Spectral analysis of complex shifted-Laplace preconditioners for the Helmholtz equation

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1. Introduction

The Helmholtz problem is defined as follows

$$-\partial_{xx}u - \partial_{yy}u - z_1 k^2(x,y)u = f, \quad \text{in} \quad \Omega,$$
$$\text{Boundary conditions} \quad \text{on} \quad \Gamma = \partial\Omega,$$

where:

- $z_1 = \alpha_1 + \mathrm{i}\beta_1$ and $k(x,y)$ is the wavenumber
- for ”solid” boundaries: Dirichlet/Neumann
- for ”fictitious” boundaries: Sommerfeld $\frac{du}{dn} - \mathrm{i}ku = 0$
- Perfectly Matched Layer (PML)
- Absorbing Boundary Layer (ABL)

Discretization

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N \times N}, \;\; b, x \in \mathbb{C}^N,$$

Discretization

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Linear system

$$Ax = b, \ \ A \in \mathbb{C}^{N \times N}, \ \ b, x \in \mathbb{C}^N,$$

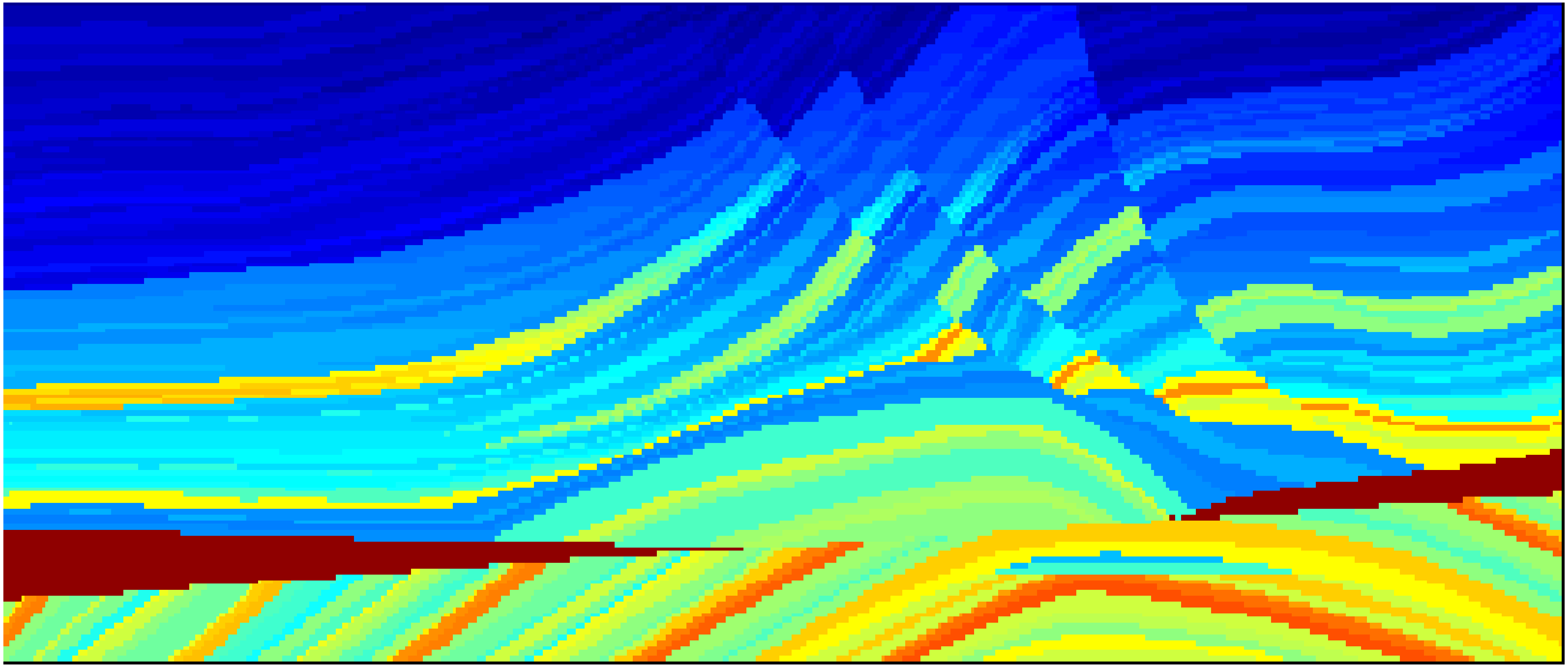
A is a sparse, highly indefinite matrix for practical values of k.
Special property $A = A^T$.

For high resolution a very fine grid is required: $30 - 60$ grid-points per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A$ is extremely large!

Characteristic properties of the problem

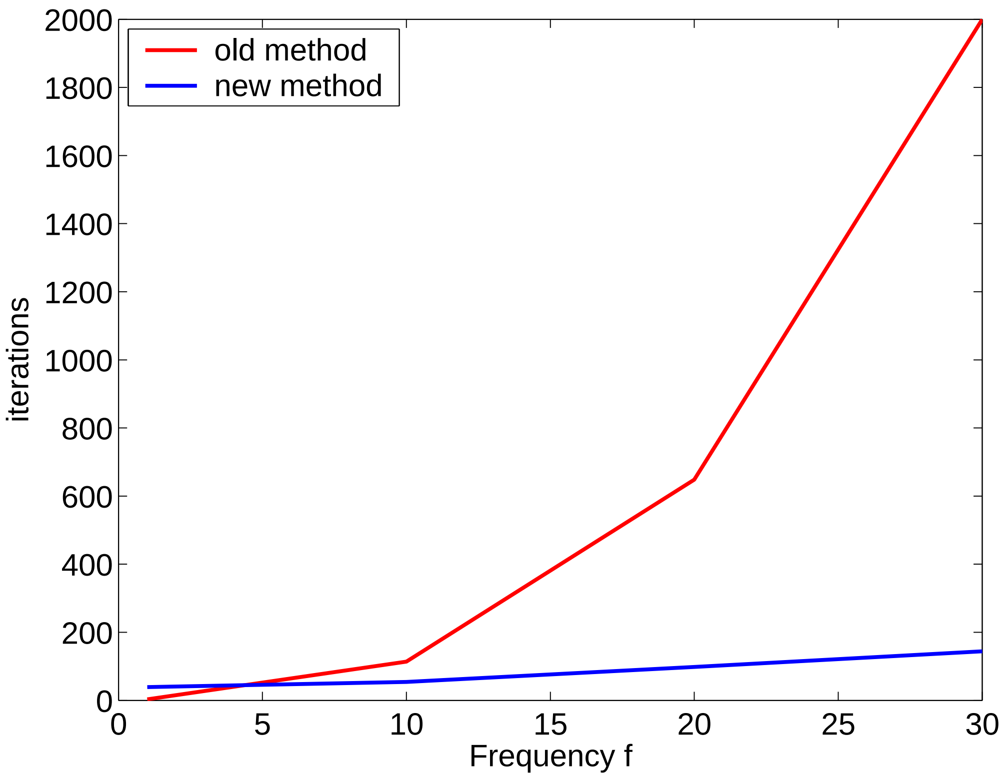
- $A \in \mathbb{C}^{N \times N}$ is sparse
- wavenumber k and grid size N are very large
- wavenumber k varies discontinuously
- real parts of the eigenvalues of A are positive and negative

Application: geophysical survey



Marmousi model (hard)

Application: geophysical survey



Marmousi model (hard)

2. Spectrum of shifted Laplacian preconditioners

Operator based preconditioner P is based on a discrete version of

$$-\partial_{xx}u - \partial_{yy}u - (\alpha_2 + \mathrm{i}\beta_2)k^2(x,y)u = f, \quad \text{in } \Omega.$$

appropriate boundary conditions

Matrix P^{-1} is approximated by an inner iteration process.

