Coupled preconditioners for the Incompressible Navier Stokes Equations

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Messages

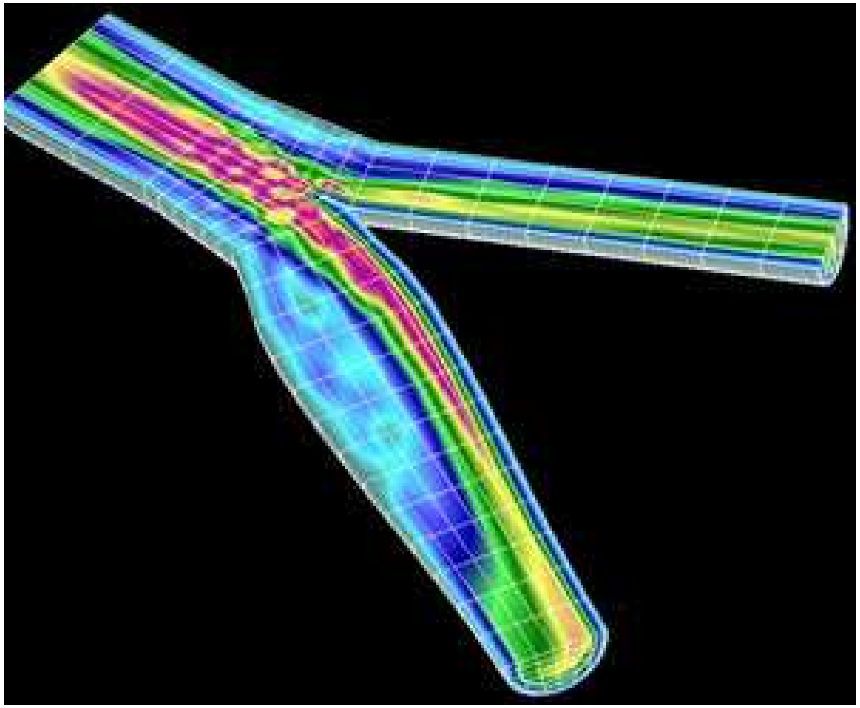
1. Incompressible Navier-Stokes are important
2. Much progress in solvers for academic test problems
3. Transfer methods to industrial problems

Outline

1. Introduction
2. Problem
3. Krylov solvers and preconditioners
4. ILU-type preconditioners
5. Block preconditioners
   - SIMPLE
   - Augmented Lagrangian
6. Maritime Applications
7. Conclusions

