POD-deflation method for highly heterogeneous porous media

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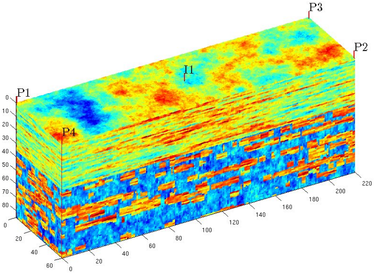
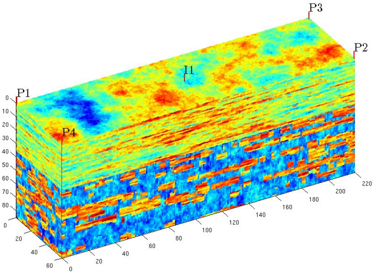
Single-phase flow, grid size 60 x 220 x 85 grid cells, contrast in permeability 10^7.



SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells, contrast in permeability 10^7.



Method	Number of iterations
ICCG	1029

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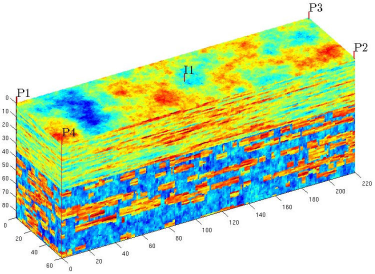

Method	Number of iterations
ICCG	1029
DICCG	1

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Problem Definition, Reservoir Simulation

Flow through a porous medium

Darcy's law + mass balance equation. [1]

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} - \nabla\cdot\left(\rho_\alpha\lambda_\alpha(\nabla p_\alpha - \rho_\alpha g\nabla z)\right) = q_\alpha\rho_\alpha.$$

- α Fluid Phase (oil (o) / water (w))
- S_α Phase Saturation
- ϕ Rock Porosity
- ρ_α Fluid Density
- p_α Phase Pressure
- K Rock Permeability
- $k_{r\alpha}$ Relative Permeabilities
- $\lambda_\alpha(S_\alpha) = \frac{Kk_{r\alpha}(S_\alpha)}{\mu_\alpha}$ Phase Mobilities
- μ_α Fluid viscosity
- g Gravity
- z Reservoir Depth
- q Sources

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Single phase incompressible

$$\mathcal{T} \mathbf{p}_w = \mathbf{q}$$

$$\mathcal{T}_{i-\frac{1}{2},j,l} = \frac{\Delta y}{\Delta x} h \lambda_{i-\frac{1}{2},j,l}$$

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Single phase compressible

$$\mathbf{J}(\mathbf{p}_w^k)\delta\mathbf{p}_w^{k+1} = -\mathbf{F}(\mathbf{p}_w^k;\mathbf{p}_w^n)$$

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Darcy's law + mass balance equation. [1]

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Single phase compressible

$$\mathbf{J}(\mathbf{p}_w^k)\delta\mathbf{p}_w^{k+1} = -\mathbf{F}(\mathbf{p}_w^k;\mathbf{p}_w^n)$$

Two phases Incompressible

$$-\nabla\cdot(\mathbf{f}_w(\mathbf{S}_w)\lambda_T\nabla\mathbf{p}_o) = \mathbf{q}$$

MRST (Lie 2013) [2]

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Conjugate Gradient Method (CG)

Single phase		Two phases
Incompressible	Compressible	Incompressible
$\mathcal{T}\mathbf{p}_w = \mathbf{q}$	$\mathbf{J}(\mathbf{p}_w^k)\delta\mathbf{p}_w^{k+1} = -\mathbf{F}(\mathbf{p}_w^k;\mathbf{p}_w^n)$	$-\nabla\cdot(\mathbf{f}_w(\mathbf{S}_w)\lambda_T\nabla\mathbf{p}_o) = \mathbf{q}$