$\alpha_2 = 0$	$\beta_2 = 0$	Laplacian	Bayliss and Turkel, 1983
$\alpha_2 = -1$	$\beta_2 = 0$	Definite Helmholtz	Laird, 2000
$\alpha_2 = 0$	$\beta_2 = -1$	Complex	Erlangga, Vuik and
$\alpha_2 = 1$	$\beta_2 = -0.5$	'Optimal'	Oosterlee, 2004, 2006

Spectrum of shifted Laplacian preconditioners

After discretization we obtain the (un)damped Helmholtz operator

$$L - z_1 M,$$

where L and M are SPD matrices and $z_1 = \alpha_1 + \mathrm{i}\beta_1$.

The preconditioner is then given by

$$L - z_2 M,$$

where $z_2 = \alpha_2 + \mathrm{i}\beta_2$ is chosen such that

- systems with the preconditioner are easy to solve,
- the outer Krylov process is accelerated significantly.

Spectrum of shifted Laplacian preconditioners

References: Manteuffel, Parter, 1990; Yserentant, 1988

Since L and M are SPD we have the following eigenpairs

$$Lv_j = \lambda_j M v_j, \text{ where, } \lambda_j \in \mathbb{R}^+$$

The eigenvalues σ_j of the preconditioned matrix satisfy

$$(L - z_1 M)v_j = \sigma_j (L - z_2 M) v_j.$$

Theorem 1
Provided that $z_2 \neq \lambda_j$, the relation

$$\sigma_j = \frac{\lambda_j - z_1}{\lambda_j - z_2} \text{ holds.}$$

Spectrum of shifted Laplacian preconditioners

Theorem 2

If $\beta_2 = 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1 \sigma_r - (\alpha_1 - \alpha_2)\sigma_i = \beta_1 .$$

Spectrum of shifted Laplacian preconditioners

Theorem 2
If $\beta_2 = 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1\sigma_r - (\alpha_1 - \alpha_2)\sigma_i = \beta_1.$$

Theorem 3
If $\beta_2 \neq 0$, the eigenvalues $\sigma_r + \mathrm{i}\sigma_i$ are on the circle in the complex plane with center c and radius R:

$$c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}, \quad R = \left|\frac{z_2 - z_1}{z_2 - \bar{z}_2}\right|.$$

Note that if $\beta_1\beta_2 > 0$ the origin is not enclosed in the circle.

Spectrum of shifted Laplacian preconditioners

Using Sommerfeld boundary conditions, it impossible to write the matrix as $L - z_1 M$ where, L and M are SPD.

Generalized matrix

$$L + \mathrm{i}C - z_1 M,$$

where L, M, and C are SPD. Matrix C contains Sommerfeld boundary conditions (or other conditions: PML, ABL).

Use as preconditioner

$$L + \mathrm{i}C - z_2 M.$$

Spectrum of shifted Laplacian preconditioners

Suppose

$$(L + \mathrm{i}C)v = \lambda_C M v$$

then

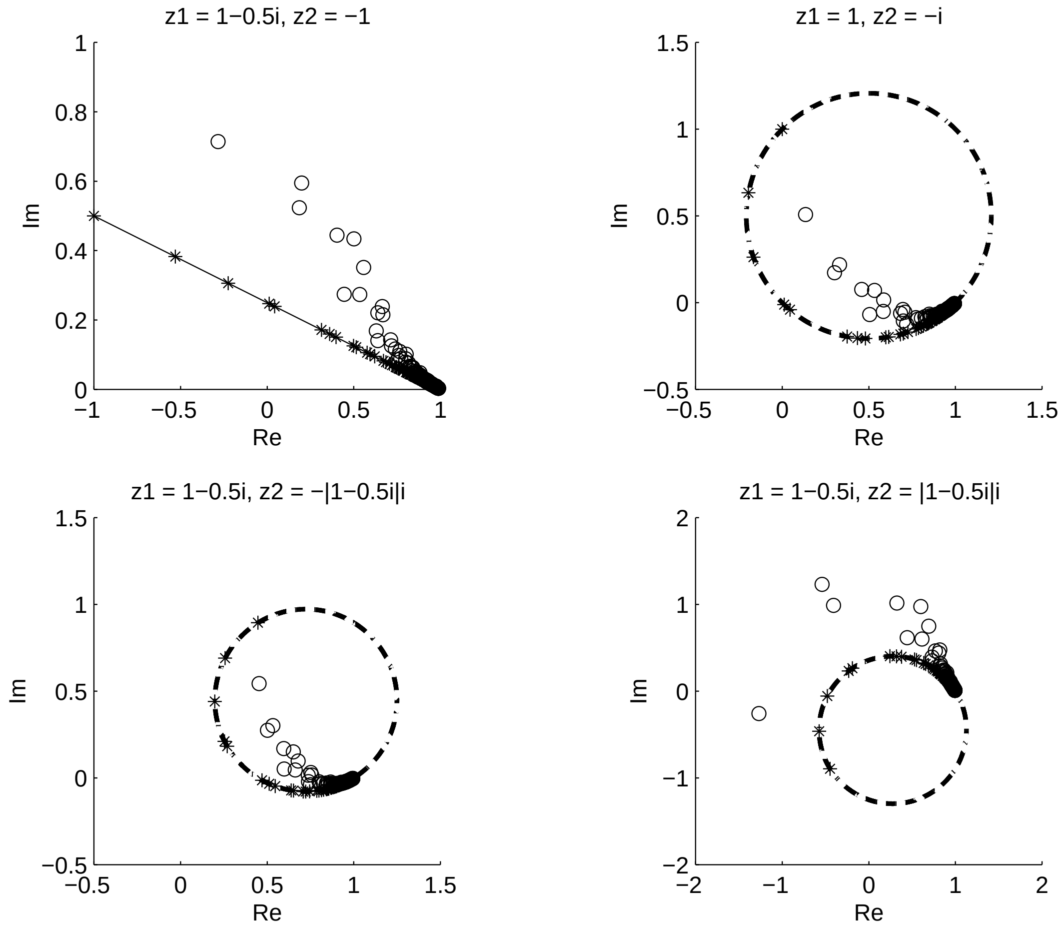
$$(L + \mathrm{i}C - z_1 M)v = \sigma_C (L + \mathrm{i}C - z_2 M)v.$$

Theorem 4

Let $\beta_2 \neq 0$ then the eigenvalues σ_C are in or on the circle with center $c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}$ and radius $R = \left| \frac{z_2 - z_1}{z_2 - \bar{z}_2} \right|$.