1. Introduction

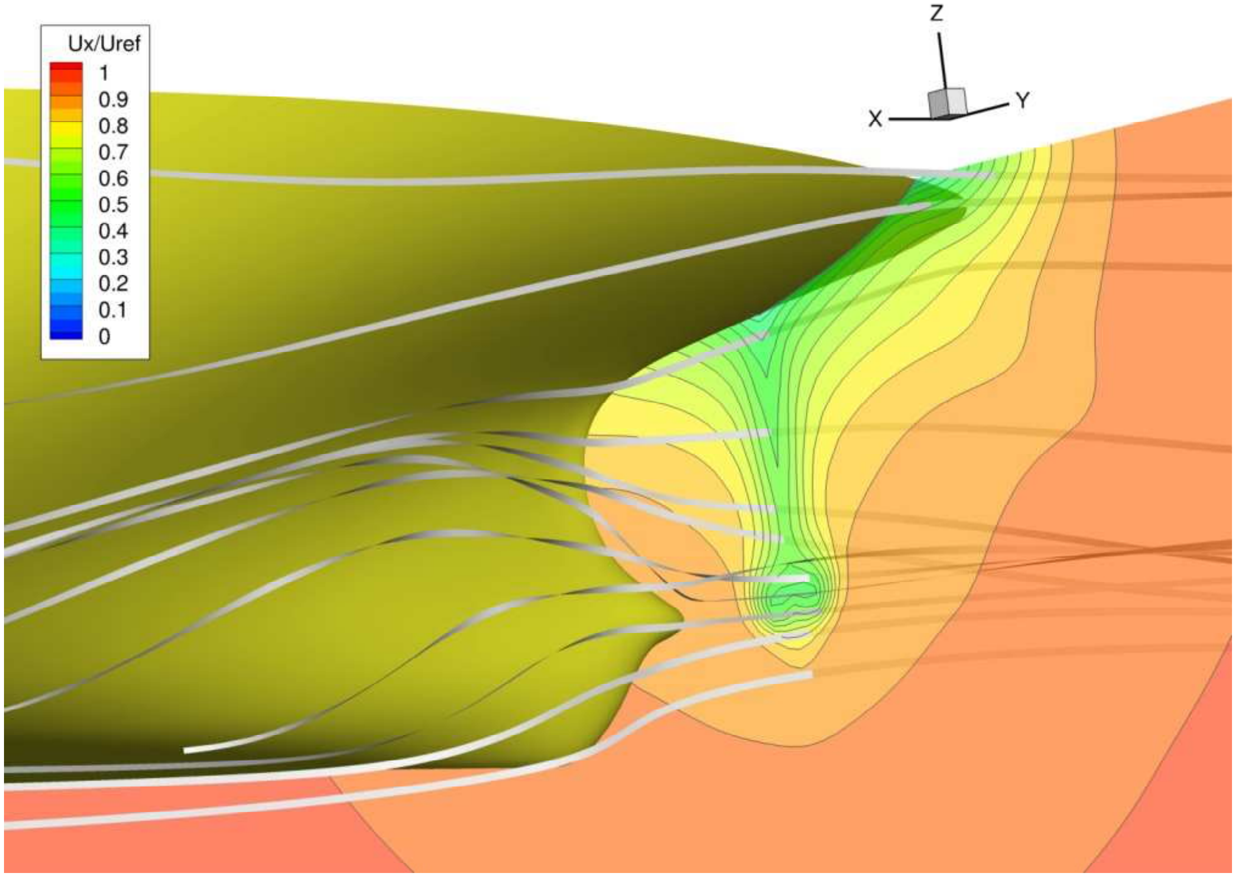
Flow in arteries



Introduction

Flooding of the Netherlands, 1953

Introduction



Streamlines around the stern and the axial velocity field in the wake.

2. Problem

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \;\; \Omega.$$

$\mathbf{u}$ is the fluid velocity vector

p is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \;\; \text{on} \;\; \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \;\; \text{on} \;\; \partial\Omega_N.$$

Linear system

Matrix form after linearization and discretization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, A is vector Laplacian matrix
- $F = \nu A + N$ in Picard linearization, N is vector-convection matrix
- $F = \nu A + N + W$ in Newton linearization, W is the Newton derivative matrix
- B is the divergence matrix
- Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.
- Saddle point problem having large number of zeros on the main diagonal

3. Krylov Solvers and preconditioners

- **Direct method:**
  To solve $\mathcal{A}x = b$,
  factorize $\mathcal{A}$ into upper U and lower L triangular matrices ($LUx = b$)
  First solve $Ly = b$, then $Ux = y$

- **Classical Iterative Schemes:**
  Methods based on matrix splitting, generates sequence of iterations
  $x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s$, where $\mathcal{A} = M - N$
  Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**
  $x_{k+1} = x_k + \alpha_k p_k$
  Some well known methods are
  CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986], GMRESR[1994], GCR[1986], IDR(s)[2007]

IDR and IDR(s) (Induced Dimension Reduction)

- Sonneveld developed IDR in the 1970's. IDR is a finite termination (Krylov) method for solving nonsymmetric linear systems.
- Analysis showed that IDR can be viewed as Bi-CG combined with linear minimal residual steps.
- This discovery led to the development of first CGS, and later of Bi-CGSTAB (by van der Vorst).

IDR and IDR(s) (continued)

- As a result of these developments the basic IDR-idea was abandoned for the Bi-CG-approach.

- Recently, Sonneveld and van Gijzen discovered that the IDR-approach was abandoned too soon and proposed a generalization of IDR: IDR(s).

- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
  SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

More information: http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html

4. ILU-type Preconditioners

A linear system $\mathcal{A}x = b$ is transformed into $P^{-1}\mathcal{A}x = P^{-1}b$ such that

- $P \approx \mathcal{A}$
- Eigenvalues of $P^{-1}\mathcal{A}$ are more clustered than $\mathcal{A}$
- $Pz = r$ cheap to compute

Several approaches, we will discuss here

- ILU preconditioner
- Preconditioned IDR(s) and Bi-CGSTAB comparison
- Block preconditioners

