Deflation with POD vectors for Porous Media Flow

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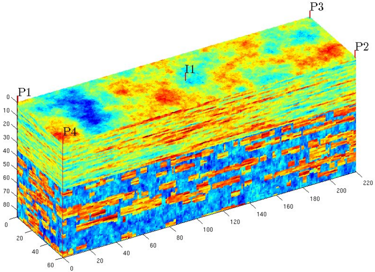
SPE 10

Single-phase flow, grid size 60 x 220 x 85 grid cells.



Method	Number of iterations
ICCG	1011
DICCG	1

Table : Number of iterations for the SPE 10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance of 10^{-7}.

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Problem Definition

Reservoir Simulation

Single-phase flow through a porous media [1]

Darcy's law + mass balance equation

$$-\nabla \cdot \left[\frac{\alpha\rho}{\mu} \vec{\mathbf{K}} (\nabla \mathbf{p} - \rho g \nabla d) \right] + \alpha\rho\phi c_t \frac{\partial \mathbf{p}}{\partial t} - \alpha\rho\mathbf{q} = 0.$$

$$c_t = (c_l + c_r),$$

α a geometric factor

ρ fluid density

μ fluid viscosity

$\mathbf{p}$ pressure

$\vec{\mathbf{K}}$ rock permeability

g gravity

d depth

ϕ rock porosity

$\mathbf{q}$ sources

c_r rock compressibility

c_l liquid compressibility

Problem Definition

Discretization

3D case, isotropic permeability, small rock and fluid compressibilities, uniform reservoir thickness and no gravity forces.

$$-\frac{h}{\mu}\frac{\partial}{\partial x}\left(k\frac{\partial \mathbf{p}}{\partial x}\right)-\frac{h}{\mu}\frac{\partial}{\partial y}\left(k\frac{\partial \mathbf{p}}{\partial y}\right)-\frac{h}{\mu}\frac{\partial}{\partial z}\left(k\frac{\partial \mathbf{p}}{\partial z}\right)+h\phi_0 c_t\frac{\partial \mathbf{p}}{\partial t}-h\mathbf{q}=0.$$

$$\boxed{\mathcal{V}\dot{\mathbf{p}}+\mathcal{T}\mathbf{p}=\mathbf{q}.}$$

$\mathbf{q}$: sources or wells in the reservoir, Peaceman well model, $\mathcal{I}_{well}$ is the well index

$$\mathbf{q}=-\mathcal{I}_{well}(\mathbf{p}-\mathbf{p}_{well})$$

Accumulation matrix

$$\mathcal{V}=Vc_t\phi_0\mathcal{I},$$

$$V=h\Delta x\Delta y\Delta z.$$

Transmissibility matrix

$$\mathcal{T}_{i-\frac{1}{2},j,l}=\frac{\Delta y}{\Delta x\Delta z}\frac{h}{\mu}k_{i-\frac{1}{2},j,l},$$

$$k_{i-\frac{1}{2},j}=\frac{2}{\frac{1}{k_{i-1,j,l}}+\frac{1}{k_{i,j,l}}}.$$

Problem Definition

Incompressible model

$$\mathcal{T}\mathbf{p} = \mathbf{q}.$$

Compressible model

$$\mathcal{V}^{n+1}\frac{(\mathbf{p}^{n+1} - \mathbf{p}^{n})}{\Delta t^{n}} + \mathcal{T}^{n+1}\mathbf{p}^{n+1} = \mathbf{q}^{n+1}.$$

Or:

$$\mathcal{F}(\mathbf{p}^{n+1}; \mathbf{p}^{n}) = 0. \tag{1}$$

Newton-Raphson

Using Newton-Raphson (NR) method, the system for the $(k+1)$-th NR iteration is:

$$\mathcal{J}(\mathbf{p}^k)\delta\mathbf{p}^{k+1} = -\mathcal{F}(\mathbf{p}^k;\mathbf{p}^n), \qquad \mathbf{p}^{k+1} = \mathbf{p}^k + \delta\mathbf{p}^{k+1},$$

where $\mathcal{J}(\mathbf{p}^k) = \frac{\partial\mathcal{F}(\mathbf{p}^k;\mathbf{p}^n)}{\partial\mathbf{p}^k}$ is the Jacobian matrix, and $\delta\mathbf{p}^{k+1}$ is the NR update at iteration step $k+1$.

$$\mathcal{J}(\mathbf{p}^k)\delta\mathbf{p}^{k+1} = \mathbf{b}(\mathbf{p}^k). \tag{2}$$