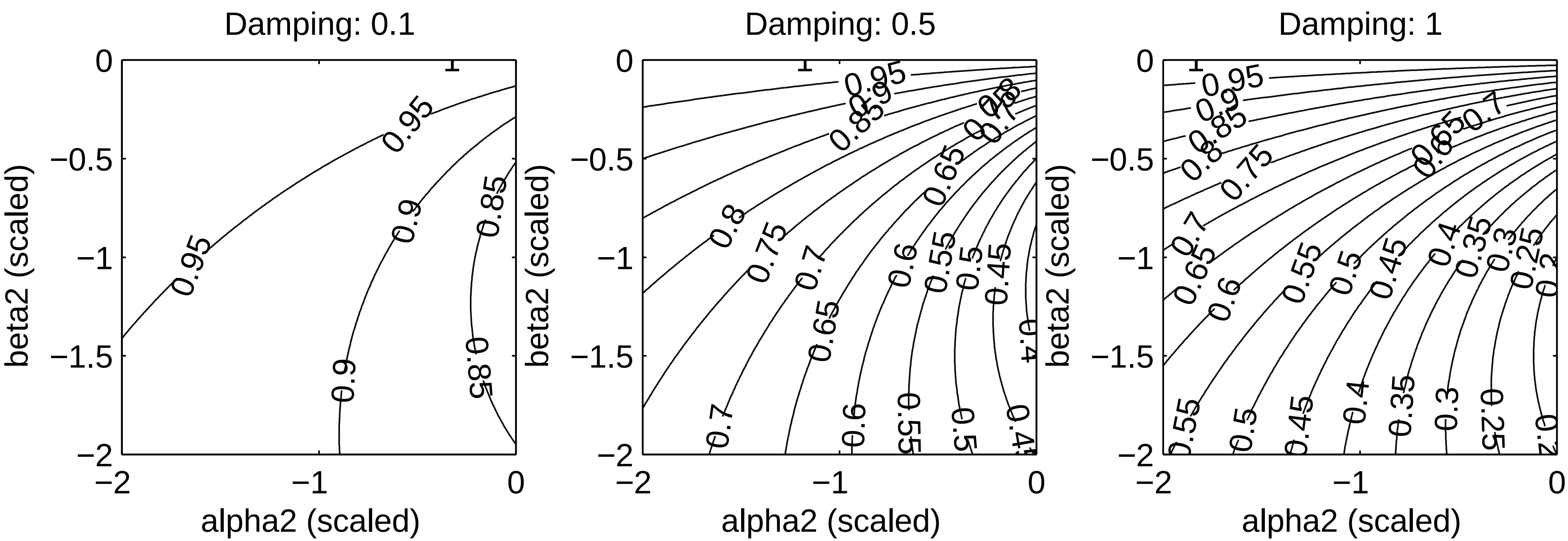
3. Shift with an SPD real part

Motivation: the preconditioned system is easy to solve.

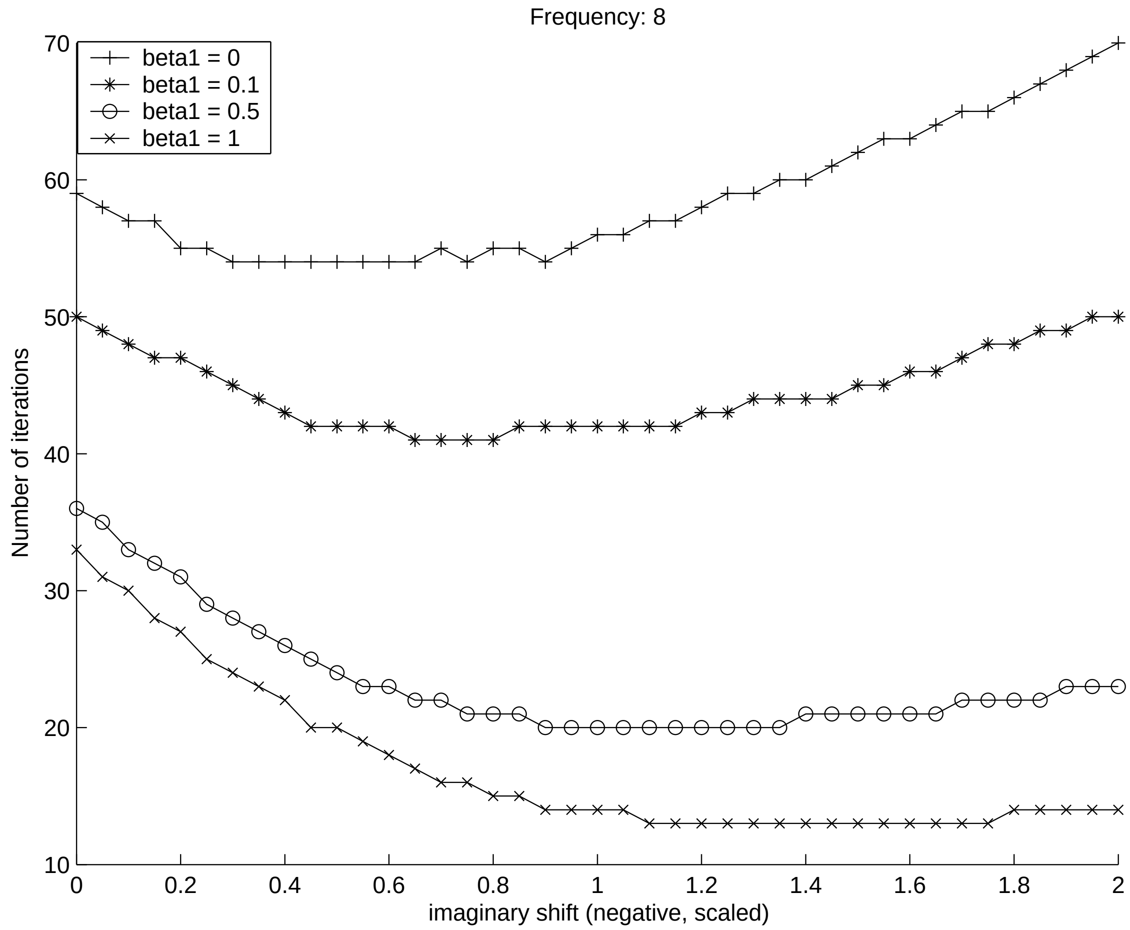


Optimization of the shift

Which choices for z_2 are optimal?



Optimal choices for z_2?



Optimal choices for z_2?

Damping	Optimal β_2	"optimal" iterations	Minimum iterations
$\beta_1 = 0$	-1	56	54
$\beta_1 = -0.1$	-1.005	42	41
$\beta_1 = -0.5$	-1.118	20	20
$\beta_1 = -1$	-1.4142	13	13

Optimal choices for z_2?