Successive approximations to obtain a more accurate solution $\mathbf{x}$ [3]

$$\mathcal{A}\mathbf{x} = \mathbf{b},$$

$$\mathbf{x}^0, \qquad \text{initial guess}$$

$$\vdots$$

$$\mathbf{x}^k = \mathbf{x}^{k-1} + \mathcal{M}^{-1}\mathbf{r}^{k-1}, \qquad \mathbf{r}^k = \mathbf{b} - \mathcal{A}\mathbf{x}^{k-1}.$$

$$min_{\mathbf{x}^k \in \mathcal{K}_k(\mathcal{A},\mathbf{r}^0)} ||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}}, \qquad ||\mathbf{x}||_{\mathcal{A}} = \sqrt{\mathbf{x}^T\mathcal{A}\mathbf{x}}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{A})} + 1}\right)^{k+1}.$$

PCG (ICCG)

Preconditioning

Improve the spectrum of $\mathcal{A}$.

$$\mathcal{M}^{-1}\mathcal{A}\mathbf{x} = \mathcal{M}^{-1}\mathbf{b}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} + 1} \right)^{k+1},$$

$$\kappa(\mathcal{M}^{-1}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

IC(0)

Let $a_{i,j} \in \mathcal{A}$ and $\ell_{i,j} \in \mathcal{L}^*$, $\mathcal{L}^*$ the matrix from the Cholesky decomposition, such that $\ell_{i,j} = 0$ if $a_{i,j} = 0$. *Cholesky Decomposition*

If $\mathcal{A} \in \mathcal{R}^{n \times n}$ is *SPD*,

$$\mathcal{A} = \mathcal{L}\mathcal{L}^T$$

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DPCG

Deflation

$$\mathcal{P} = \mathcal{I} - \mathcal{A}\mathcal{Q}, \qquad \mathcal{P} \in \mathbb{R}^{n\times n}, \qquad \mathcal{Q} \in \mathbb{R}^{n\times n},$$
$$\mathcal{Q} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T, \qquad \mathcal{Z} \in \mathbb{R}^{n\times k}, \qquad \mathcal{E} \in \mathbb{R}^{k\times k},$$
$$\mathcal{E} = \mathcal{Z}^T\mathcal{A}\mathcal{Z} \text{ (Tang 2008, [4]).}$$

Convergence

Deflated system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} + 1} \right)^{k+1}.$$

Deflated and preconditioned system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} + 1} \right)^{k+1}.$$

$$\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A}) \leq \kappa_{eff}(\mathcal{P}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Deflation vectors

Recycling deflation (Clemens 2004, [5]).

$$\mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}],$$

x^i's are solutions of the system.
Multigrid and multilevel (Tang 2009, [6]).
The matrices $\mathcal{Z}$ and $\mathcal{Z}^T$ are the restriction and prolongation matrices of multigrid methods.
Subdomain deflation (Vuik 1999,[7]).

Proposal

- Use solution of the system with diverse well configurations as deflation vectors (Recycling deflation).
- Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).

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Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis for a given data set (Markovinović 2009 [8], Astrid 2011 [9])

$$\Phi = [\phi_1, \phi_2, ...\phi_l] \in \mathbb{R}^{n\times l}$$

ϕ_i, basis functions.

- Get the snapshots
$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, ...\mathbf{x}_m].$$
- l eigenvectors of $\mathcal{R}$ satisfying:
$\frac{\sum_{j=1}^{l}\lambda_j}{\sum_{j=1}^{m}\lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1.$

$$\mathcal{R} := \frac{1}{m}\mathcal{X}\mathcal{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i\mathbf{x}_i^T.$$

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Numerical experiments

Heterogeneous permeability (Neumann boundary conditions).

Single-phase flow

$nx = ny = 64$ grid cells.

Neumann boundary conditions.

15 snapshots, 4 linearly independent.

W1 = W2 = W3 = W4 = -1 bars,
W5 = +4 bars.

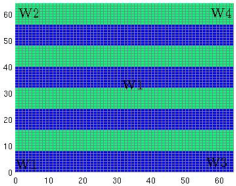


Figure: Heterogeneous permeability layers.

