The Krylov accelerated SIMPLE(R) method for incompressible flow problems

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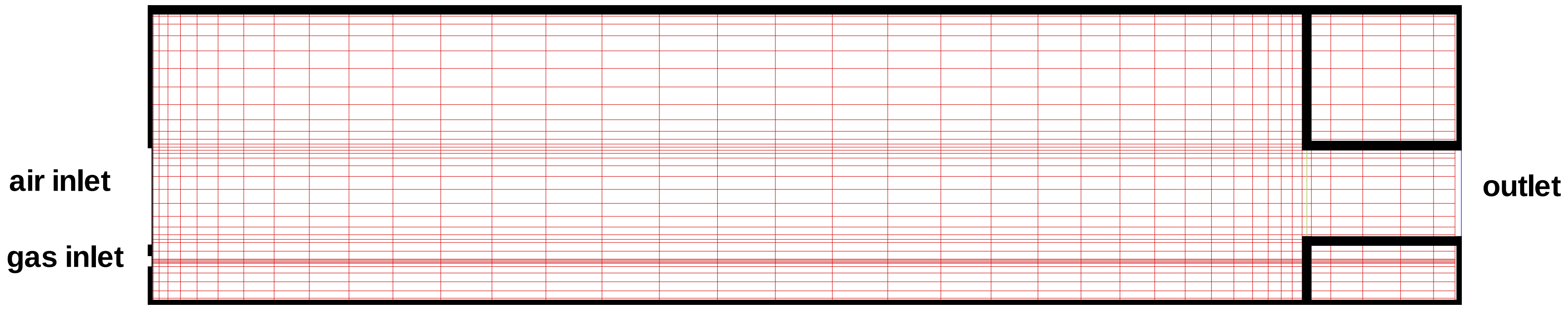
Gas-fired glass melting furnace Combustion process

The symmetry plane of the furnace Grid: $42 \times 37 \times 27$



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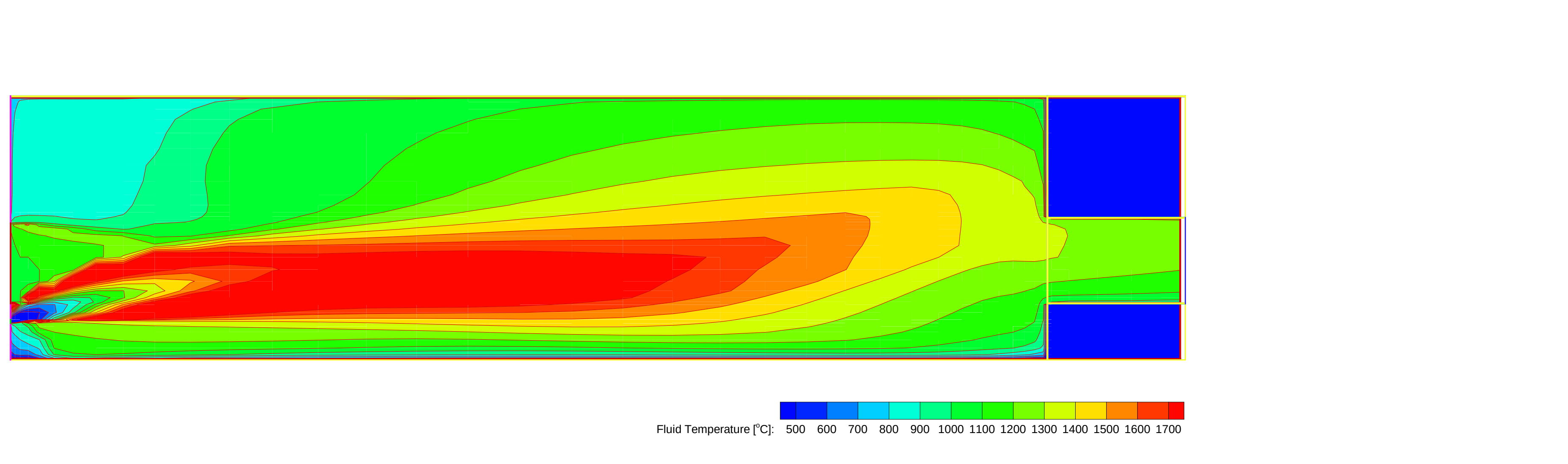
1. Introduction

Gas-fired glass melting furnace

Combustion process

The symmetry plane of the furnace Grid: $42 \times 37 \times 27$



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Mathematical model

3D incompressible Navier-Stokes

Turbulence ($k - \varepsilon$)

Combustion

Chemistry (one step global reaction)