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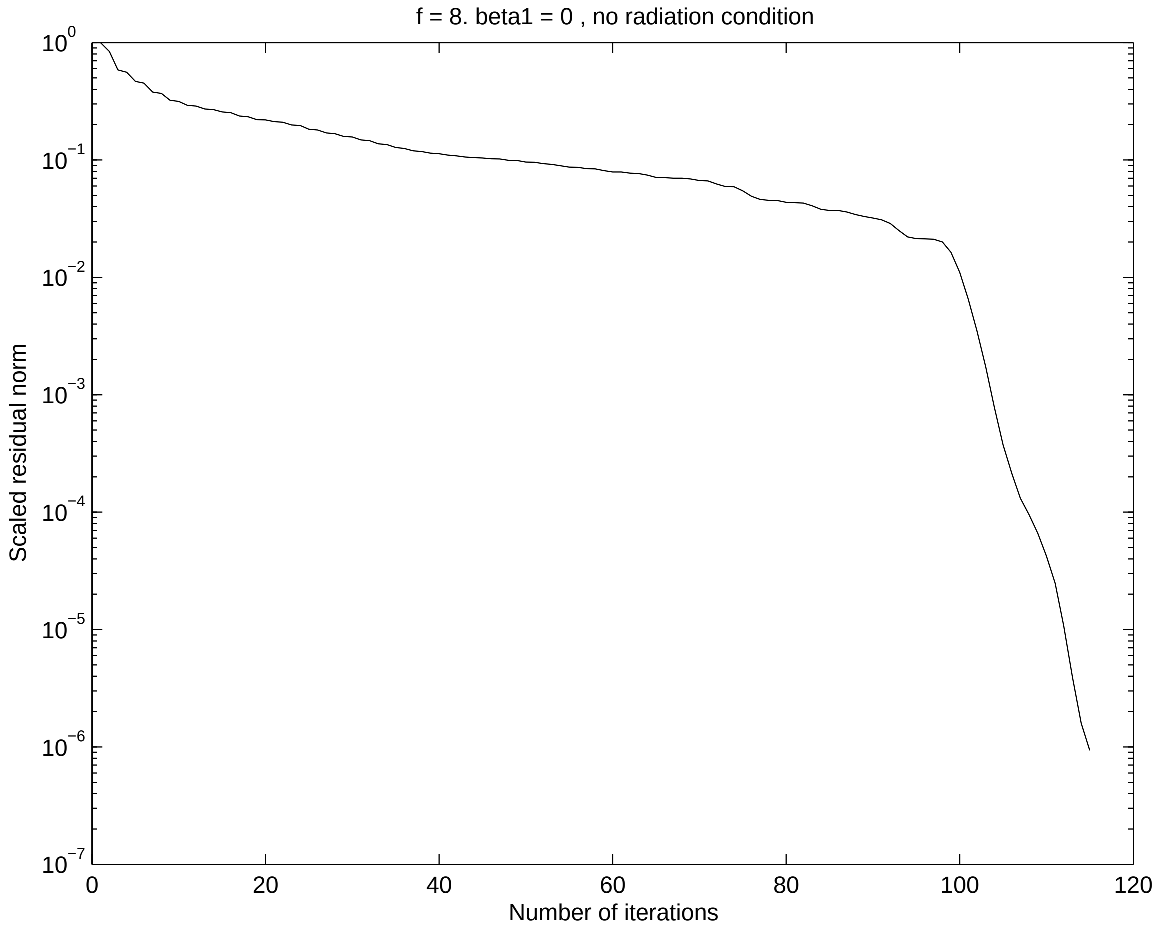
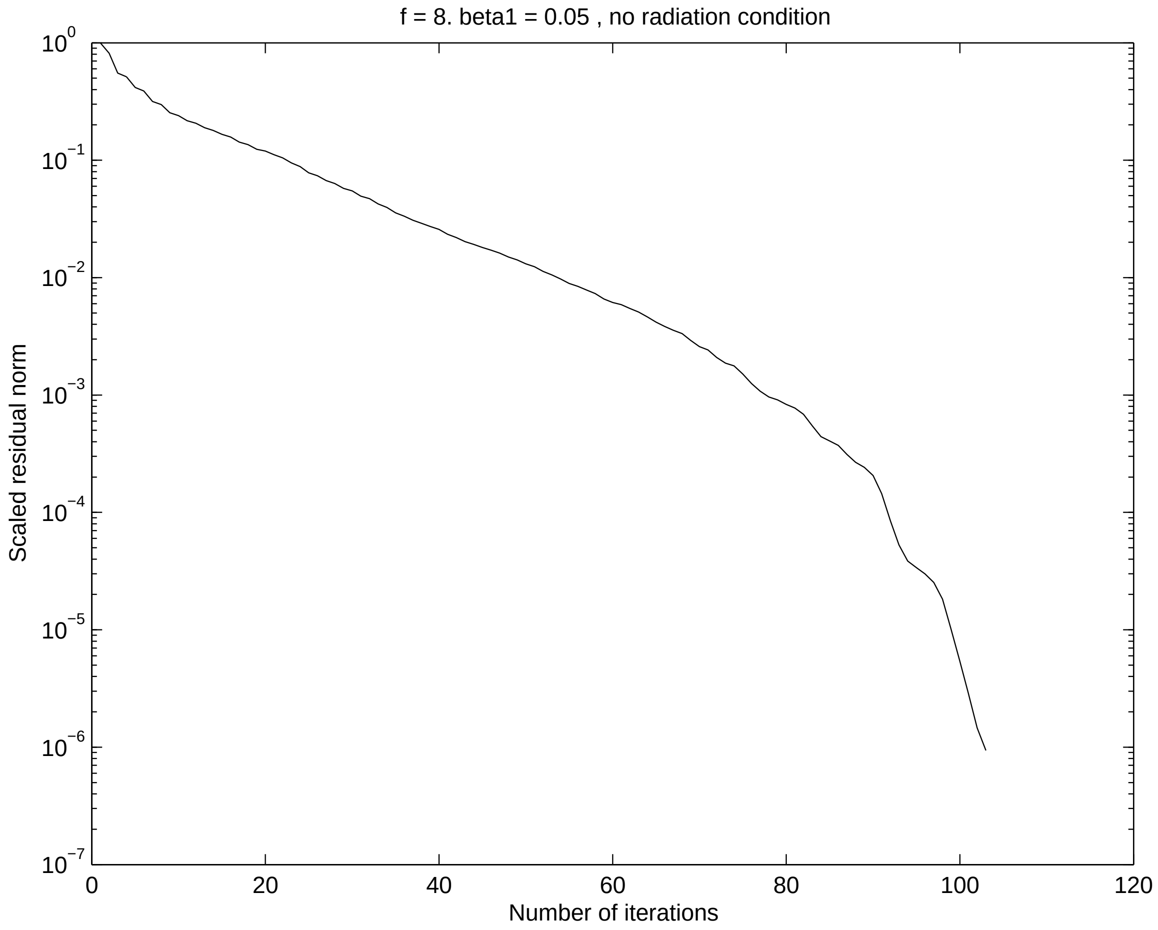
$\beta_1 = -1$	-1.4142	13	13

	Number of iterations				
h f	100/2 2	100/4 4	100/8 8	100/16 16	100/32 32
$\beta_1 = 0$	14	25	56	116	215
$\beta_1 = -0.1$	13	22	42	63	80
$\beta_1 = -0.5$	11	16	20	23	23
$\beta_1 = -1$	9	11	13	13	23