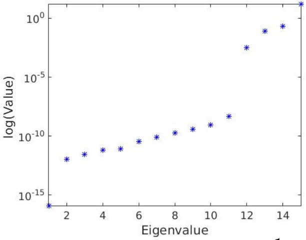


Figure: Eigenvalues of $\mathcal{R} = \frac{1}{m}\mathcal{X}\mathcal{X}^T$.

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
ICCG	90	131	65*	64*
$DICCG_{15}$	500*	500*	500*	500*
$DICCG_{4}$	1	1	1*	1*
$DICCG_{POD_4}$	1	1	1*	1*

Table: Iterations, tol 10^{-7}

Numerical experiments (Heterogeneous permeability)

Condition number of a SPD matrix.

$$\kappa_2(\mathcal{A}) = \frac{\lambda_{max}(\mathcal{A})}{\lambda_{min}(\mathcal{A})}$$

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$\kappa(A)$	2.6×10^3	2.4×10^5	2.4×10^7	2.4×10^9
$\kappa(M^{-1}A)$	206.7	8.3×10^3	8.3×10^5	8.3×10^7
$\kappa_{eff}(M^{-1}PA)$	83.27	6×10^3	1×10^6	6×10^7

Table: Condition number for various permeability contrasts between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$.

Relative error

$$e = \frac{||\mathbf{x} - \mathbf{x}^k||_2}{||\mathbf{x}||_2} \leq \kappa_2(A)\epsilon, \qquad \text{with } \mathbf{x} \text{ the true solution and } \mathbf{x}^k \text{ the approximation.}$$

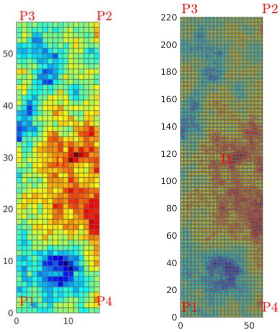
Taking $e = 10^{-7}$,

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$tol = \frac{e}{\kappa_2(M^{-1}A)} = \frac{10^{-7}}{\kappa_2(M^{-1}A)}$	5×10^{-9}	1×10^{-10}	1×10^{-12}	1×10^{-14}
$tol = \frac{e}{\kappa_{eff}(M^{-1}PA)} = \frac{10^{-7}}{\kappa_{eff}(M^{-1}PA)}$	1×10^{-8}	2×10^{-10}	1×10^{-12}	2×10^{-14}

Table: Tolerance needed for various permeability contrast between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$, for an error of $e = 10^{-7}$.

Numerical experiments (SPE 10)

SPE 10 model, 2nd layer



(a) 16 x 56 (b) 60 x 220

Figure: Permeability field, 16 x 56 and 60 x 220 grid cells (upscaled with MRST [2]).

Grid size	16 x 56	30 x 110	46 x 166	60 x 220
Contrast ($\times 10^7$)	1.04	2.52	2.6	2.8

Table: Contrast in permeability for different grid sizes ($\sigma_{max}/\sigma_{min}$).

Condition number	
$\kappa(A)$	2.2×10^6
$\kappa(M^{-1}A)$	377
$\kappa_{eff}(M^{-1}PA)$	82.7

Table: Table with the condition number of the SPE10 model, grid size of 16 x 56.

Numerical experiments(SPE 10)

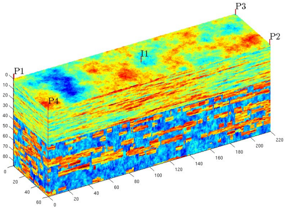
SPE 10 model, 2nd layer

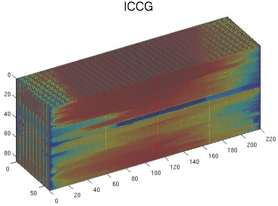
Tol (snapshots)	Method	16 x 56	30 x 110	46 x 166	60 x 220
	ICCG	34	73	126	159
10^{-1}	$DICCG_4$	33	72	125	158
10^{-3}	$DICCG_4$	18	38	123	151
10^{-5}	$DICCG_4$	11	21	27	55
10^{-7}	$DICCG_4$	1	1	1	1

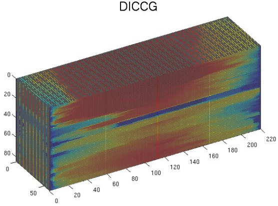
Table: Number of iterations for ICCG and DICCG, diverse tolerance for the snapshots, different grid sizes. $DICCG_4$ is computed with 4 deflation vectors.

