Accuracy Enhancement and Filtering for Visualisation of Discontinuous Solutions

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Motivation and Background

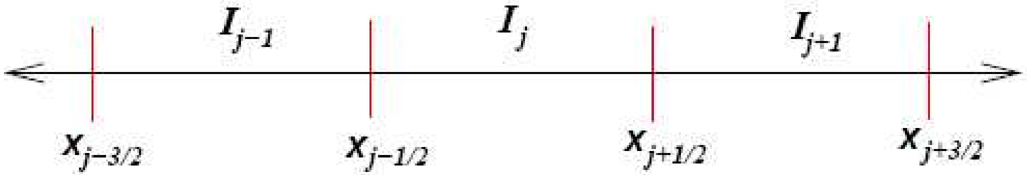
- Discontinuous Galerkin Method
- Post-Processing for Accuracy Enhancement
- Applications in Visualisation

Issues and challenges

- non-uniform mesh
- derivative post-processing
- $\Rightarrow$ one-sided post-processing $\Leftarrow$

Summary

1D Discontinuous Galerkin Formulation



Define a Mesh and an Approximation Space:

$I_j = (x_j - \frac{\triangle x_j}{2}, x_j + \frac{\triangle x_j}{2}), \quad j = 1, \cdots, N$ and $V_h = \{\phi_j^{(l)}(x) \in \mathbb{P}^k|_{I_j}, \ j = 1, \cdots, N\}$

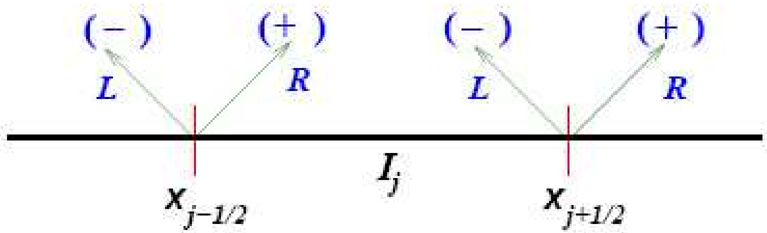
Consider $u_t + f(u)_x = 0.$

Weak Formulation: Find $u_h(x,t) \in V_h$ such that

$$\int_{I_j} (u_h)_t v dx = \int_{I_j} f(u_h) v_x dx - f((u_h)_{j+\frac{1}{2}}) v_{j+\frac{1}{2}} + f((u_h)_{j-\frac{1}{2}}) v_{j-\frac{1}{2}}$$

for all $v \in V_h$.

1D Discontinuous Galerkin Formulation



Numerical Scheme:

$$\int_{I_j} (u_h)_t v dx = \int_{I_j} f(u_h) v_x dx - \hat{f}_{j+1/2} v^-_{j+1/2} + \hat{f}_{j-1/2} v^+_{j-1/2}$$

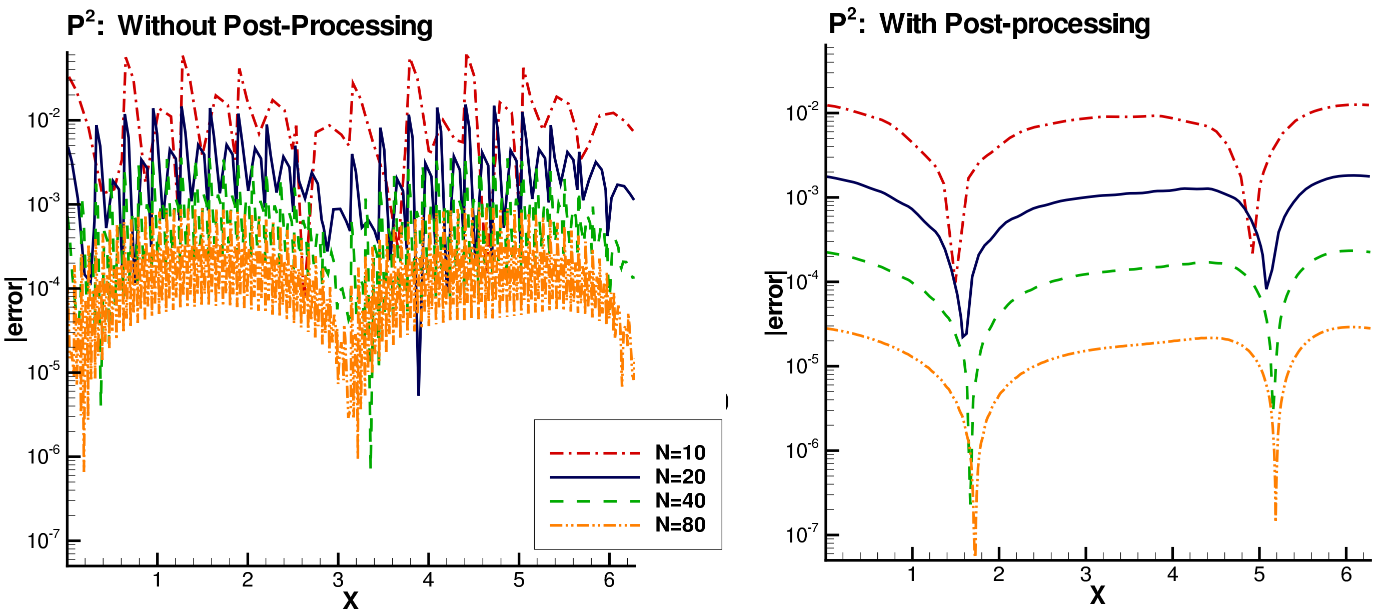
$\forall v \in V_h$.

- Use upwind monotone flux
- Take v from inside the cell

DG solution: $u_h(x,t) = \sum_{l=0}^{k} u_i^{(l)}(t)\phi_i^{(l)}(x)$ if $x \in I_i$.

Can we improve an existing DG approximation?

1-D Variable Coefficient



$u_t+(a(x)u)_x=f(x,t)$
$a(x)=2+\sin(x)$
$u(x,0)=\sin(3x)$
x in $(0,2\pi)$, $T=12.5$

Post-Processing to Improve and Approximation

The post-processor:

$$u^{\star} = K_h^{2(k+1),k+1} \star u_h$$

Why do we post-process?

- Errors in DG solution are highly oscillatory
- Post-processing filters out oscillations around the exact solution
- Result is a solution that has increased smoothness and accuracy

Post-Processor

`B. Cockburn, M. Luskin, C.-W. Shu, A. Süli, Math Comp. (2003)`

- Discontinuous Galerkin approximation errors:

$$||u_h - u||_{-l} = \mathcal{O}(h^{2k+1}),$$

whereas in the L_2-norm we have

$$||u_h - u||_2 = \mathcal{O}(h^{k+1}).$$

- Post-processor extracts this information.

$$u^*(x) = K_h * u_h$$

- Works for a locally uniform mesh:
  $\longrightarrow$ Translation invariant
  $\longrightarrow$ Post-Processor is local

Negative Order Sobolev Norm

The negative order norm is given by

$$||u||_{-\ell,\Omega} = \sup_{\phi \in \mathcal{C}_0^\infty} \frac{\int_\Omega u(x)\phi(x)dx}{||\phi||_{\ell,\Omega}}, \quad \ell \geq 1,$$

which is just a seminorm divided by the usual Sobolev norm.

Example: For the function $u_N = \sin(2\pi N x), \quad \Omega = (-1,1), \quad \ell \geq 1$, the negative order norm is

$$||u_N||_{-\ell,\Omega} = \frac{1}{(2\pi N)^\ell}$$

The negative order norm tells us that $\sin(2\pi N x)$ oscillates around zero fairly regularly.