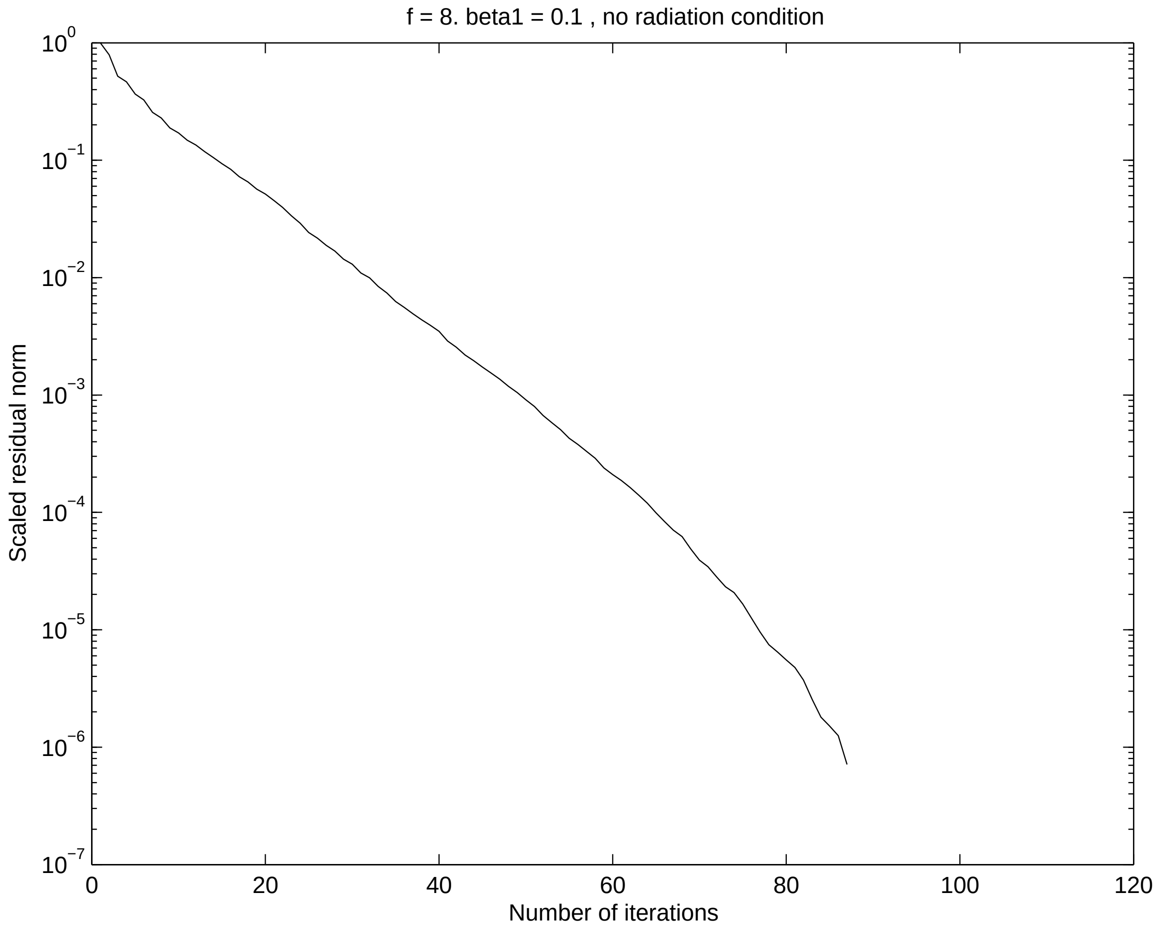
Superlinear convergence of GMRES



Superlinear convergence of GMRES



Superlinear convergence of GMRES



4. General Shifted Laplacian preconditioner

No restriction on α_2

For the outer loop $\alpha_2 = 1$ and $\beta_2 = 0$ is optimal. Convergence in 1 iteration. **But**, the inner loop does not converge with multi-grid (original problem).

However, it appears that multi-grid works well for $\alpha_2 = 1$ and $\beta_2 = -1$

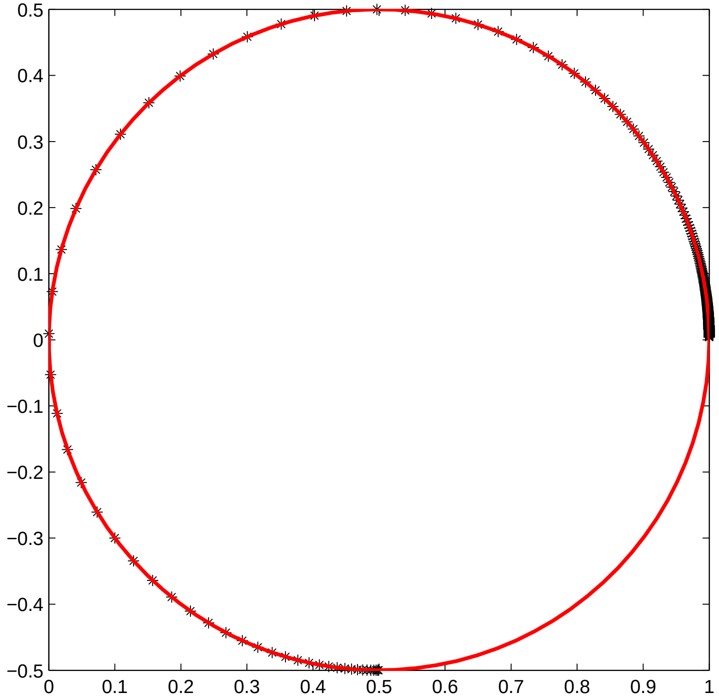
and the convergence of the outer loop is much faster than for the choice

$\alpha_2 = 0$ and $\beta_2 = -1$.

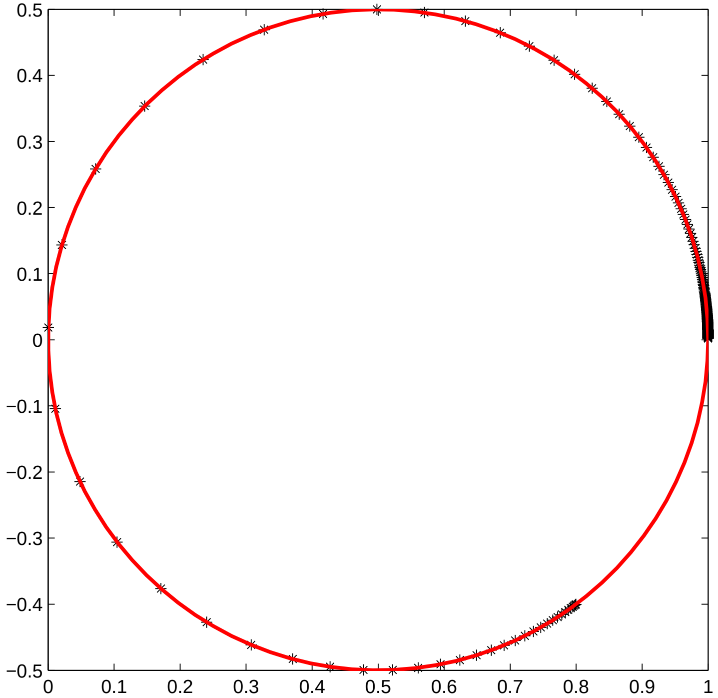
Eigenvalues for Complex preconditioner $k = 100$ *and* $\alpha_2 = 1$