Numerical experiments

SPE 10 model, 85 layers







Method	Number of iterations
ICCG	1029
DICCG	1

Table: Number of iterations for the SPE10 benchmark (85 layers) for the ICCG and DICCG methods, relative tolerance 10^{-7}.

Numerical experiments

Compressible SPE 10 problem

60*x*220*x*85 grid cells.

Neumann boundary conditions.

W1 = W2 = W3 = W4 = 100 bars, W5 = 600 bars.

Initial pressure 200 bars.

Contrast in permeability of $3x10^7$.

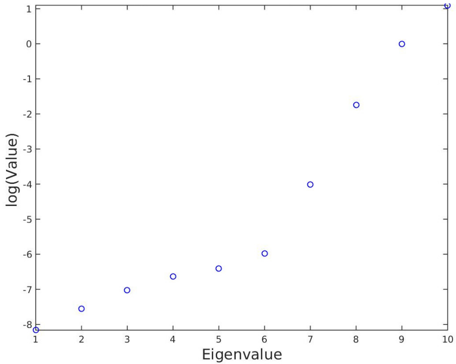


Figure: Eigenvalues of the data snapshot correlation matrix.

Numerical experiments

Compressible SPE 10 problem

1^{st} NR Iteration					
Total ICCG (only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG
10173	$DICCG_{10}$	1770	1134	2904	28
10173	$DICCG_{POD_4}$	1770	1554	3324	32

Table: Average number of linear iterations for the first NR iteration, full SPE 10 benchmark.

2^{nd} NR Iteration					
Total ICCG (only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG
10231	$DICCG_{10}$	1830	200	2030	20
10231	$DICCG_{POD_4}$	1830	200	2030	20

Table: Average number of linear iterations for the second NR iteration, full SPE 10 benchmark.

Numerical experiments

Layered problem, two phases, water injection through wells, 35 x 35 cells

Well	Water Sat	Oil Sat	Pressure
P_1	0	1	50 bars
P_2	0	1	50 bars
P_3	0	1	50 bars
P_4	0	1	50 bars
I	1	0	200 bars

Table: Wells properties.

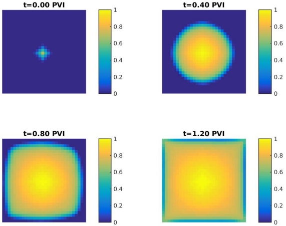


Figure: Pressure fields.

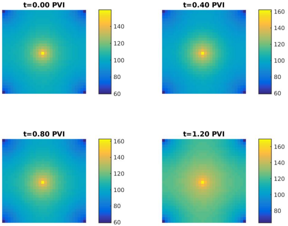


Figure: Water Saturation.

Numerical experiments

Layered problem, two phases, water injection through wells, 35 x 35 cells

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG + DICCG	% of total ICCG
10^1	24394	$DICCG_{POD_{10}}$	502	1869	2371	10
10^1	24394	$DICCG_{POD_5}$	502	2477	2979	12
10^2	27364	$DICCG_{POD_{10}}$	551	1906	2457	9
10^2	27364	$DICCG_{POD_5}$	551	2583	3134	11
10^3	27092	$DICCG_{POD_{10}}$	529	2033	2562	9
10^3	27092	$DICCG_{POD_5}$	529	2430	2959	11

Table: No capillary pressure included.

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG + DICCG	% of total ICCG
10^1	23810	$DICCG_{POD_{10}}$	502	2641	3143	13
10^1	23810	$DICCG_{POD_5}$	502	2683	3185	13
10^2	27629	$DICCG_{POD_{10}}$	551	2719	3270	12
10^2	27629	$DICCG_{POD_5}$	551	2793	3344	12
10^3	23962	$DICCG_{POD_{10}}$	517	2872	3389	14
10^3	23962	$DICCG_{POD_5}$	517	2744	3261	14

Table: Capillary pressure included.