SILU preconditioners

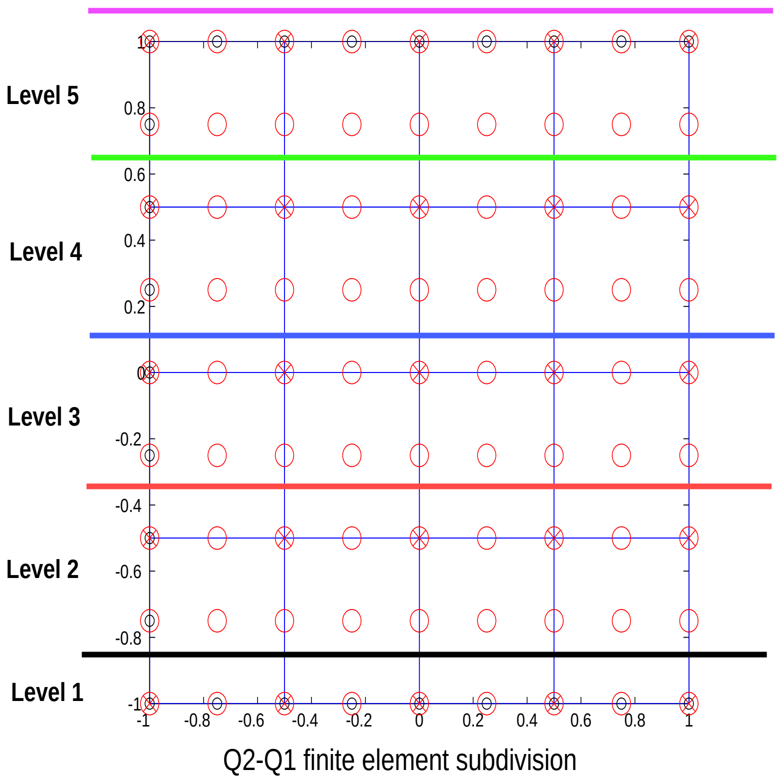
New renumbering Scheme

- Renumbering of grid points:
  - Sloan algorithm [Sloan - 1986]
  - Cuthill McKee algorithms [Cuthill McKee - 1969]
- The unknowns are reordered by p-last or p-last per level methods
  - In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually it produces a large profile but avoids breakdown of LU decomposition.
  - In **p-last per level reordering**, unknowns are reordered per level such that at each level, the velocity unknowns are followed by the pressure unknowns.

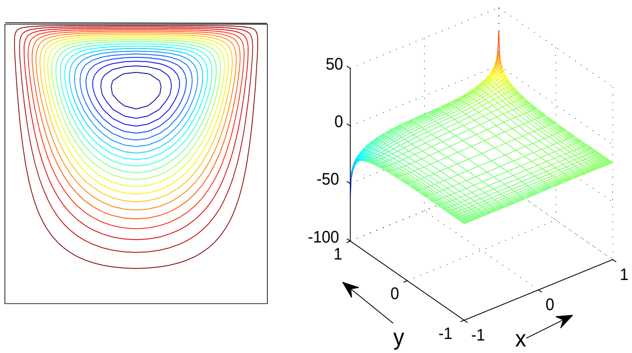
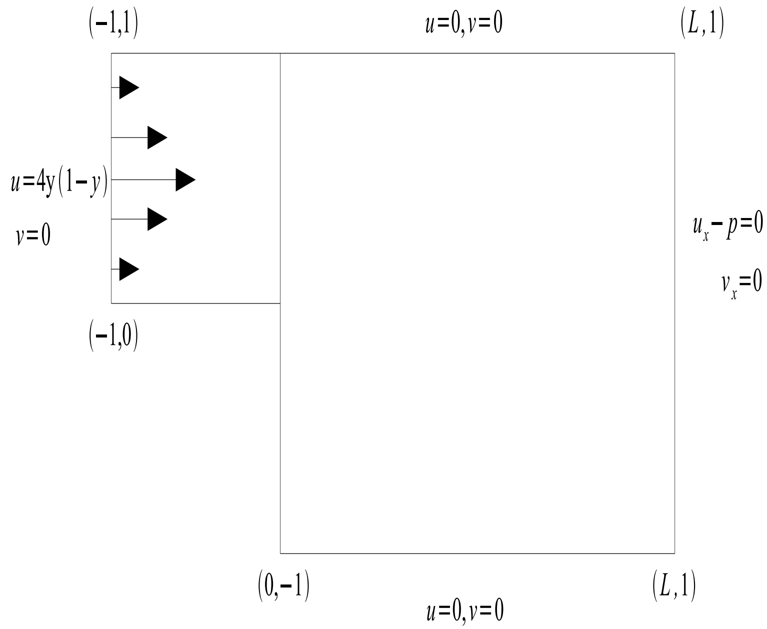
I.N. Konshin, M.A. Olshanskii, Yu.V. Vassilevski, ILU preconditioners for non-symmetric saddle point matrices with application to the incompressible Navier-Stokes equations, SIAM J.Sci.Comp., 37 (2015), A2171-A2197

SILU preconditioner

4×4 Q2-Q1 grid



Numerical experiments (SILU preconditioner)

Driven cavity flow problem	Backward facing step problem
	

The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol$.

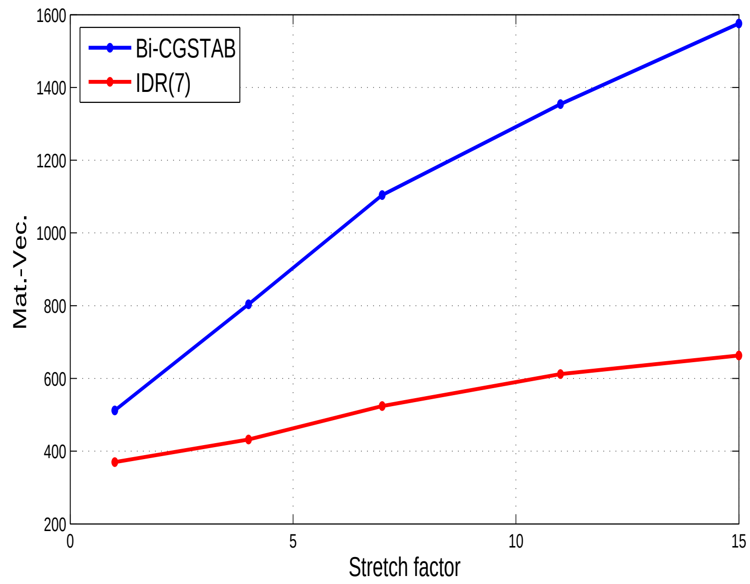
Numerical experiments (SILU preconditioners)