Spectrum is independent of the grid size

$\beta_2 = -1$



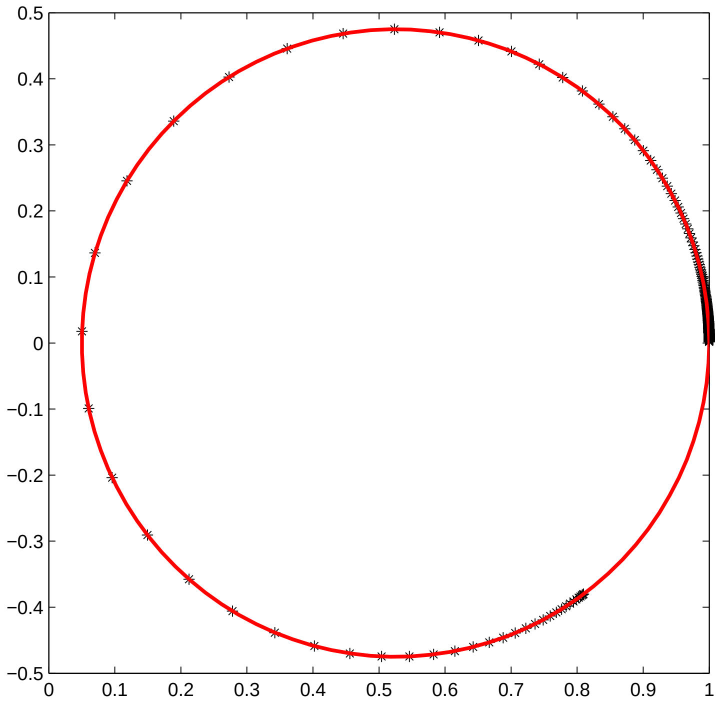
$\beta_2 = -0.5$



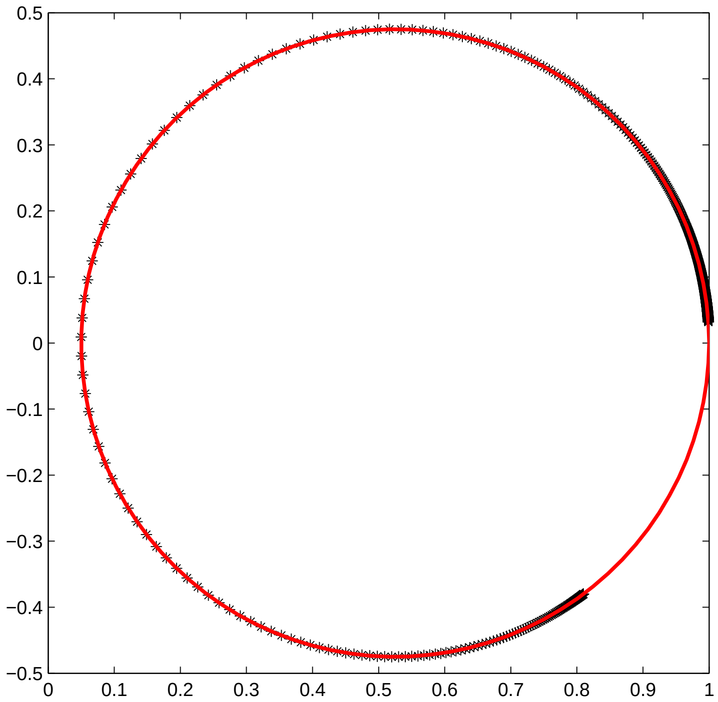
Eigenvalues for $\beta_1 = -0.025$ *(damping) and* $\alpha_2 = -1$, $\beta_2 = -0.5$

Spectrum is independent of the grid size and the choice of k.