Radiative heat transfer

NO_x postprocessor

Soot formation

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Results for the IFRF furnace

The IFRF furnace (Grid $24 \times 20 \times 16$)

method	$niter$	CPU time (hours)
SIMPLE	2047	4.8
SIMPLER	2415	6.9
GCR-SIMPLE	623	2.4
GCR-SIMPLER	578	2.0

Discretization

Incompressible Stokes equation

$$\begin{aligned} -\nu \Delta \mathbf{u} + \text{grad} p &= \mathbf{f}, \\ \text{div} \mathbf{u} &= 0. \end{aligned}$$

Finite volumes, staggered grid

$$\begin{pmatrix} \mathbf{Q}_1 & \mathbf{O} & \mathbf{O} & \mathbf{G}_1 \\ \mathbf{O} & \mathbf{Q}_2 & \mathbf{O} & \mathbf{G}_2 \\ \mathbf{O} & \mathbf{O} & \mathbf{Q}_3 & \mathbf{G}_3 \\ \mathbf{G}_1^T & \mathbf{G}_2^T & \mathbf{G}_3^T & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Solution of the linear system

$$\begin{pmatrix} \mathbf{Q} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Difficulties due to zero block

- Traditional iterative solvers fail
- SIMPLE(R) converges slowly — Patankar
- Krylov method and ILU preconditioner — Dahl, Wille, Segal, Vuik
- Multigrid acceleration — Gjesdal, Wesseling, Wittum
- Saddle point preconditioner — Elman, Silvester, Wathen

2. SIMPLE(R) methods

$\mathbf{D} = diag(\mathbf{Q})$ and $\mathbf{R} = -\mathbf{G}^T\mathbf{D}^{-1}\mathbf{G}$

SIMPLE algorithm

1. Choose an initial estimate p^*.
2. Solve $\mathbf{Q}u^* = b_1 - \mathbf{G}p^*$.
3. Solve $\mathbf{R}\delta p = b_2 - \mathbf{G}^T u^*$.
4. Compute $u = u^* - \mathbf{D}^{-1}\mathbf{G}\delta p$
   and $p := p^* + \delta p$.
5. If not converged take $p^* = p$ and go to 2.

Systems are solved by a TDMA solver, use of relaxation parameters

Patankar, Spalding, Wittum, Van Doormaal, Raithby, Ferziger, Peric

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Algebraic view of SIMPLE

Definitions

$$\mathbf{A} = \begin{pmatrix} \mathbf{Q} & \mathbf{G} \\ \mathbf{G}^T & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \mathbf{I} & -\mathbf{D}^{-1}\mathbf{G} \\ 0 & \mathbf{I} \end{pmatrix}$$

Problem

$$\mathbf{A}x = b$$

Right-preconditioned system

$$\mathbf{AB}y = b, x = \mathbf{B}y$$

Algebraic view of SIMPLE (continued)

$$\mathbf{AB} = \begin{pmatrix} \mathbf{Q} & \mathbf{G} - \mathbf{Q}\mathbf{D}^{-1}\mathbf{G} \\ \mathbf{G}^T & R \end{pmatrix}$$

Splitting method (Gauss-Seidel)

$$\mathbf{AB} = \mathbf{M} - \mathbf{N}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{Q} & 0 \\ \mathbf{G}^T & R \end{pmatrix}$$

SIMPLE method

$$x^{k+1} = x^k + \mathbf{B}\mathbf{M}^{-1}(b - \mathbf{A}x^k)$$

distributive iterative method Hackbusch, Wittum, Wesseling

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SIMPLER method

1. Solve $\mathbf{R}p^k = b_2 - \mathbf{G}^T\mathbf{D}^{-1}((\mathbf{D}-\mathbf{Q})u^k + b_1)$.
2. Solve $\mathbf{Q}u^* = b_1 - \mathbf{G}p^k$.
3. Solve $\mathbf{R}\delta p = b_2 - \mathbf{G}^T u^*$.
4. Compute $u^{k+1} = u^* - \mathbf{D}^{-1}\mathbf{G}\delta p$
   and $p^{k+1} := p^k + \delta p$.
5. If not converged go to 1.