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Bramble & Schatz, Math.  Comp.  (1977)
Mock & Lax, Comm.  Pure Appl.  Math (1978)
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Post-Processor Kernel

- Independent of the partial differential equation.
- Applied only at the final time.
- Filters out oscillations in the error.

Kernel Properties

- Compact Support $\Rightarrow$ Computationally advantages
- Reproduces polynomials of degree $2k$ by convolution. $\Rightarrow$ Accuracy is not lost.
- Linear combination of B-splines.

Post-Processor

- Use Negative order norms $\Rightarrow$ Tells us how oscillatory a function is (difficult to compute).
- Use Convolution $\Rightarrow$ “Filters” out these oscillations
- B-splines $\Rightarrow$ Gives the convolution kernel nice properties.
- Make assumptions on the approximation and the mesh.

`Result:` A post-processor that filters out oscillations in the error and improves the order of accuracy.

Kernel Construction

Post-processed solution: $u^\star(x) = K_h^{2(k+1),k+1} \star u_h$.

$$K_h^{2(k+1),k+1}(x) = \frac{1}{h}\sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1}\psi^{(k+1)}\left(\frac{x}{h}-\gamma\right)$$

$h = \triangle x_i$ for all i, and $c_\gamma^{2(k+1),k+1} \in \mathbb{R}$.

B-spline recursion formula:

$$\psi^{(1)} = \chi_{[-1/2,1/2]},$$

$$\psi^{(k+1)} = \frac{1}{k}\left[\left(x+\frac{k+1}{2}\right)\psi^{(k)}\left(x+\frac{1}{2}\right)+\left(\frac{k+1}{2}-x\right)\psi^{(k)}\left(x-\frac{1}{2}\right)\right], \quad k \geq 1.$$

Convolution Coefficients

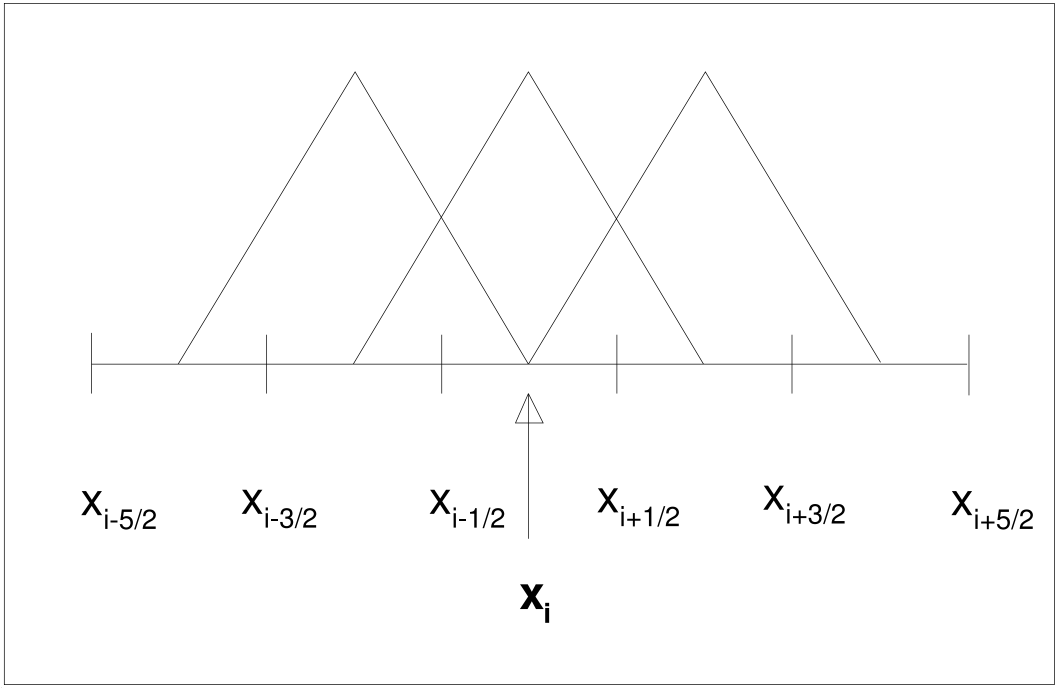
To find $c_\gamma,\ \gamma = -k, \cdots, k:$
Use $K_h^{2(k+1),k+1} \star x^m = x^m$ for $m = 1, \cdots, x^{2k}$

$$\begin{bmatrix} \int \psi^{(k+1)}(x-y-k)\,dy & \cdots & \int \psi^{(k+1)}(x-y+k)\,dy \\ \int \psi^{(k+1)}(x-y-k)y\,dy & \cdots & \int \psi^{(k+1)}(x-y+k)y\,dy \\ \int \psi^{(k+1)}(x-y-k)y^2\,dy & \cdots & \int \psi^{(k+1)}(x-y+k)y^2\,dy \\ \cdots & \cdots & \cdots \\ \int \psi^{(k+1)}(x-y-k)y^{2k}\,dy & \cdots & \int \psi^{(k+1)}(x-y+k)y^{2k}\,dy \end{bmatrix} \begin{bmatrix} c_{-k} \\ \cdots \\ c_0 \\ \cdots \\ c_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & x^{k+1} & \cdots & x^{2k} \end{bmatrix}^T$$

Example: Kernel B-splines

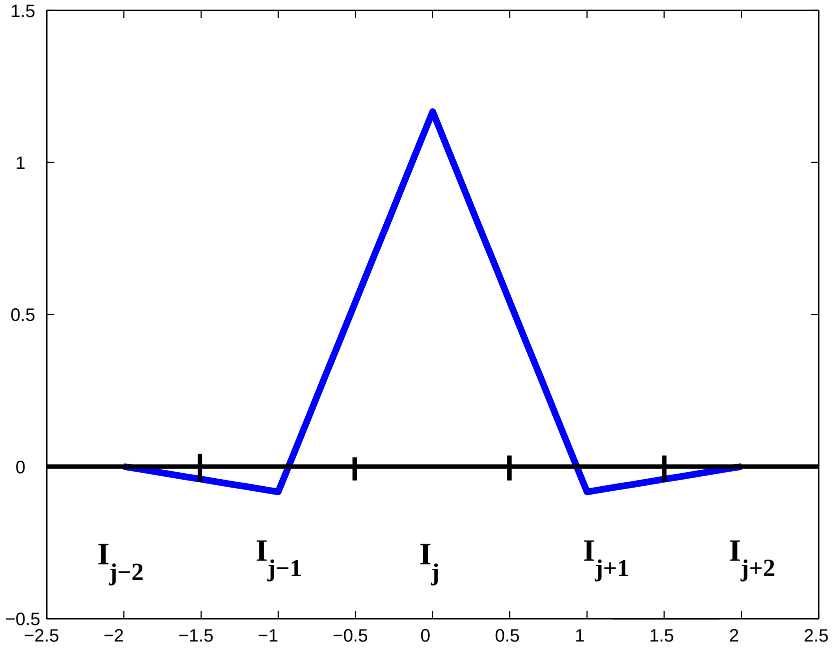
Second Order Approximation



$$\psi^{(2)}(x+1) \qquad \psi^{(2)}(x) \qquad \psi^{(2)}(x-1)$$

Kernel for Linear Approximation

Find $c_\gamma,\ \gamma = -1, 0, 1$: Use $K_h^{4,2} \star p = p$ for $p = 1, x, x^2$



$$K^{4,2}(x) = \frac{-1}{12}\psi^{(2)}(x-1) + \frac{7}{6}\psi^{(2)}(x) - \frac{1}{12}\psi^{(2)}(x+1)$$