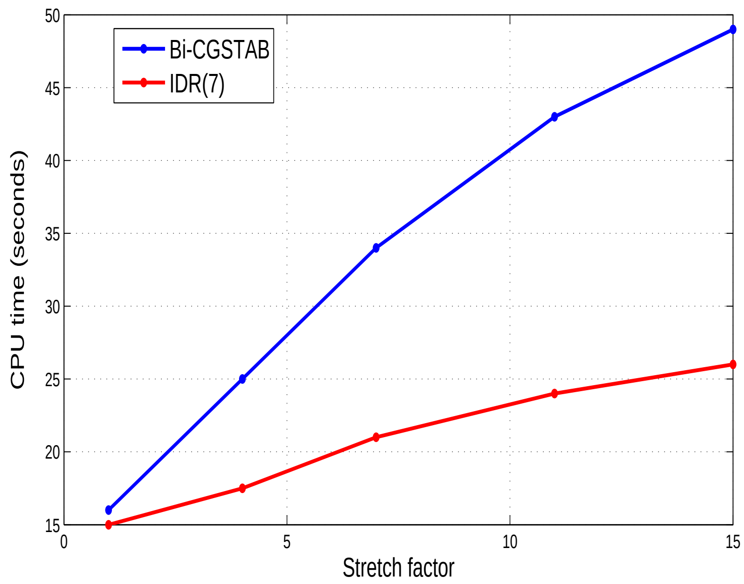
Stokes Problem in a square domain with Bi-CGSTAB, $accuracy = 10^{-6}$, Sloan renumbering

	$Q2 - Q1$		$Q2 - P1$	
Grid size	p-last	p-last per level	p-last	p-last per level
16×16	36(0.11)	25(0.09)	44(0.14)	34(0.13)
32×32	90(0.92)	59(0.66)	117(1.08)	75(0.80)
64×64	255(11.9)	135(6.7)	265(14)	165(9.0)
128×128	472(96)	249(52)	597(127)	407(86)

Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for increasing stretch factor for the driven cavity Stokes problem.





Numerical Experiments (IDR(s) vs Bi-CGSTAB(l))

SILU preconditioned: Comparison of iterative methods

Driven Cavity Stokes problem, stretch factor 10

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
128×128	1104(36.5)	4	638(24.7)	6
256×256	5904(810)	6	1749(307)	8

Channel flow Stokes problem, length 100

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
64×64	1520(12)	4	938(8.7)	8
128×128	NC	6	8224(335)	8

5. Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1}B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

Preconditioners are based on combination of these blocks involve:

$Fz_1 = r_1$ The velocity subsystem

$$S \longrightarrow \hat{S}$$

$\hat{S}z_2 = r_2$ The pressure subsystem

Block preconditioners

Block triangular preconditioners

$$P_t = \mathcal{U}_{bt} = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}$$

- Pressure convection diffusion (PCD) [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$, Q_p is the pressure mass matrix
- Least squares commutator (LSC) [Elman et al, 2002]
  $\hat{S} = -(BQ_u^{-1}B^T)(BQ_u^{-1}FQ_u^{-1}B^T)^{-1}(BQ_u^{-1}B^T)$, Q_u is the velocity mass matrix
- Augmented Lagrangian approach (AL) [Benzi and Olshanskii, 2006]
  F is replaced by $F_\gamma = F + \gamma B W^{-1} B^T$
  $\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1}), \; W = \hat{Q}_p$
- Benzi, Golub, and Liesen, Numerical Solution of Saddle Point Problem, Acta Numerica, 2005

Block preconditioners (SIMPLE)

SIMPLE-type preconditioners[Vuik et al-2000]

SIMPLE	SIMPLER
$z = \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}r$	$z = \mathcal{U}_{bt}^{-1}\mathcal{L}_b^{-1}r$
	$z = z + \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$
$M_u = D$	$M_l = M_u = D$, $D = diag(F)$
$\hat{S} = BD^{-1}B^T$	$\hat{S} = BD^{-1}B^T$
One Poisson solve	Two Poisson solves
One velocity solve	Two velocity solves

Lemma: In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical .

Improvements in SIMPLE-type preconditioners

We use approximate solvers for subsystems, so flexible Krylov solvers are required (GCR, FGMRES, GMRESR)

MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.
LSC: $\hat{S} \approx -(B\hat{Q_u}^{-1}B^T)(B\hat{Q_u}^{-1}\underbrace{F\hat{Q_u}^{-1}}B^T)^{-1}(B\hat{Q_u}^{-1}B^T)$

assuming $F\hat{Q_u}^{-1} \approx I$ (e.g. time dependent problems with a small time step)

$\hat{S} = -B\hat{Q_u}^{-1}B^T$

MSIMPLER uses this approximation for the Schur complement and updates scaled with $\hat{Q_u}^{-1}$.

-Convergence better than other variants of SIMPLE
-Cheaper than SIMPLER (in construction) and LSC (per iteration)

Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-6} (SEPRAN) using Q2-Q1 hexahedrons

Grid	SIMPLE	LSC	MSIMPLER
iter. (t_s) $\frac{\text{in-it-}u}{\text{in-it-}p}$			
$8 \times 8 \times 16$	44(4) $\frac{97}{342}$	16(1.9) $\frac{41}{216}$	14(1.4) $\frac{28}{168}$
$16 \times 16 \times 32$	84(107) $\frac{315}{1982}$	29(51) $\frac{161}{1263}$	17(21) $\frac{52}{766}$
$24 \times 24 \times 48$	99(447) $\frac{339}{3392}$	26(233) $\frac{193}{2297}$	17(77) $\frac{46}{1116}$
$32 \times 32 \times 40$	132(972) $\frac{574}{5559}$	37(379) $\frac{233}{2887}$	20(143) $\frac{66}{1604}$

Numerical Experiments (comparison)

3D Lid driven cavity problem (tetrahedrons):The Navier-Stokes problem is solved with accuracy 10^{-4}, a linear system at each Picard step is solved with accuracy 10^{-2} using preconditioned Krylov subspace methods. Bi-CGSTAB is used as inner solver in block preconditioners(SEPRAN)

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 16$			
20	30(20)	20(16)	144(22)
50	57(37)	37(24)	234(35)
100	120(81)	68(44)	427(62)
$32 \times 32 \times 32$			
20	38(234)	29(144)	463(353)
50	87(544)	53(300)	764(585)
100	210(1440)	104(654)	1449(1116)

Numerical Experiments (comparison)

2D Lid driven cavity problem on 64×64 stretched grid: The Stokes problem is solved with accuracy 10^{-6}. PCG is used as inner solver in block preconditioners (SEPRAN) .

Stretch factor	LSC	MSIMPLER	SILU
	GCR iter.	GCR iter.	Bi-CGSTAB iter.
1	20	17	96
8	49	28	189
16	71	34	317
32	97	45	414
64	145	56	NC
128	NC	81	NC

The Augmented Lagrangian method

$$\begin{bmatrix} F & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$ is transformed into

$$\begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \hat{f} \\ g \end{bmatrix} \quad \text{or} \quad \mathcal{A}_{AL} \mathbf{x} = \hat{b},$$

with $\hat{f} = f + \gamma B^T W^{-1} B\, g$, where W is a non-singular matrix.

The *Ideal* AL preconditioner proposed for $\mathcal{A}_{AL}$ is

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix}.$$

The Augmented Lagrangian method

$$\mathcal{A}_{AL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & B^T \\ B & 0 \end{bmatrix} \qquad (S_{AL} = -B(F + \gamma B^T W^{-1} B)^{-1} B^T)$$

$$\mathcal{P}_{IAL} = \begin{bmatrix} F + \gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{bmatrix} \qquad (F_\gamma = F + \gamma B^T W^{-1} B)$$

- The Schur complement S_{AL} of $\mathcal{A}_{AL}$ is approximated by $-\frac{1}{\gamma}W$.
- The block F_γ becomes increasingly ill-conditioned with $\gamma \to \infty$.
- In practice it is often chosen as $\gamma = 1$, or $\gamma = O(1)$, and $W = \hat{Q}_P$.
- Open question: fast solution methods for systems with F_γ, which is denser than F and consists of mixed derivatives.

[1] M. Benzi and M.A. Olshanskii. An augmented Lagrangian-based approach to the Oseen problem. *SIAM J. Sci. Comput.*, 28:2095-2113, 2006.

The Augmented Lagrangian method

The transformed coefficient matrix $\mathcal{A}_{AL} = \left[\begin{smallmatrix} F+\gamma B^T W^{-1} B & B^T \\ B & 0 \end{smallmatrix}\right]$ and the ideal AL precondition $\mathcal{P}_{IAL} = \left[\begin{smallmatrix} F+\gamma B^T W^{-1} B & 0 \\ B & -\frac{1}{\gamma} W \end{smallmatrix}\right]$ includes (in 2D)

- the convection-diffusion block: $F = \left[\begin{smallmatrix} F_{11} & O \\ O & F_{11} \end{smallmatrix}\right]$,
- the (negative) divergence matrix: $B = \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]$,
- the modified pivot block $F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}$.