Numerical experiments

Layered problem, two phases, water injection through the boundary, 24 x 24 x 24 cells.

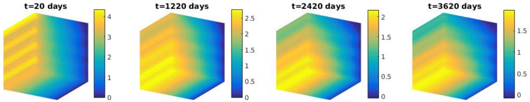
	Water	Oil
$S_{0,x\neq 0,L_x}$	0	1
$S_{x=0}$	1	0
$S_{x=L_x}$	0	1

Table: Saturations.

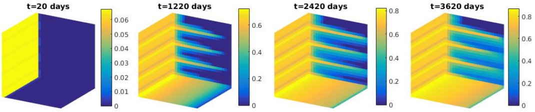
Property	Value	Units
$Q_{x=0}$	0.4	m^3/day
$P_{0,x\neq 0,L_x}$	100	*bars*
$P_{x=L_x}$	0	*bars*

Table: Control.

Pressure field



Water Saturation



Numerical experiments

Layered problem, two phases, water injection through the boundary, 24 x 24 x 24 cells.

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG +DICCG	% of total ICCG
10^1	19426	$DICCG_{POD_{10}}$	654	1969	2623	14
10^1	19426	$DICCG_{POD_5}$	654	2258	2912	15
10^2	22577	$DICCG_{POD_{10}}$	762	2340	3102	14
10^2	22577	$DICCG_{POD_5}$	762	2714	3476	15
10^3	21832	$DICCG_{POD_{10}}$	594	2086	2680	12
10^4	18483	$DICCG_{POD_{10}}$	529	1868	2397	13

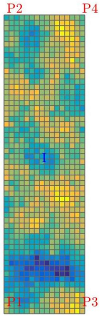
Table: Gravity included.

$\frac{\sigma_2}{\sigma_1}$	Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG +DICCG	% of total ICCG
10^1	17224	$DICCG_{POD_{10}}$	660	3431	4091	24
10^1	17224	$DICCG_{POD_5}$	660	3658	4318	25
10^2	20562	$DICCG_{POD_{10}}$	763	3468	4231	21
10^2	20562	$DICCG_{POD_5}$	763	3596	4359	21
10^3	18514	$DICCG_{POD_{10}}$	605	2894	3499	19
10^3	18514	$DICCG_{POD_5}$	605	2943	3548	19

Table: Capillary pressure ($C = 10 * (1 - S)$) and gravity included

Numerical experiments

SPE 10, first layer, 16 x 56 grid cells



Well	Water Sat	Oil Sat	Pressure
P_1	0	1	275 bars
P_2	0	1	275 bars
P_3	0	1	275 bars
P_4	0	1	275 bars
I	1	0	1100 bars

Table: Wells properties.

Total ICCG	Method	ICCG Snapshots	DICCG	Total ICCG + DICCG	% of total ICCG
11313	$DICCG_{POD_{10}}$	567	1615	2182	19
11313	$DICCG_{POD_5}$	567	2235	2802	25

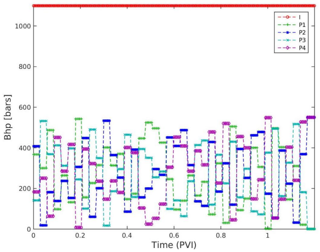
Table: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the SPE 10 problem, 16 x 56 grid cells (upscaled with MRST[2]).

Numerical experiments

SPE 10, first layer, 16 x 56 grid cells, training example

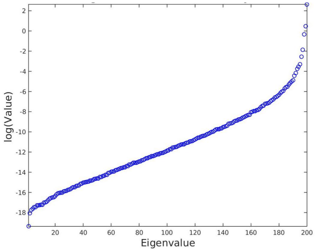
Well	Water Sat	Oil Sat	Pressure
P_1	0	1	$rand(0-275)$ bars
P_2	0	1	$rand(0-275)$ bars
P_3	0	1	$275 - P_1$ bars
P_4	0	1	$275 - P_2$ bars
I	1	0	1100 bars

Table: Wells properties.