Implementing the Post-processor

For element $I_j = (x_{j-1/2,j+1/2})$:

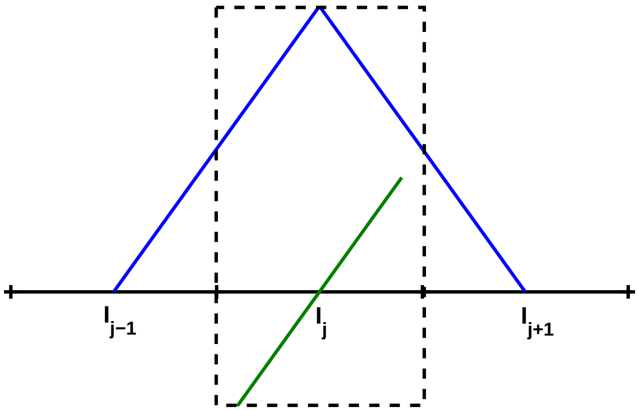
$$\Rightarrow u^\star(x) = \sum_i \sum_{l=0}^{k} u_i^l \sum_{\gamma=-k}^{k} c_\gamma^{2(k+1),k+1} \int \psi^{(k+1)}\left(\frac{x-y}{h} - \gamma\right) \phi_i^{(l)}(y)\, dy.$$

where $i = j - p', \cdots, j + p',\ p' = \lceil \frac{3k+1}{2} \rceil$

k	1	2	3
p'	2	3	5

Note: p' is the number of elements needed on each side of the element being post-processed.

Example: Implementing $k = 1$ case



green line = DG approximation on one element.

blue line = kernel. The kernel is introducing smoothness at the element boundaries.

Convolution Kernel:

$$K^{4,2}(x) = \frac{-1}{12}\psi^{(2)}(x-1) + \frac{7}{6}\psi^{(2)}(x) - \frac{1}{12}\psi^{(2)}(x+1)$$

Discontinuous Galerkin Solution: $u_h(x) = u_j^{(0)}\phi_j^{(0)} + u_j^{(1)}\phi_j^{(1)}$ on element $I_j = (x_{j-1/2}, x_{j+1/2})$.

$2-D$ *Kernel*

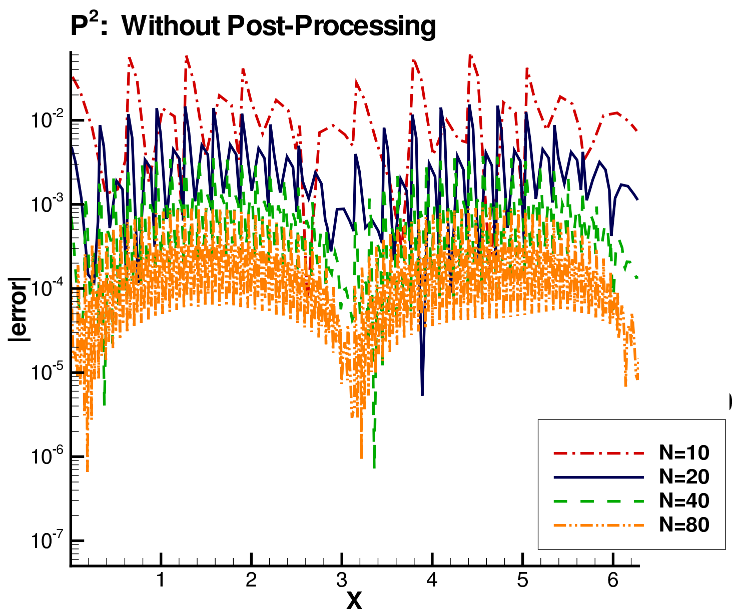
The $2-$D case is simply a tensor product of the $1-$D case.

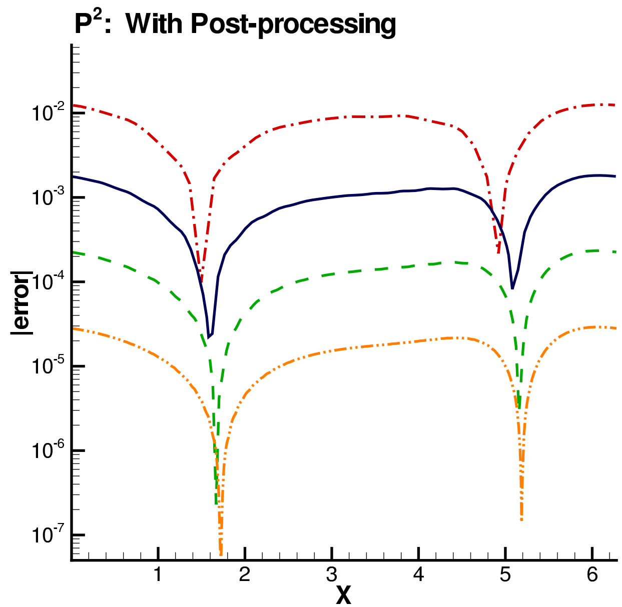
Kernel:

$$K_h = \frac{1}{h_x h_y} \sum_{\gamma_x=-k}^{k} \sum_{\gamma_y=-k}^{k} c_{\gamma_x} c_{\gamma_y} \psi^{(k+1)}\left(\frac{x}{h_x} - \gamma_x\right) \psi^{(k+1)}\left(\frac{y}{h_y} - \gamma_y\right)$$

We can use *either* a tensor product of polynomials, $\mathbb{Q}^k$ - ($\{1, x, y, xy\}$), or the usual polynomial basis, $\mathbb{P}^k$ - ($\{1, x, y\}$).

1-D Variable Coefficient





$$u_t+(a(x)u)_x=f(x,t)$$
$$a(x)=2+\sin(x)$$
$$u(x,0)=\sin(3x)$$
$$x \text{ in } (0,2pi),\ T=12.5$$

$1-D$ *Variable Coefficient Equation*

Ryan, Shu, Atkins, SISC (2005)

	$u_h(x, 12.5)$		$u^*(x, 12.5)$	
mesh	L^2 error	order	L^2 error	order
		$\mathbb{P}^1$		
10	1.83E-02	—	7.82E-02	—
20	4.35E-03	2.07	1.08E-03	2.86
40	1.07E-03	2.03	1.39E-04	2.96
		$\mathbb{P}^2$		
10	8.61E-04	—	1.34E-04	—
20	1.07E-04	3.01	2.34E-06	5.84
40	1.34E-05	3.00	4.55E-08	5.69

$$u_t + (au)_x = f$$

$$\begin{aligned} a(x) &= 2 + \sin(x) \\ u(x,0) &= \sin(3x) \\ u(0,t) &= u(2\pi, t) \\ T &= 12.5 \end{aligned}$$

TUDelft

Streamline Calculation: Filtering Entire Field

- Obtain numerical approximation
- Post-Process the approximation
- We can then choose our time integrator for the streamline calculation (such as RK-4)