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SIMPLER as distributive iterative method

Define $\mathbf{B}_L$ and $\mathbf{M}_L$ as follows:

$$\mathbf{B}_L = \begin{pmatrix} \mathbf{I} & 0 \\ -\mathbf{G}^T\mathbf{D}^{-1} & \mathbf{I} \end{pmatrix} \text{ and } \mathbf{M}_L = \begin{pmatrix} \mathbf{Q} & \mathbf{G} \\ 0 & \mathbf{R} \end{pmatrix}$$

The SIMPLER method is:

$$x^{k+1} = x^k + \mathbf{B}_R\mathbf{M}_R^{-1}\mathbf{B}_L^{-1}\mathbf{T}\mathbf{B}_R^{-1}\mathbf{M}_L^{-1}\mathbf{B}_L(b - \mathbf{A}x^k),$$

where $\mathbf{T}$ is the block diagonal part of $\mathbf{M}_L + \mathbf{M}_R - \mathbf{A}$.

Symmetric Block Gauss-Seidel

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3. GCR acceleration

LSQR	Paige and Saunders
GMRES	Saad and Schultz
CGS	Sonneveld
Bi-CGSTAB	Van der Vorst and Sonneveld

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LSQR	Paige and Saunders
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GCR	Eisenstat, Elman and Schultz
GMRESR	Van der Vorst and Vuik

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GCR-SIMPLE

$r^0 = b - \mathbf{A}x^0$

for k = 0, 1,...,$ngcr$

$$s^{k+1} = \mathbf{BM}_k^{-1} r^k$$
$$v^{k+1} = \mathbf{A}s^{k+1}$$

for i = 1, 2,...,k

$$v^{k+1} = v^{k+1} - (v^{k+1}, v^i)v^i, \qquad s^{k+1} = s^{k+1} - (v^{k+1}, v^i)s^i$$

end for

$$v^{k+1} = v^{k+1}/\|v^{k+1}\|_2, \qquad s^{k+1} = s^{k+1}/\|v^{k+1}\|_2$$
$$x^{k+1} = x^k + (r^k, v^{k+1})s^{k+1}$$
$$r^{k+1} = r^k - (r^k, v^{k+1})v^{k+1}$$

end for

Diagonal scaling

Dirichlet boundary conditions (velocity)

$$u_P = g_P$$

Add c_{max} to the main diagonal, add $c_{max} g_P$ to the right-hand side

GCR-SIMPLE: bad results

Diagonal scaling $\Rightarrow$ GCR-SIMPLE: good results

Navier-Stokes

$$\mathbf{Q}(u)u + \mathbf{G}p = b_1$$

for $k = 0, 1, 2 \ldots, niter$

solve $\mathbf{A}(x^k)x^{k+1} = b$ with GCR-SIMPLE(R)

end for

1. solve u and p with GCR-SIMPLE(R)
2. Solve the turbulent quantities, temperature and concentrations with TDMA
3. Solve for radiative heat transfer
4. Repeat this procedure until convergence

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4. Numerical experiments

Some properties of GCR-SIMPLE(R)

2D Navier-Stokes flow between two flat plates

Method	SIMPLER		GCR-SIMPLER	
	$niter$	CPU	$niter$	CPU
LINE TDMA	78	7.4	33	7
PLANE TDMA	139	16.9	33	9.9

Results using LINE TDMA and PLANE TDMA, Grid (40×20)

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Relaxation factors

rel. factor	SIMPLER		GCR-SIMPLER	
	$niter$	CPU	$niter$	CPU
1	no conv.		33	9.9
0.9	80	10.3	78	23.3
0.8	139	16.9	130	33.9
0.7	205	24.0	162	42.0
0.6	281	32.3	220	56.4

Results for various relaxation factors
Grid (40×20)

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$ngcr$ (restart value)

$ngcr$	Grid (40×20)		Grid (40×40)	
	$niter$	CPU	$niter$	CPU
2	43	10.3	96	39.9
3	33	9.9	67	35.6
4	30	10.4	59	37.4
6	21	9.8	38	33.4
8	17	9.9	31	35
14	11	10.6	14	27.5

Results of the GCR-SIMPLER method for various values of $ngcr$

Grid size dependence

Grid size	SIMPLER		GCR-SIMPLER	
	$niter$	CPU	$niter$	CPU
20×20	61	5.2	29	5.9
40×20	139	16.9	33	9.9
80×20	303	68.5	80	40.2