$k = 100$



$k = 400$

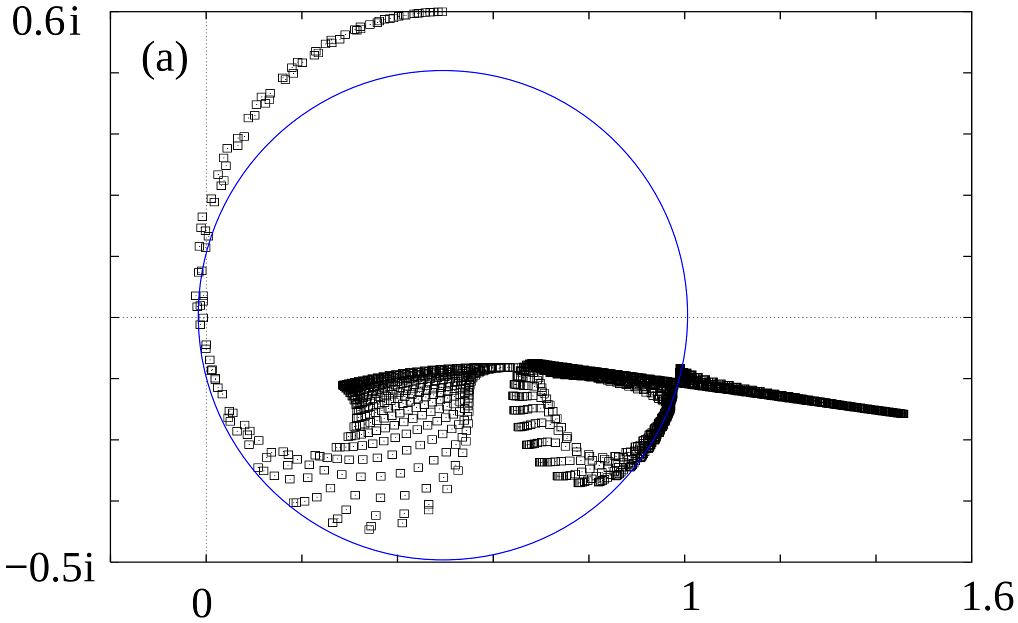


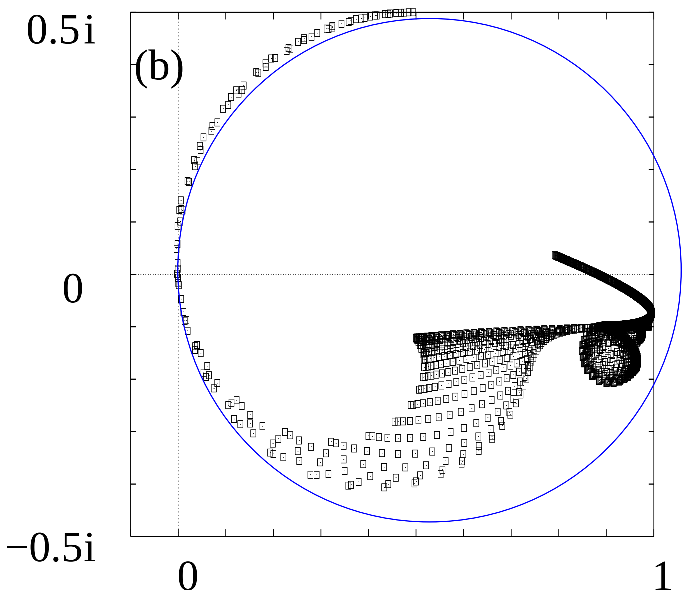
5. Numerical experiments

Multi-grid components

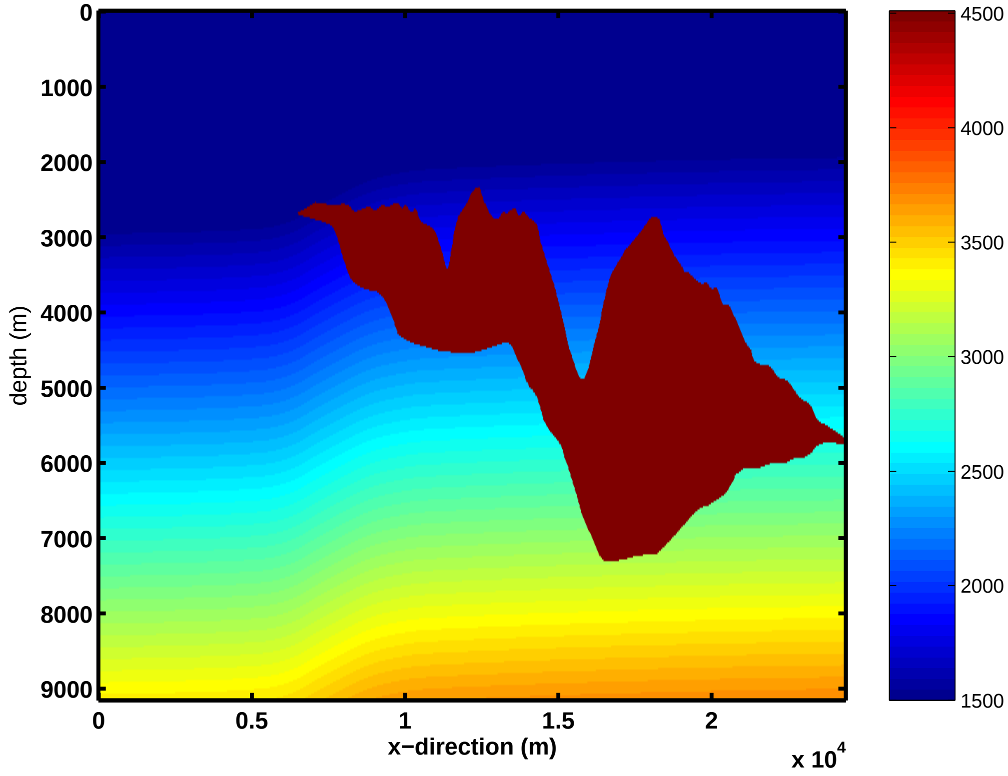
- geometric multi-grid
- ω-JAC smoother
- matrix dependent interpolation, restriction operator full weighting
- Galerkin coarse grid approximation
- F(1,1)-cycle
- P^{-1} is approximated by *one* multi-grid iteration
- in 3D semi-coarsening is used

Spectrum with inner iteration





Sigsbee model



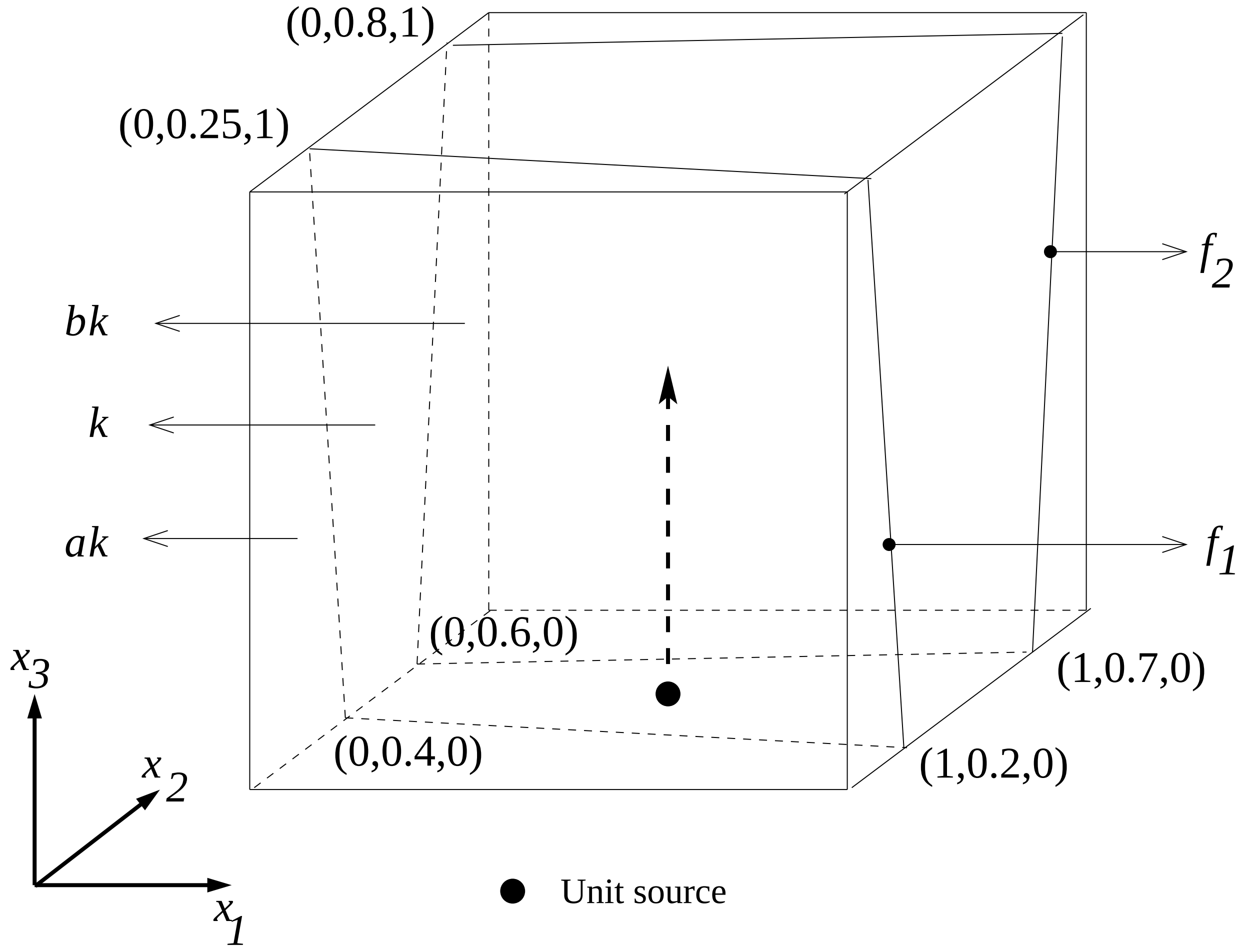
Sigsbee model

$dx = dz = 22.86$ m; $D = 24369 \times 9144$ m^2; grid points 1067×401.