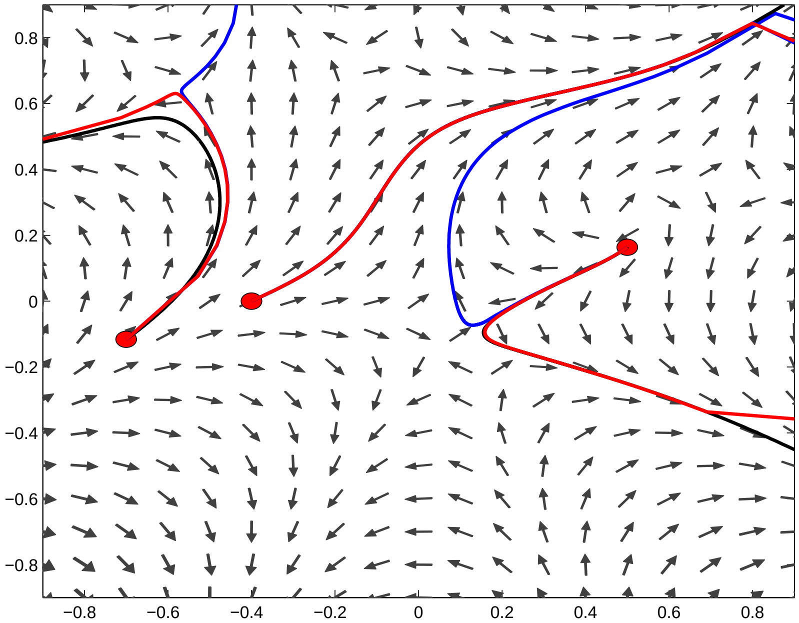
$$\begin{aligned} \frac{d}{dt}\vec{x}(t) &= \vec{F}(\vec{x}(t)) \\ \vec{x}(t=0) &= \vec{x}_0 \end{aligned}$$

- The post-processor increases smoothness of the approximation to help obtain the correct streamline.

Applications in Filtering for Streamline Visualisation

Example Field: Scheuerman, Tricoche, and Hagen, IEEE Vis (1999).

Steffan, Curtis, Kirby, and Ryan, IEEE-TVCG (2008).



$$
\begin{aligned}
z &= x + \imath y \\
u &= Re(r) \\
v &= -Im(r).
\end{aligned}
$$

$$
\begin{aligned}
r = \; & (z - (0.74 + 0.35\imath))(z - (0.68 - 0.59\imath) \\
& (z - (-0.11 - 0.72\imath))(\bar{z} - (-0.58 + 0. \\
& (\bar{z} - (0.51 - 0.27\imath))(\bar{z} - (-0.12 + 0.84
\end{aligned}
$$

$$(u, v)^T = \vec{F}(x, y), \quad \Omega = [-1, 1] \times [-1, 1]$$

Streamline Calculation: Filtering Entire Field

U component

	L^2 error		L^∞ error	
N	BEFORE	AFTER	BEFORE	AFTER
	$\mathbb{P}^1$			
20	1.2642E-02	4.8779E-04	1.3028E-01	2.0830E-03
40	4.4291E-03	3.8597E-05	4.8341E-02	1.7929E-04
80	1.3054E-03	2.7114E-06	1.7165E-02	1.3033E-05
	$\mathbb{P}^2$			
20	2.2576E-04	6.8329E-06	1.8986E-03	1.3061E-05
40	5.0880E-05	1.4086E-07	5.4698E-04	2.6435E-07
80	8.4056E-06	2.4689E-09	9.9905E-05	4.6007E-09

Applications in Filtering for Visualization

Streamline Calculation: Filtering Entire Field

Limitations:

- Uniform quadrilateral mesh $\cdots$ What about $3-D$?
  $\Rightarrow$ For 1 & 2-D use a characteristic length.$\Leftarrow$
- Higher order streamline integrator - need derivative information.
  $\longrightarrow$ Use smoother splines.
- Maintaining Boundary Values.
- Post-Processing entire field can be expensive (R.M. Kirby, Utah).

Nonuniform Mesh: Characteristic Length

Curtis, Kirby, Ryan, and Shu, SISC (2007).

Post-processing solution on cell I_j.

- Let L be the characteristic length used in the post-processor, where $L = \max_{i=1,\cdots,N} \triangle x_i$.

$$C_L(i,l,k,x) = \frac{1}{L}\int_{I_{i+j}} \psi^{(k+1)}\left(\frac{y-x}{L} - \gamma\right)\left(\frac{y - x_{i+j}}{\triangle x_{i+j}}\right)^l dy,$$

- Find post-processed solution on I_j :

$$u^\star(x) = \sum_{i=-p'}^{p'} \sum_{l=0}^{k} u_{(i+j)}^{(l)} C_L(i,l,k,x)$$

Streamline Calculation: Filtering Entire Field

Limitations:

- Uniform quadrilateral mesh $\cdots$ What about $3-D$?
  $\longrightarrow$ For 1 & 2-D use a characteristic length.
- $\Rightarrow$Higher order streamline integrator - need derivative information.$\Leftarrow$
  $\longrightarrow$ Use smoother splines.
- Maintaining Boundary Values.
- Post-Processing entire field can be expensive (R.M. Kirby, Utah).

Accuracy Improvement for Derivatives

Two methods

- Calculating the derivative of the post-processing polynomial directly.

  `Ryan, Shu, Atkins, SISC (2005)`

$$\Rightarrow \mathcal{O}(h^{2k+2-d})$$

- $\Rightarrow$ Using higher-order B-splines in the convolution kernel together with divided differences of the numerical solution. $\Leftarrow$

  `Thomee, Math. Comp. (1977)`

  `Cockburn & Ryan, JCP (2009)`

$$\Rightarrow \mathcal{O}(h^{2k+1})$$

Accuracy Improvement for Derivatives: *Higher Order Splines*

$$\frac{d^s u^*}{dx^s}(x) = \frac{1}{h}\int_{-\infty}^{\infty} \tilde{K}^{s,2(k+1),k+1}\left(\frac{y-x}{h}\right)\partial_h^s u_h(y,T)\,dy.$$

for the s^{th} derivative.

- Uses higher order B-splines than post-processed solution.
- Kernel has a wider support.

Kernel:

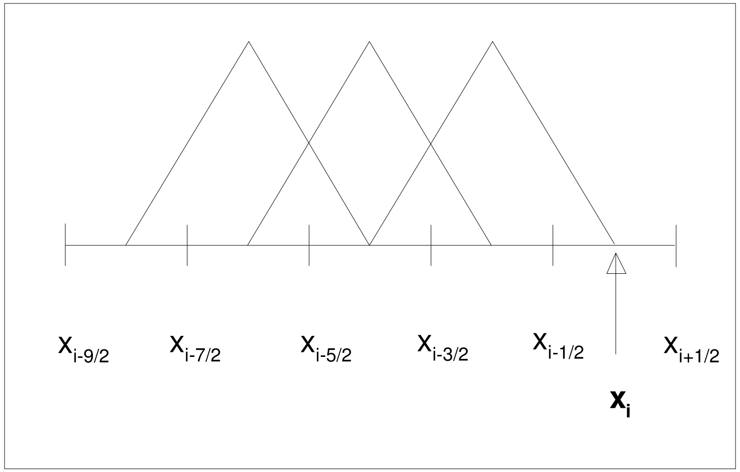
$$\tilde{K}^{s,2(k+1),k+1} = \sum_{\gamma=-k}^{k} \tilde{c}_\gamma\, \psi^{(k+s+1)}(x-\gamma).$$

Streamline Calculation: Filtering Entire Field

Limitations:

- Uniform quadralateral mesh $\cdots$ What about $3-D$?
  $\longrightarrow$ For 1 & 2-D use a characteristic length.
- Higher order streamline integrator - need derivative information.
  $\longrightarrow$ Use smoother splines.
- $\Longrightarrow$ Maintaining Boundary Values. $\Longleftarrow$
- Post-Processing entire field can be expensive (R.M. Kirby, Utah).