One approximation of F_γ is $\widetilde{F}_\gamma = \left[\begin{smallmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & O \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{smallmatrix}\right]$, which leads to the modified AL preconditioner $\mathcal{P}_{MAL}$ for $\mathcal{A}_{AL}$.

The Augmented Lagrangian method

$$\mathcal{P}_{IAL} = \begin{bmatrix} F_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(F_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & \gamma B_1^T W^{-1} B_2 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

$$\mathcal{P}_{MAL} = \begin{bmatrix} \widetilde{F}_\gamma & 0 \\ B & -\frac{1}{\gamma}W \end{bmatrix} \qquad \left(\widetilde{F}_\gamma = \begin{bmatrix} F_{11}+\gamma B_1^T W^{-1} B_1 & 0 \\ \gamma B_2^T W^{-1} B_1 & F_{11}+\gamma B_2^T W^{-1} B_2 \end{bmatrix}\right)$$

- systems with $\widetilde{F}_\gamma$ are easier to be solved, compared to F_γ.
- the number of iterations by using the ideal and modified AL preconditioners are both independent of the mesh refinement, and nearly independent of the Reynolds (viscosity) number.
- by using the modified AL preconditioner, there exists an optimal value of γ, which minimises the number of Krylov subspace iterations. The optimal γ is problem dependent, but mesh size independent.

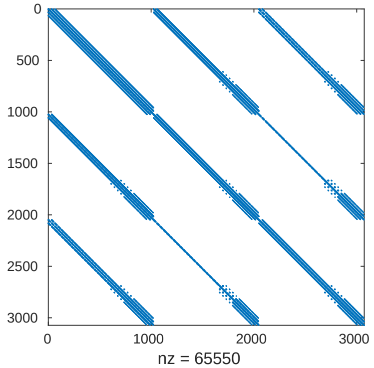
Numerical experiments (Lid driven cavity)

Re	100	400	1000	$2500^{\star}$	$5000^{\star}$
		modified AL preconditioner			
Newton iterations:	6	7	7	8	9
GCR iterations:	8	14	21	33	50
total time:	14.8	26.2	74.6	194.2	277.1
		modified 'grad-div' preconditioner			
Newton iterations:	6	7	8	9	9
GCR iterations:	10	17	28	53	77
total time:	8.5	15.7	32.7	119.1	167.9
		modified SIMPLER preconditioner			
Newton iterations:	10	$8^{\star}$	$8^{\star}$	11	15
GCR iterations:	43	82	84	80	90
total time:	68.3	102.9	232.8	203.2	561.6

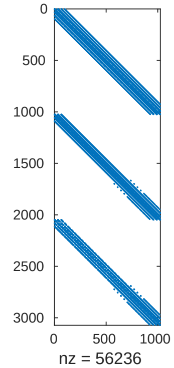
Numerical experiments

BFS ($Re = 100$) grids:	24×12	48×24	72×36	96×48
$\mathcal{P}_{MAL}$ with $\widetilde{\gamma}_{opt} = 40$				
Iter.$_{\text{Picard}}$	48	48	48	48
Iter.$_{\text{Linear}}$	13	13	13	13
$\mathcal{P}_t$				
Iter.$_{\text{Picard}}$	48	48	48	48
Iter.$_{\text{Linear}}$	66	66	66	66
LDC ($Re = 5000$) grids:	16^2	32^2	64^2	128^2
$\mathcal{P}_{MAL}$ with $\widetilde{\gamma}_{opt} = 50$				
Iter.$_{\text{Picard}}$	116	200	191	135
Iter.$_{\text{Linear}}$	14	20	25	32
$\mathcal{P}_t$				
Iter.$_{\text{Picard}}$	116	199	189	134
Iter.$_{\text{Linear}}$	22	50	134	>400

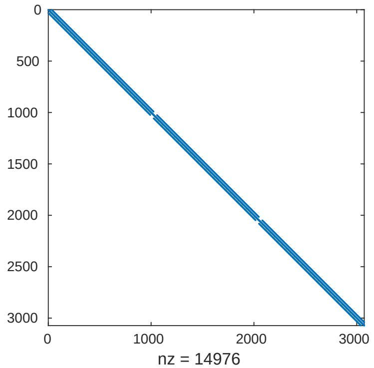
Numerical experiments sparseness of the matrices