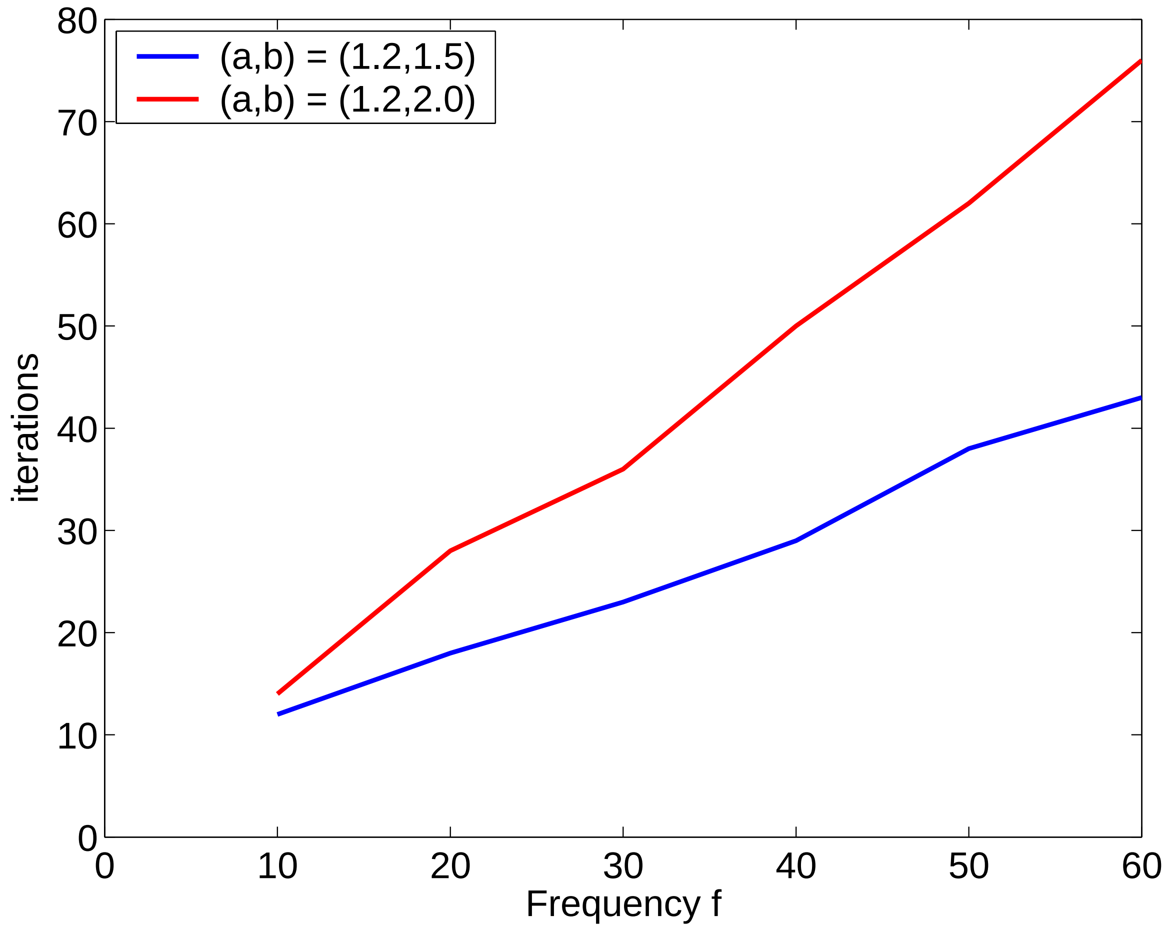
Bi-CGSTAB	5 Hz		10 Hz	
	CPU (sec)	Iter	CPU (sec)	Iter
NO preco	3128	16549	1816	9673
With preco	86	48	92	58

Note:
- Without preconditioner, number of iterations $> 10^4$,
- With shifted Laplacian preconditioner, only 58 iterations.

3D wedge problem



Numerical results for 3D wedge problem



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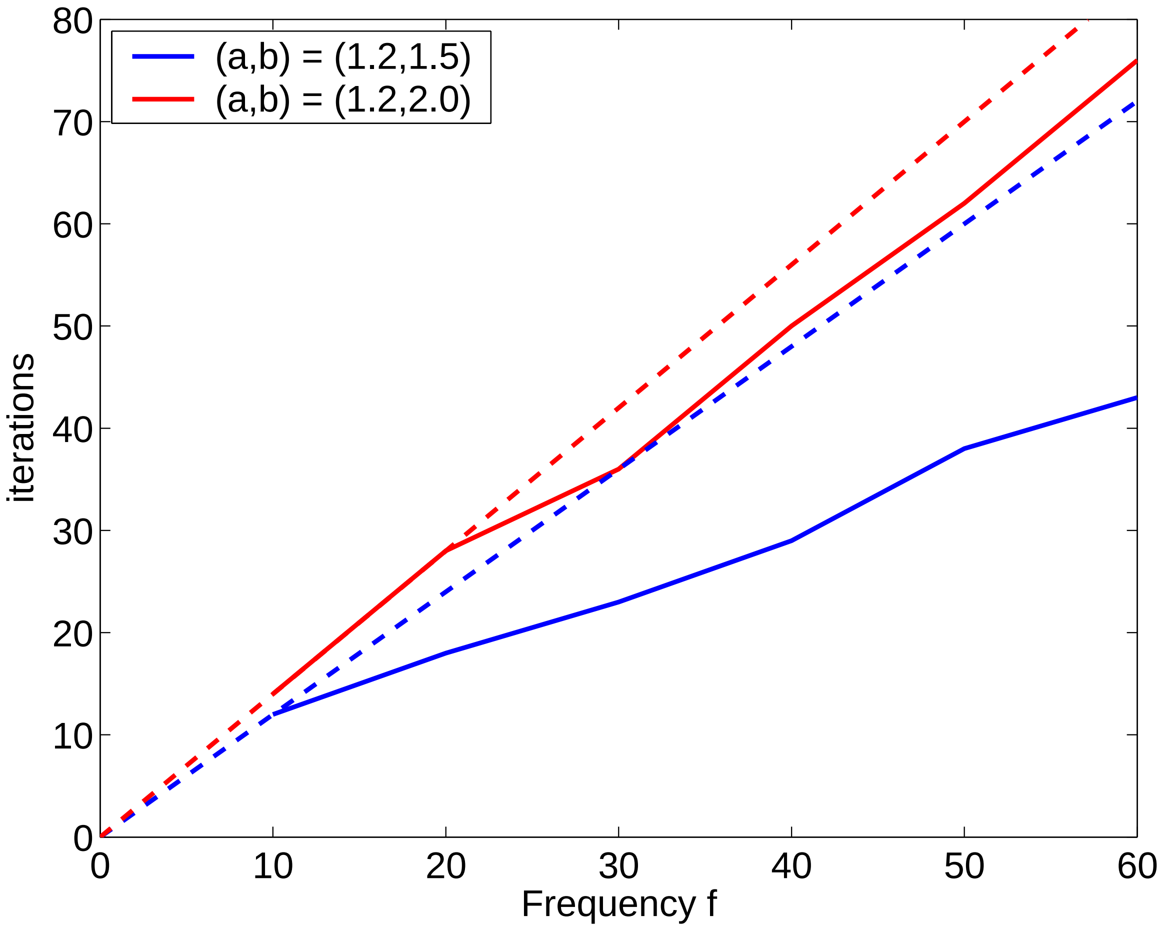

6. Conclusions

- The shifted Laplacian operator leads to robust preconditioners for the 2D and 3D Helmholtz equations with various boundary conditions.
- For real shifts the eigenvalues of the preconditioned operator are on a straight line.
- For complex shifts the eigenvalues of the preconditioned operator are on a circle.
- The proposed preconditioner (shifted Laplacian + multi-grid) is independent of the grid size and linearly dependent of k.
- With physical damping the proposed preconditioner is also independent of k.

Further information/research

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