(Old) Left Post-Processor



Ryan and Shu, MAA (2003)

$$u^\star(x) = \sum_{j=-2p'}^{0} \sum_{l=0}^{k} u_{i+j}^{(l)}\, C(j,l,k,x)$$

where $p' = \lceil (3k+1)/2 \rceil \leq 2k$
and $u^\star \in \mathbb{P}^{2k+1}$

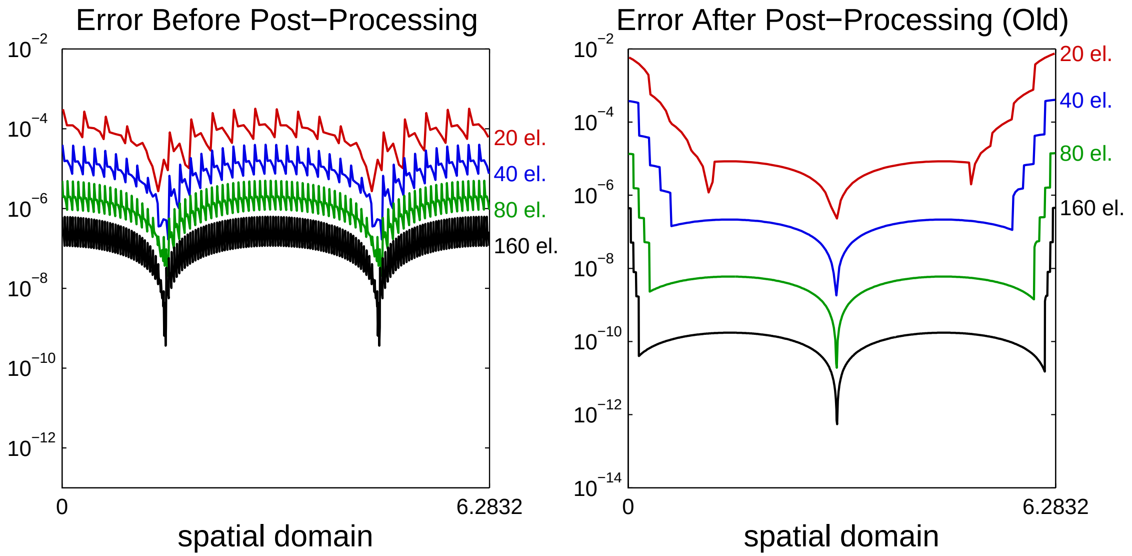
$$C(j,l,k,x) = \frac{1}{h} \sum_{\gamma=-2k-1}^{-k} c_\gamma^{2(k+1),k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}(\eta)\,(\xi_i+\eta+\gamma-j)^l\, dy$$

For $k = 1$:

$$K(x) = \frac{11}{12}\psi^{(2)}(x+3) - \frac{17}{6}\psi^{(2)}(x+2) + \frac{35}{12}\psi^{(2)}(x+1)$$

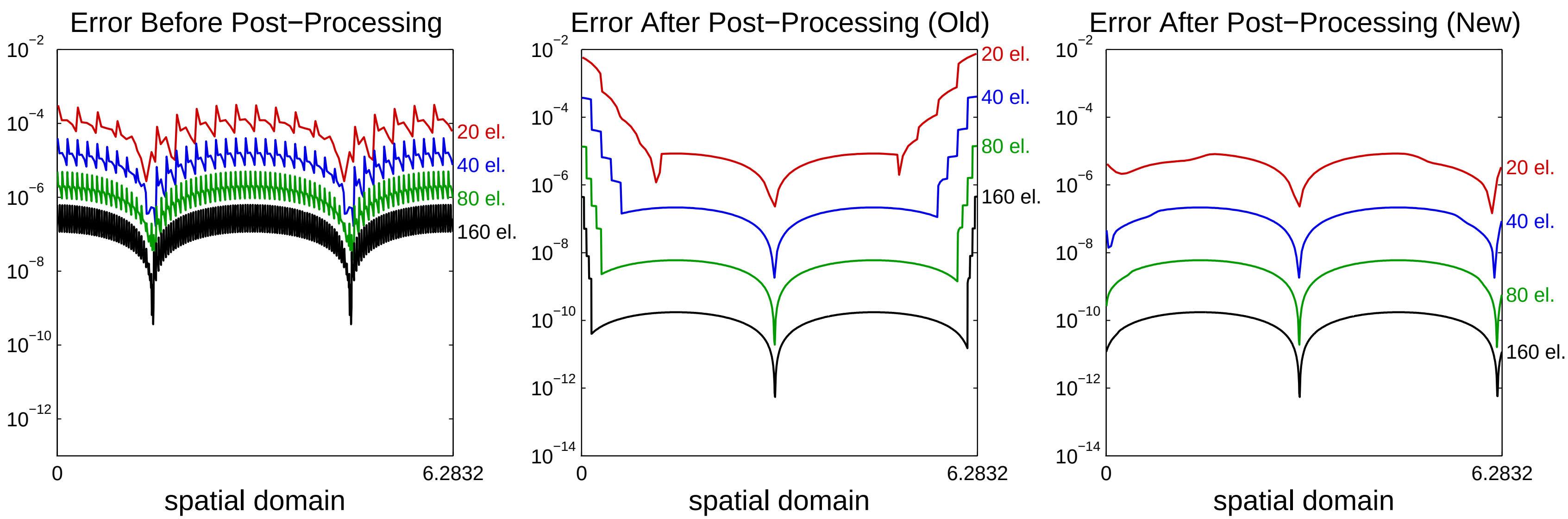
Old One-Sided Post-Processing: : $u_t + u_x = 0$, *periodic BC*

Problem 1: discontinuities are not eliminated (stair-stepping)
Problem 2: the errors at the boundary can be worse than before



New One-Sided Post-Processing: : $u_t + u_x = 0$, *periodic BC*

Problem 1: not all discontinuities are eliminated (stair-stepping)
Problem 2: the errors at the boundary can be worse than before



These problems can be solved through a new type of one-sided post-processing (following slides)

New One-Sided Post-Processing

The discontinuities can be avoided by using kernel nodes that depend continuously on the evaluation point through the shift function $\lambda(\bar{x})$:

$$u_h^\star(\bar{x}) = \sum_{\gamma=0}^{2k} c_\gamma(\bar{x}) \int_I \psi_h^{(k+1)}\Big(x - \underbrace{(\lambda(\bar{x}) + \gamma)}_{\text{kernel node}}\Big)\, u_h(\bar{x} - x)\, dx.$$

van Slingerland, Ryan, & Vuik (2009).

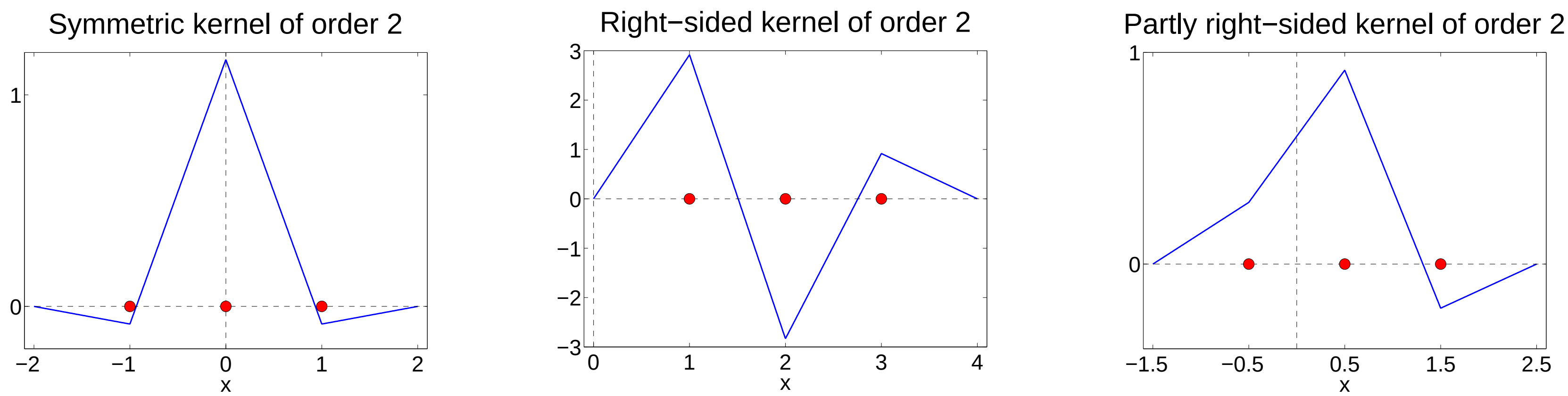
New One-Sided Post-Processing

The discontinuities can be avoided by using kernel nodes that depend continuously on the evaluation point through the shift function $\lambda(\bar{x})$:

$$u_h^\star(\bar{x}) = \sum_{\gamma=0}^{2k} c_\gamma(\bar{x}) \int_I \psi_h^{(k+1)}\Big(x - \underbrace{(\lambda(\bar{x}) + \gamma)}_{\text{kernel node}}\Big)\, u_h(\bar{x} - x)\, dx.$$

Three examples (the kernel nodes are indicated by the red circles):

$\lambda(\bar{x}) = -k$ | $\lambda(\bar{x}) = \frac{k+1}{2}$ | $\lambda(\bar{x}) = -0.5$



Use in the domain interior. | Use at the left boundary | Use near the left boundary

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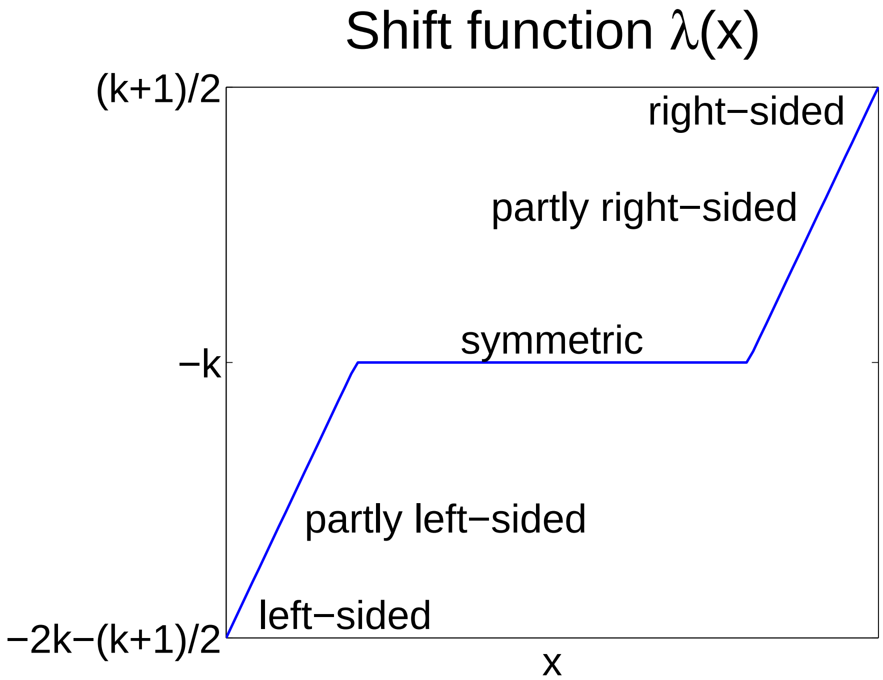
New One-Sided Post-Processing

The discontinuities can be avoided by using kernel nodes that depend continuously on the evaluation point through the shift function $\lambda(\bar{x})$:

$$u_h^\star(\bar{x}) = \sum_{\gamma=0}^{2k} c_\gamma(\bar{x}) \int_I \psi_h^{(k+1)}\big(x - \underbrace{(\lambda(\bar{x}) + \gamma)}_{\text{kernel node}}\big)\, u_h(\bar{x} - x)\, dx.$$



New One-Sided Post-Processing

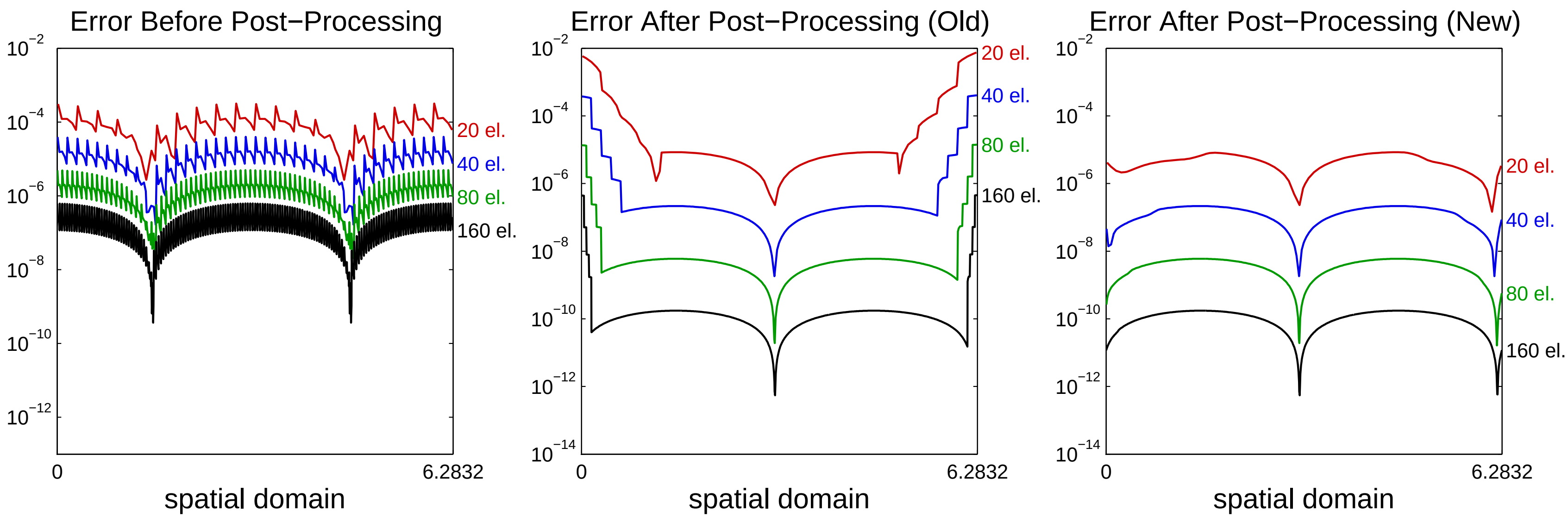
The accuracy near the boundary can be improved by using extra kernel nodes in that region.

$$\underbrace{u_h^\star(\bar{x}) = \theta(\bar{x}) \underbrace{u^\star_{h,2k+1}(\bar{x})}_{\text{filtering with } 2k+1 \text{ nodes}} + (1-\theta(\bar{x})) \underbrace{u^\star_{h,4k+1}(\bar{x})}_{\text{filtering with } 4k+1 \text{ nodes}}}_{\text{smooth convex combination}}$$

- In the interior: $\theta(\bar{x}) = 1$ (old filter suffices)
- Near the boundary: $\theta(\bar{x}) = 0$ (extra accuracy through extra nodes)
- Transition regions: choose θ smooth

New One-Sided Post-Processing: $u_t + u_x = 0$, *periodic BC*

The new post-processor improves both the convergence rate and the absolute value of the errors for a problem with a periodic BC