Total	Method	DICCG	% of total
11362	$\text{DICCG}_{POD_{30}}$	1734	16
11362	$\text{DICCG}_{POD_{10}}$	2498	23

Table: Comparison between the ICCC and DICCG methods of the average number of linear iterations for the SPE 10 problem, 16 x 56 grid cells.



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Conclusions

- Solution is reached in 1 iteration for DICCG method for the single phase incompressible problem.
  TU Delft Report: [10].

- A reduction of $\approx 80\%$ of the ICCG number of iterations is achieved with DICCG method for the solution of the pressure equation, for the two phase layered and SPE 10 problems and for the full SPE 10 compressible problem ($\approx 10^6$ grid cells).
  JCAM paper: G.B. Diaz Cortes, C. Vuik and J.D. Jansen, 2017 [11]

- Number of iterations for the DICCG method does not depend on the contrast between the coefficients or the grid size.
  TU Delft Report: [12].

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References I

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Lemma 1. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix, and $\mathbf{x}$ is the solution of:

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Let $\mathbf{x}_i, \mathbf{b}_i \in \mathbb{R}^n$, $i = 1, ..., m$, be vectors linearly independent ($l.i.$) and

$$\mathbf{A}\mathbf{x}_i = \mathbf{b}_i.$$

The following equivalence holds

$$\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \qquad \Leftrightarrow \qquad \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i.$$

Proof $\Rightarrow$ $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \Rightarrow \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i.$

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{m} \mathbf{A} c_i \mathbf{x}_i = \mathbf{A}(c_1 \mathbf{x}_1 + ... + c_m \mathbf{x}_m) =$$

$$= \mathbf{A} c_1 \mathbf{x}_1 + ... + \mathbf{A} c_m \mathbf{x}_m = c_1 \mathbf{b}_1 + ... + c_m \mathbf{b}_m = \sum_{i=1}^{m} c_i \mathbf{b}_i = \mathbf{b}.$$

Lemma 2. If the the deflation matrix $\mathbf{Z}$ is constructed with a set of m vectors

$$\mathbf{Z} = \begin{bmatrix} \mathbf{x}_1 & \dots & \dots & \mathbf{x}_m \end{bmatrix},$$

such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ *l.i.*, then the solution of system (34) is obtained with one iteration of DCG.

$$\mathbf{x} = \mathbf{Q}\mathbf{b} + \mathbf{P}^T \hat{\mathbf{x}}.$$

$$\begin{aligned}
\mathbf{Q}\mathbf{b} &= \mathbf{Z}\mathbf{E}^{-1}\mathbf{Z}^T \left(\sum_{i=1}^{m} c_i \mathbf{b}_i \right) = \mathbf{Z}(\mathbf{Z}^T\mathbf{A}\mathbf{Z})^{-1}\mathbf{Z}^T \left(\sum_{i=1}^{m} c_i \mathbf{A}\mathbf{x}_i \right) \\
&= \mathbf{Z}(\mathbf{Z}^T\mathbf{A}\mathbf{Z})^{-1}\mathbf{Z}^T(\mathbf{A}\mathbf{Z}\mathbf{c}) = \mathbf{Z}\mathbf{c} \\
&= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4 + c_5\mathbf{x}_5 = \sum_{i=1}^{m} c_i \mathbf{x}_i = \mathbf{x}.
\end{aligned}$$

$$\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{P}\mathbf{b} = (\mathbf{I} - \mathbf{A}\mathbf{Q})\mathbf{b} = \mathbf{b} - \mathbf{A}\mathbf{Q}\mathbf{b} = \mathbf{b} - \mathbf{A}\mathbf{x} = 0$$

$$\mathbf{P}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}\mathbf{P}^T\hat{\mathbf{x}} = 0 \quad \Rightarrow \quad \mathbf{P}^T\hat{\mathbf{x}} = 0$$

$$\mathbf{x} = \mathbf{Q}\mathbf{b} + \mathbf{P}^T\hat{\mathbf{x}} = \mathbf{Q}\mathbf{b} = \mathbf{x}.$$