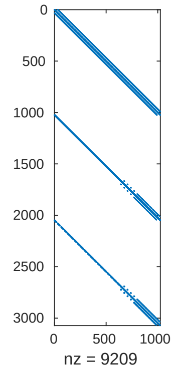
(a) Q_γ



(b) G_γ



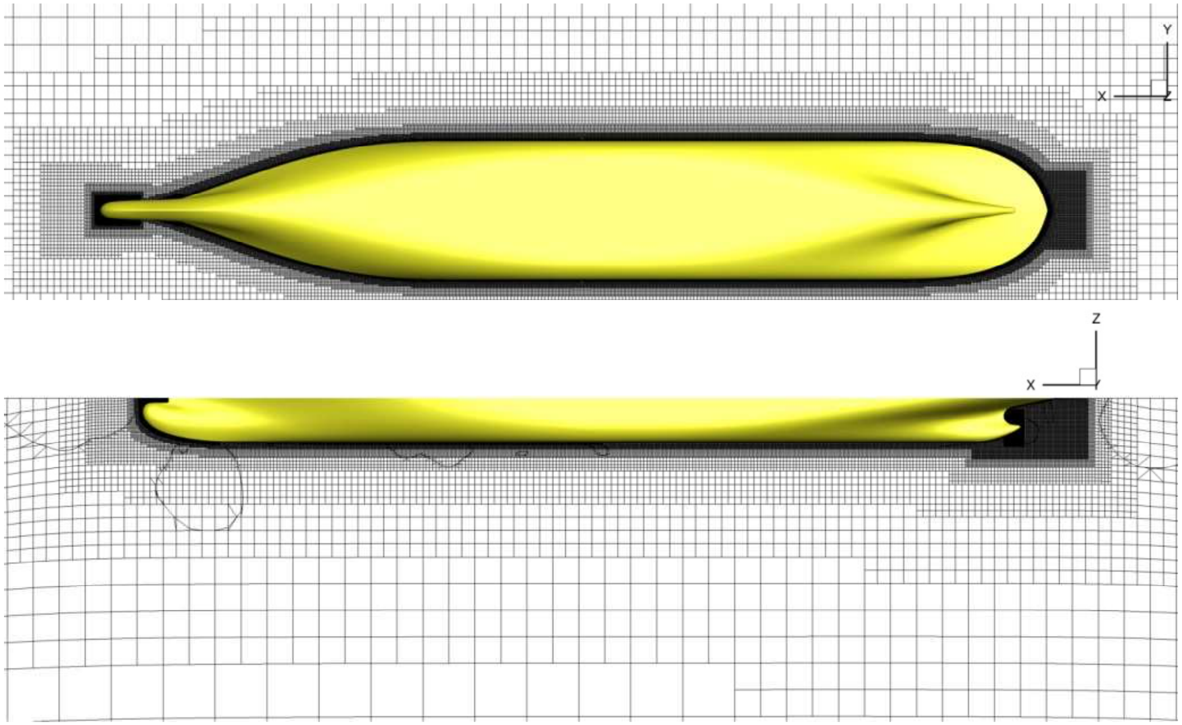
(c) Q



(d) G

6. Maritime Applications

Container vessel (unstructured grid)



RaNS equations

k-ω turbulence model

$y^+ \approx 1$

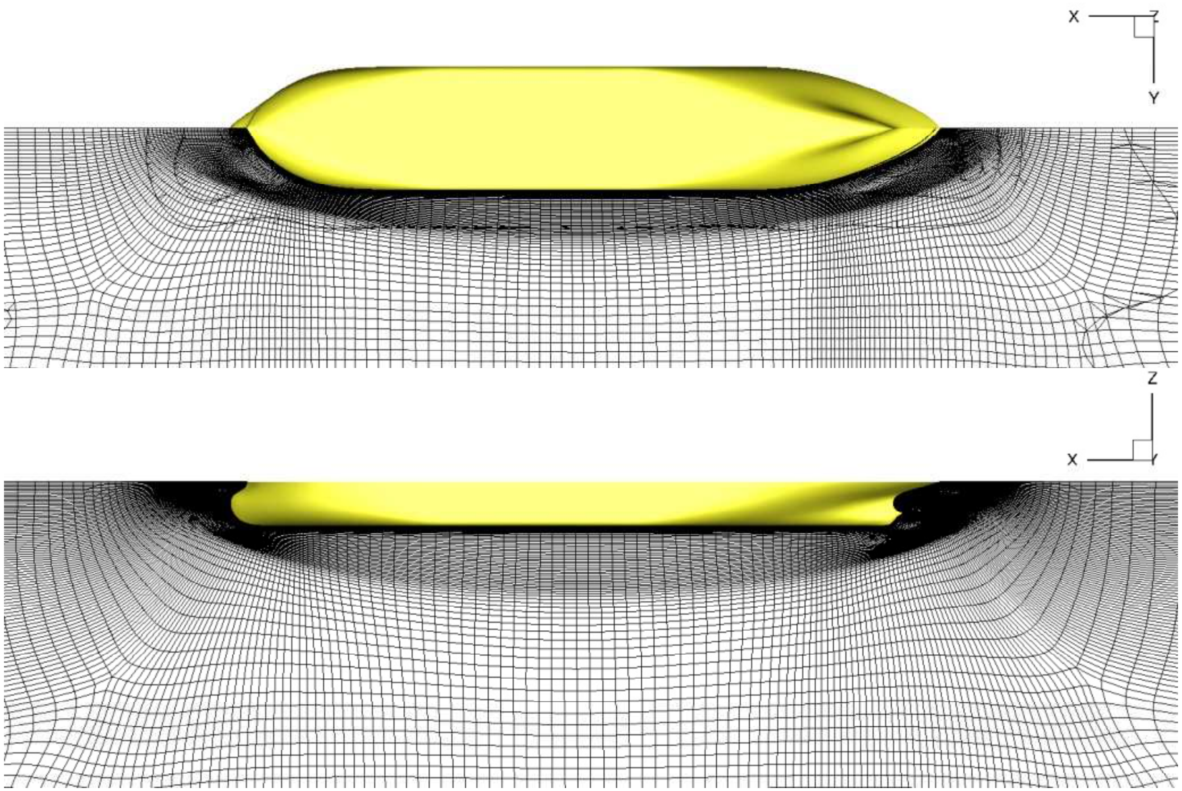
Model-scale:

$\mathrm{Re} = 1.3 \cdot 10^7$

13.3m cells

max aspect ratio $1 : 1600$

Tanker (block-structured grid)



Model-scale:

$\mathrm{Re} = 4.6 \cdot 10^6$

2.0m cells

max aspect ratio $1 : 7000$

Full-scale:

$\mathrm{Re} = 2.0 \cdot 10^9$

2.7m cells

max aspect ratio $1 : 930\,000$

Discretization

Co-located, cell-centered finite volume discretization of the steady Navier-Stokes equations with Picard linearization leads to linear system:

$$\begin{bmatrix} Q_1 & 0 & 0 & G_1 \\ 0 & Q_2 & 0 & G_2 \\ 0 & 0 & Q_3 & G_3 \\ D_1 & D_2 & D_3 & C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ g \end{bmatrix} \qquad \text{for brevity:} \begin{bmatrix} Q & G \\ D & C \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

with $Q_1 = Q_2 = Q_3$.

$\Rightarrow$ Solve system with FGMRES and SIMPLE-type preconditioner
Turbulence equations (k-ω model) remain segregated

SIMPLE-method

Given u^k and p^k:

1. solve $Qu^* = f - Gp^k$
2. solve $(C - DQ^{-1}G)p' = g - Du^* - Cp^k$
3. compute $u' = -Q^{-1}Gp'$
4. update $u^{k+1} = u^* + u'$ and $p^{k+1} = p^k + p'$

with the SIMPLE approximation $Q^{-1} \approx \mathrm{diag}(Q)^{-1}$.
$\Rightarrow$ “Matrix-free”: only assembly and storage of Q and $(C - DQ^{-1}G)$. For D, G and C the action suffices.

SIMPLER: additional pressure prediction

Given u^k and p^k, start with a pressure prediction:

1. solve $(C - D\,\mathrm{diag}(Q)^{-1}G)p^* = g - Du^k - D\,\mathrm{diag}(Q)^{-1}(f - Qu^k)$
2. continue with SIMPLE using p^* instead of p^k

Container vessel

Tables show number of non-linear iterations and wall clock time needed to converge to machine precision, starting from uniform flow.

Model-scale $\mathrm{Re} = 1.3 \cdot 10^7$, max cell aspect ratio $1 : 1600$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		# its	Wall clock	# its	Wall clock
13.3m	128	3187	5h 26mn	427	3h 27mn

Tanker

Model-scale $\mathrm{Re} = 4.6 \cdot 10^6$, max cell aspect ratio $1 : 7000$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		its	Wall clock	its	Wall clock
0.25m	8	1379	25mn	316	29mn
0.5m	16	1690	37mn	271	25mn
1m	32	2442	57mn	303	35mn
2m	64	3534	1h 29mn	519	51mn

Full-scale $\mathrm{Re} = 2.0 \cdot 10^9$, max cell aspect ratio $1 : 930\,000$

grid	CPU cores	SIMPLE		KRYLOV-SIMPLER	
		its	Wall clock	its	Wall clock
2.7m	64	29 578	16h 37mn	1330	3h 05mn

7. Conclusions

- *MSIMPLER is at present the fastest of all SIMPLE-type preconditioners.*
- *In our experiments, MSIMPLER proved to be cheaper than SILU, especially when the problem is solved with high accuracy.*
- *MSIMPLER shows better performance than LSC. Both have similar convergence characteristics.*
- *For academic problems, Modified Augmented Lagrangian (MAL) and grad-div are nearly independent of the grid size and Reynolds number*
- *MAL/grad-div are faster than (M)SIMPLER*
- *Future research: MAL/grad-div for industrial (Maritime) applications*

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