New One-Sided Post-Processing: $u_t + u_x = 0$, *periodic BC*

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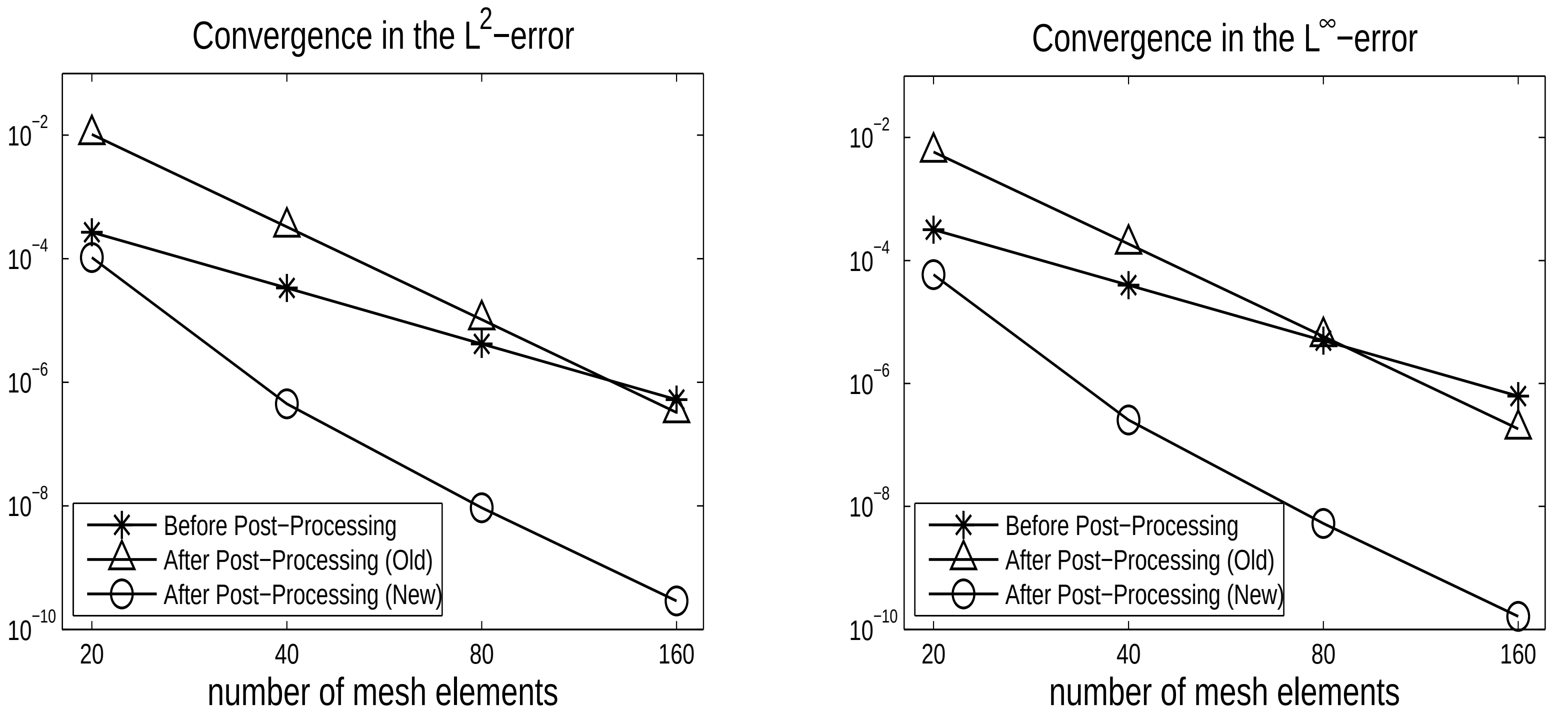


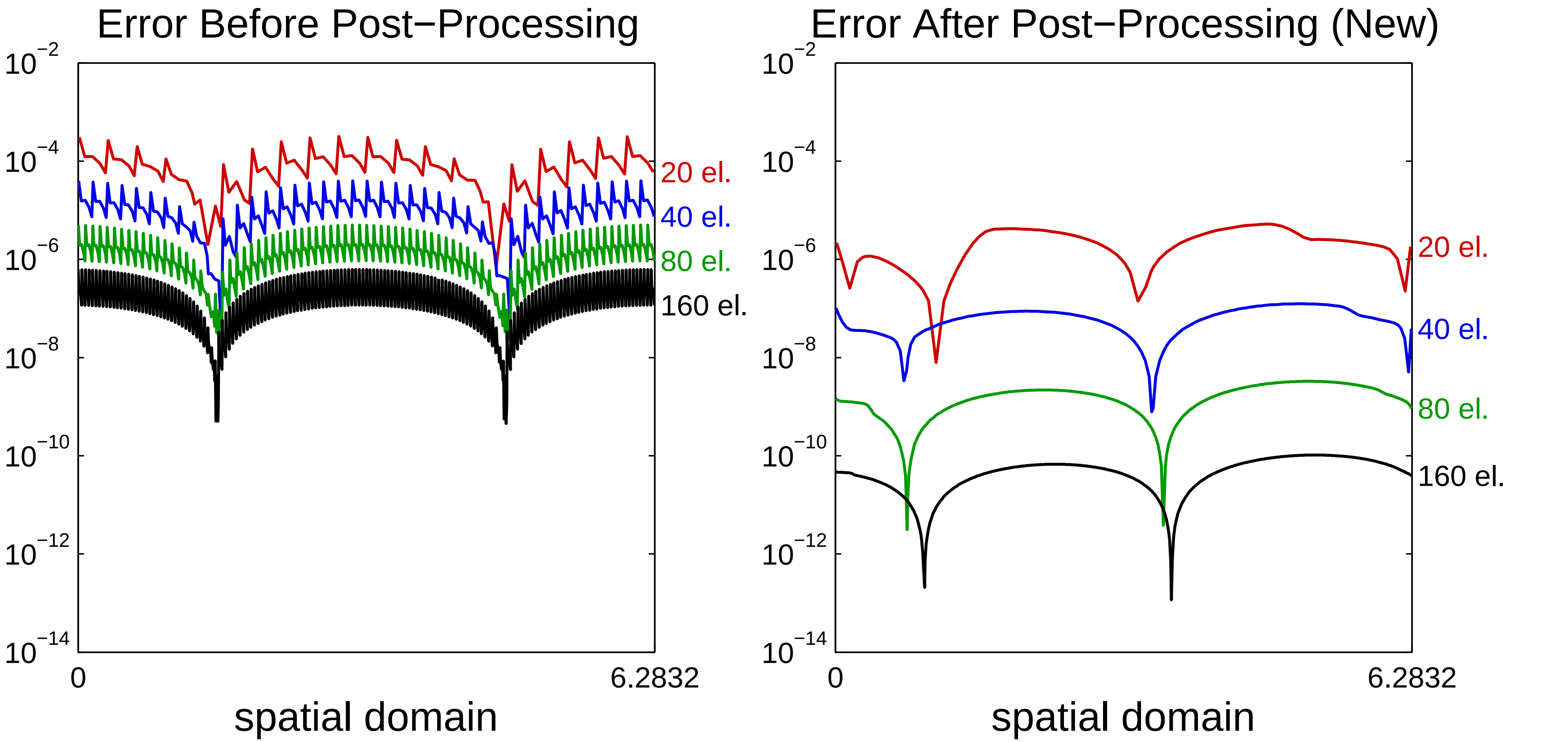
New One-Sided Post-Processing: $u_t + u_x = 0$, *periodic BC*

The new post-processor improves both the convergence rate and the absolute value of the errors for a problem with a periodic BC

