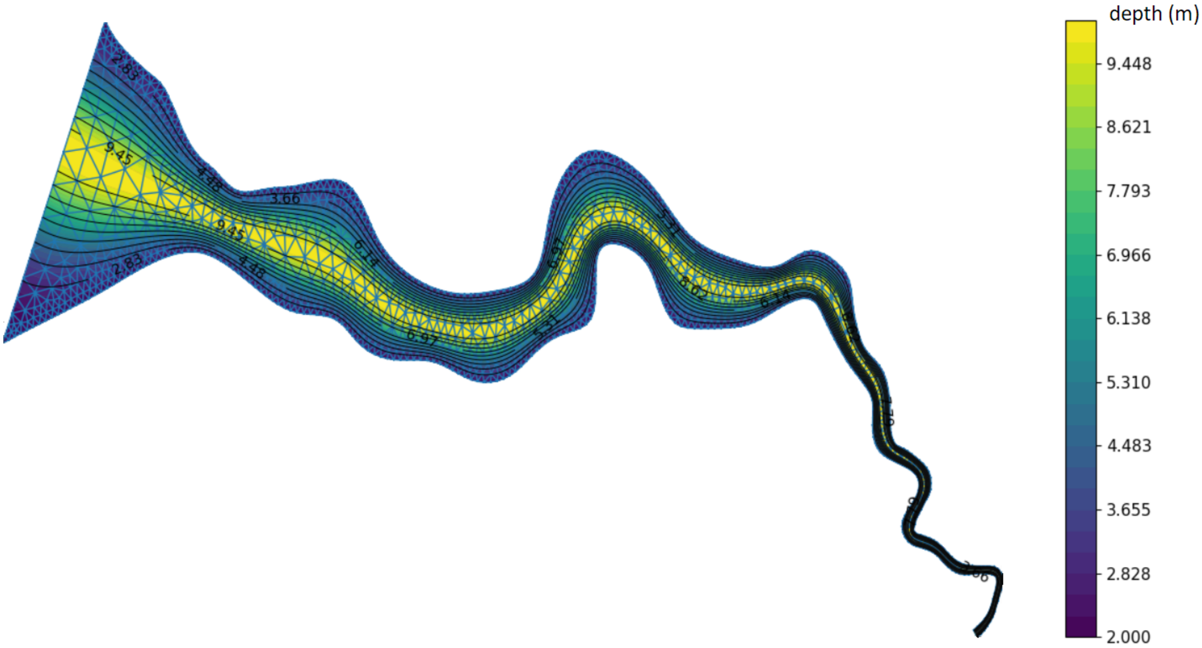

Figure 1: Example of an idealised geometry and bathymetry of the Scheldt estuary.

the polynomial degree and the mesh size. They have also been shown to be optimal for solving a wide class of problems. In this study, splines will be utilized for two purposes: for building a geometric model for an estuary (see Figure 1), as well as for solving the Helmholtz equation on that geometric model.

The master student is expected to be interested in learning about and implementing these finite element techniques for the above purpose. The implementation does not need to start from scratch, it can be performed on top of several libraries that already contain some of the required finite element functionality (e.g., GeoPDEs (De Falco *et al.*, 2011), G+Smo (Bressan and Mokriš, 2017), NGSolve (Schöberl, 2014)). The library and the specifics of the project can be tailored to suit the student's background; this will be decided based on the literature review and in discussion with the supervisors.

Time schedule: The thesis project comprises of two parts, a literature review phase (~3 months) followed by a research phase. Before beginning the research phase, the student will present the findings of their literature review (state of the art, research questions to be investigated in the project, an outline of the research plan, etc.).

Contact

This project is supervised by Dr. Deepesh Toshniwal and Ir. Marco Rozendaal. If you are interested please send an email to either D.Toshniwal@tudelft.nl or M.P.Rozendaal@tudelft.nl.

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