Results for various grid sizes

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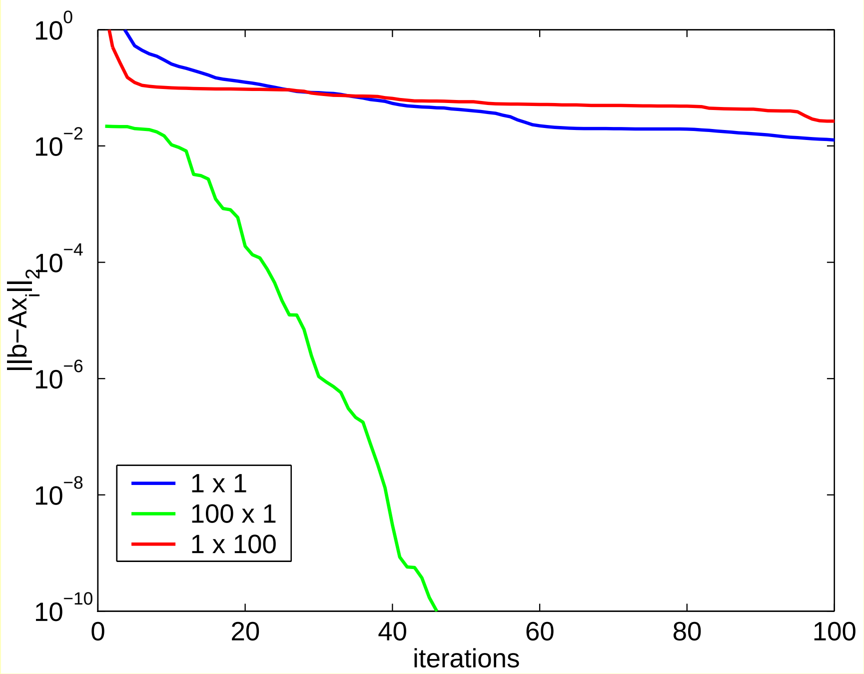
Results for various grid sizes