	Before		**After (Old)**		**After (New)**	
mesh	L^2-error	order	L^2-error	order	L^2-error	order
	Polynomial Degree k = 2					
20	2.683e-04	-	4.003e-03	-	1.301e-05	-
40	3.352e-05	3.00	2.108e-04	4.25	3.767e-07	5.11
80	4.190e-06	3.00	5.464e-06	5.27	1.056e-08	5.16
160	5.238e-07	3.00	1.254e-07	5.45	3.090e-10	5.10
	Polynomial Degree k = 3					
20	5.176e-06	-	1.304e-04	-	3.757e-07	-
40	3.236e-07	4.00	4.712e-06	4.79	6.634e-10	9.15
80	2.023e-08	4.00	3.406e-08	7.11	2.957e-12	7.81
160	1.264e-09	4.00	1.999e-10	7.41	1.287e-14	7.84

New One-Sided Post-Processing: $u_t + u_x = 0$, *Dirichlet BC*

The new post-processor improves both the convergence rate and the absolute value of the errors for a problem with a Dirichlet BC



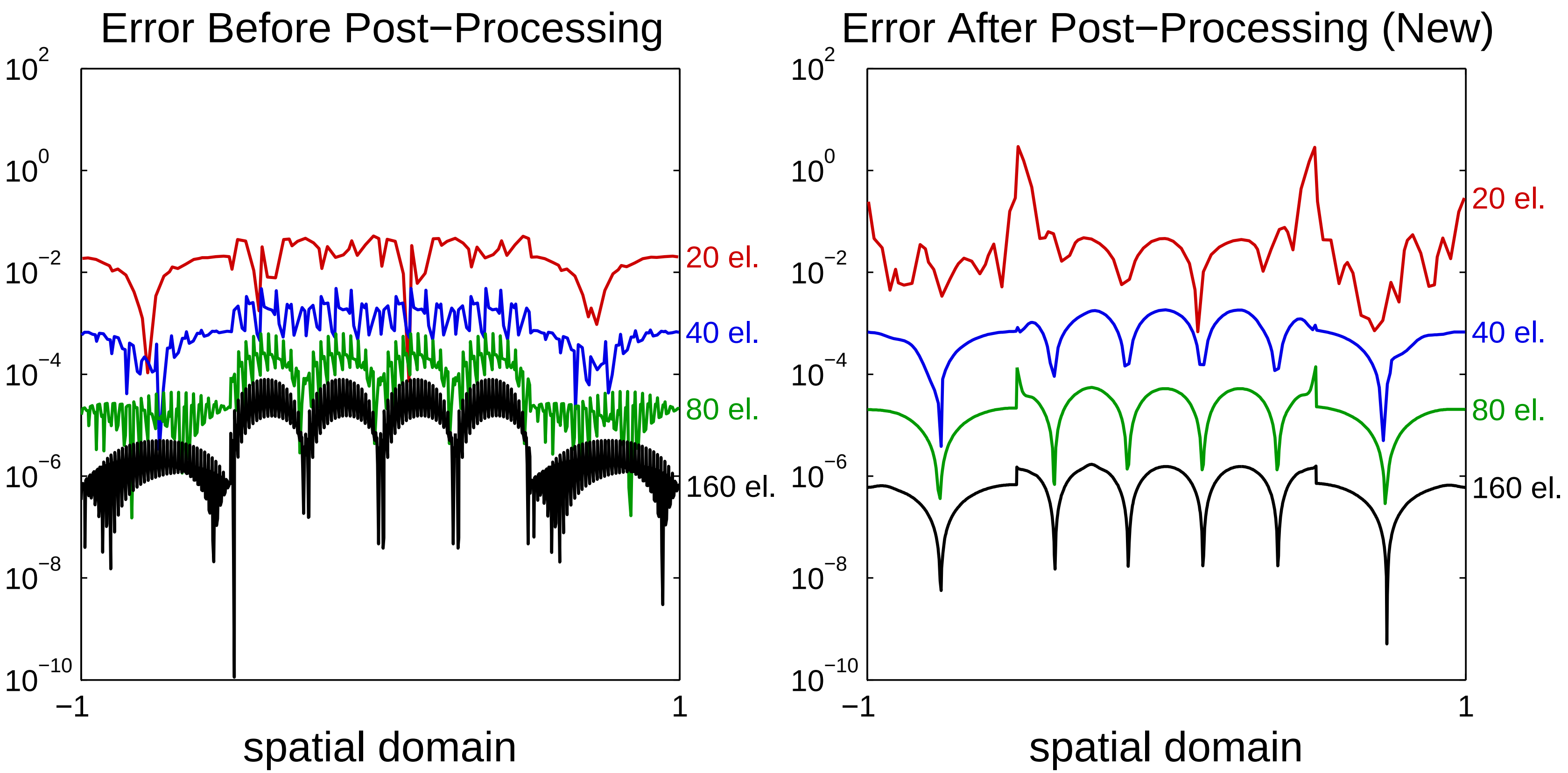
New One-Sided Post-Processing: $u_t + u_x = 0$, *Dirichlet BC*

The new post-processor improves both the convergence rate and the absolute value of the errors for a problem with a Dirichlet BC

	Before		**After (Old)**		**After (New)**	
mesh	L^2-error	order	L^2-error	order	L^2-error	order
	Polynomial Degree k = 2					
20	2.681e-04	-	4.003e-03	-	6.984e-06	-
40	3.352e-05	3.00	2.108e-04	4.25	1.850e-07	5.24
80	4.190e-06	3.00	5.464e-06	5.27	4.798e-09	5.27
160	5.238e-07	3.00	1.254e-07	5.45	1.498e-10	5.00
	Polynomial Degree k = 3					
20	5.176e-06	-	1.304e-04	-	3.751e-07	-
40	3.236e-07	4.00	4.712e-06	4.79	6.396e-10	9.20
80	2.023e-08	4.00	3.406e-08	7.11	2.867e-12	7.80
160	1.264e-09	4.00	1.999e-10	7.41	3.079e-14	6.54

New One-Sided Post-Processing: $u_t + au_x = 0$, *a discontinuous*

For this problem with two stationary shocks, the post-processor requires a sufficiently fine mesh



New One-Sided Post-Processing: $u_t + au_x = 0$, *a discontinuous*

For this problem with two stationary shocks, the post-processor requires a sufficiently fine mesh

	Before		After (Old)		After (New)	
mesh	L^2-error	order	L^2-error	order	L^2-error	order
	Polynomial Degree k = 2					
20	3.646e-02	-	6.808e+00	-	5.709e-01	-
40	2.052e-03	4.15	1.672e-01	5.35	1.249e-03	8.84
80	2.173e-04	3.24	6.027e-03	4.79	4.166e-05	4.91
160	2.682e-05	3.02	8.414e-05	6.16	1.181e-06	5.14
	Polynomial Degree k = 3					
20	1.085e-03	-	3.579e+00	-	2.270e-01	-
40	6.602e-05	4.04	1.865e-02	7.58	2.640e-03	6.43
80	4.132e-06	4.00	6.502e-04	4.84	5.205e-06	8.99
160	2.584e-07	4.00	2.623e-06	7.95	4.670e-09	10.12

Summary

- Using B-splines allows us to induce smoothness on the DG field and enhance accuracy.
- We can obtain this improvement from order k+1 to order 2k+1 for smoothly varying meshes as well as derivatives of the DG solution.
- Recent extensions allow us to have the improvement in accuracy near the boundaries as well.
  - The kernel is adjusted according to the point we would like to post-process.
  - Near the boundary, we use more kernel nodes.
- We can use this post-processing technique as a visualisation tool to maintain more accurate streamlines.

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