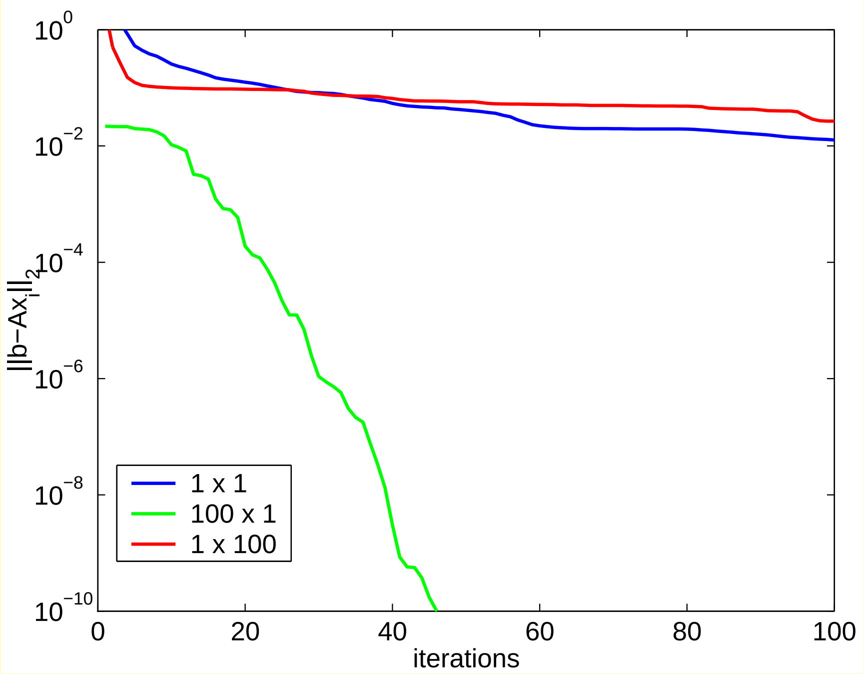
Some results from MATLAB experiments

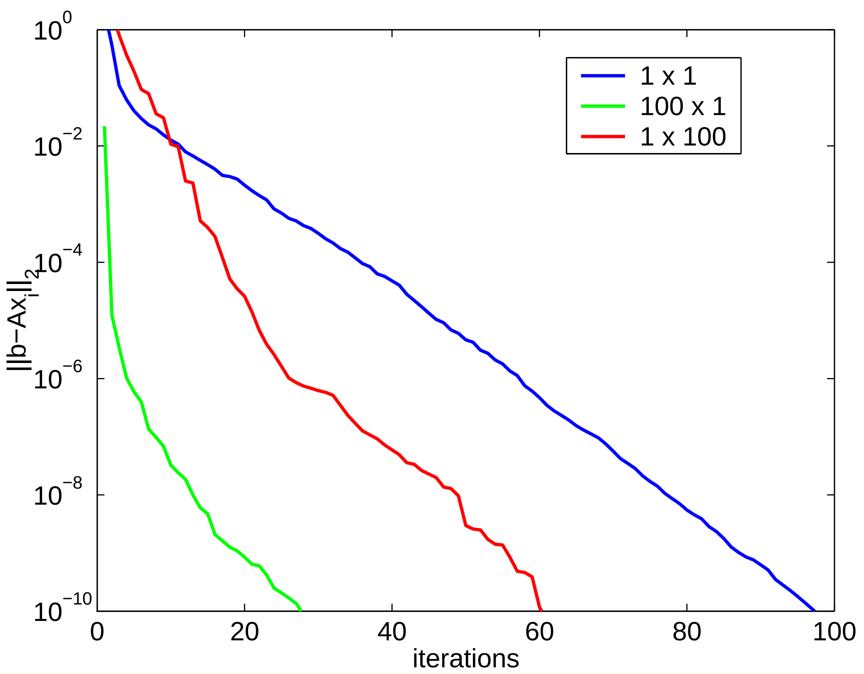
The convergence behavior for GCR-ILU



Some results from MATLAB experiments

The convergence behavior for GCR-ILU and GCR-SIMPLER





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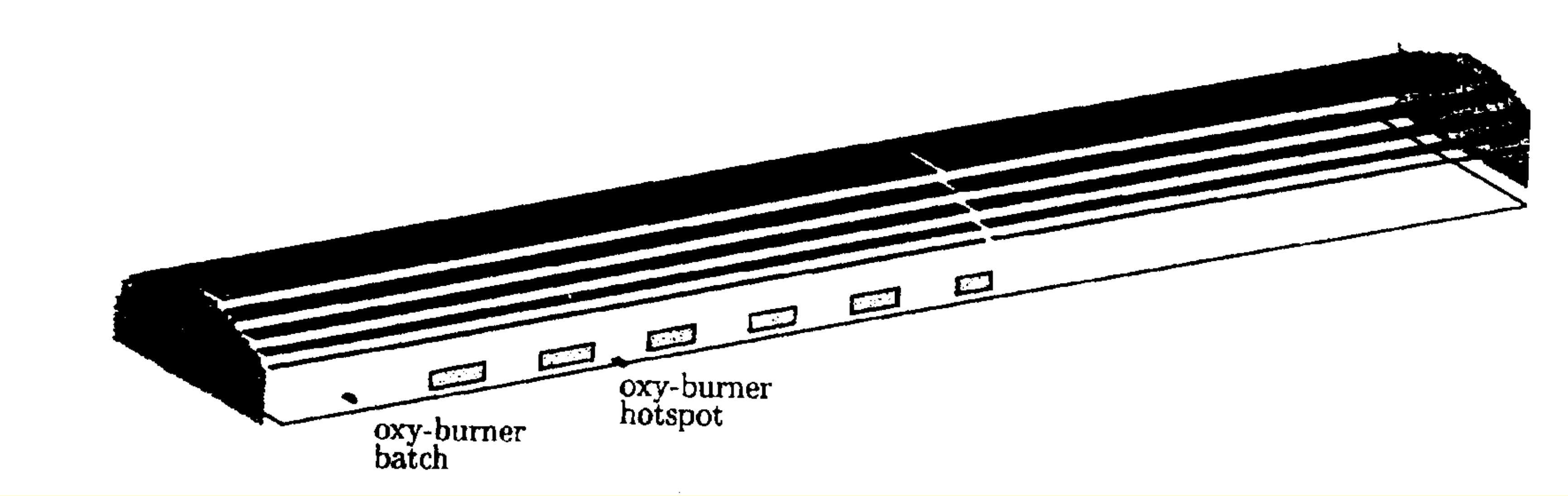
The Ford Nashville furnace

combustion chamber dimensions: $34.7 \times 10.1 \times 2.3\ m$



grid $130 \times 40 \times 40$ = 208000 points

GCR-SIMPLER: 3390 iteration, CPU time ≈ 3.3 days

SIMPLER: not converged after 7.5 days

Memory requirements

Memory requirements for various problems measured in Megabytes

problem	SIMPLER	GCR-SIMPLER ($ngcr = 3$)
Plates (120×120)	31	39
IFRF ($42 \times 37 \times 27$)	52	78
Ford ($130 \times 40 \times 40$)	202	333

5. Conclusions

- GCR-SIMPLER is an efficient and robust method to simulate incompressible flows (glass-melting furnaces)
- GCR-SIMPLER allows large relaxation factors
- The GCR acceleration can easily be added in an existing CFD code
- GCR-SIMPLER requires more memory

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Further information

C. Vuik, A. Saghir and G.P. Boerstoel
The Krylov accelerated SIMPLE(R) method for flow problems in industrial furnaces
International J. for Numer. Methods in Fluids, 33, pp. 1027-1040, 2000.

http://ta.twi.tudelft.nl/nw/users/vuik/pub.html

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