Conjugate Gradient Method (CG)

Successive approximations to obtain a more accurate solution $\mathbf{x}$ [2]

$$\mathcal{A}\mathbf{x} = \mathbf{b},$$

$$\mathbf{x}^0, \qquad \text{initial guess} \qquad \mathbf{r}^{k-1} = \mathbf{b} - \mathcal{A}\mathbf{x}^{k-1}.$$

$$min_{\mathbf{x}^k \in \kappa_k(\mathcal{A},\mathbf{r}^0)} ||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}}, \qquad ||\mathbf{x}||_{\mathcal{A}} = \sqrt{\mathbf{x}^T \mathcal{A}\mathbf{x}}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{A})} + 1} \right)^k .$$

Preconditioning

Improve the spectrum of $\mathcal{A}$.

$$\mathcal{M}^{-1}\mathcal{A}\mathbf{x} = \mathcal{M}^{-1}\mathbf{b}.$$

Convergence

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} - 1}{\sqrt{\kappa(\mathcal{M}^{-1}\mathcal{A})} + 1} \right)^k , \qquad \kappa(\mathcal{M}^{-1}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Deflated PCG

DPCG history

- 1987 Nicolaides and Dostal
  First versions of DPCG
- 1999 Vuik, Meijerink, Segal
  DPCG applied to reservoir simulations (Shell)
- 2004 Nabben, Vuik
  Theory and porous media flow
- 2008 Nabben, Tang, Vuik, ...
  Theory comparison: DPCG, MG and Domain Decomposition, bubbly flow

Deflated PCG

DPCG history

- 2008 Nabben Erlangga
  Convection diffusion, Helmholtz, MLK method
- 2010 Jönsthövel, Vuik
  Mechanical problems, parallel computing
- 2014 Nabben, Sheikh, Lahaye, Vuik, Garcia
  MLK/ADEF method Helmholtz equation
- 2016 Diaz, Jansen, Vuik
  Porous media flow, Model Order Reduction (MOR)

DPCG

Deflation

$$\mathcal{P} = \mathcal{I} - \mathcal{A}\mathcal{Q}, \qquad \mathcal{P} \in \mathbb{R}^{n\times n}, \qquad \mathcal{Q} \in \mathbb{R}^{n\times n},$$
$$\mathcal{Q} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T, \qquad \mathcal{Z} \in \mathbb{R}^{n\times k}, \qquad \mathcal{E} \in \mathbb{R}^{k\times k},$$
$$\mathcal{E} = \mathcal{Z}^T\mathcal{A}\mathcal{Z} \text{ (Tang 2008, [3]).}$$

Convergence

Deflated system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{P}\mathcal{A})} + 1} \right)^k .$$

Deflated and preconditioned system

$$||\mathbf{x} - \mathbf{x}^k||_{\mathcal{A}} \leq 2||\mathbf{x} - \mathbf{x}^0||_{\mathcal{A}} \left(\frac{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} - 1}{\sqrt{\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A})} + 1} \right)^k .$$

$$\kappa_{eff}(\mathcal{M}^{-1}\mathcal{P}\mathcal{A}) \leq \kappa_{eff}(\mathcal{P}\mathcal{A}) \leq \kappa(\mathcal{A}).$$

Deflation vectors

Recycling deflation (Clemens 2004, [4]).

$$\mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}],$$

$\mathbf{x}^i$'s are solutions of the system.

Multigrid and multilevel (Tang 2009, [5]).
The matrices $\mathcal{Z}$ and $\mathcal{Z}^T$ are the restriction and prolongation matrices of multigrid methods.

Subdomain deflation (Vuik 1999,[6]).

Deflation Vectors

Model Order Reduction (MOR)

Many modern mathematical models of real-life processes pose challenges when used in numerical simulations, due to complexity and large size.

Model order reduction aims to lower the computational complexity of such problems by a reduction of the model's associated state space dimension or degrees of freedom, an approximation to the original model is computed. (Vuik 2005, [7])

- Proper Orthogonal Decomposition (POD)
- Reduced Basis Method (RBM)
- Principal Component Analysis (PCA)
- Singular Value Decomposition (SVD)

Deflation vectors

Proposal

Use solution of the system with diverse well configurations '*snapshots*' as deflation vectors (Recycling deflation).

Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).

Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis for a given data set (Markovinović 2009 [8], Astrid 2011, [9])

- Get the snapshots

$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m].$$

- Form $\mathcal{R}$

$$\mathcal{R} := \frac{1}{m}\mathcal{X}\mathcal{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_i \mathbf{x}_i^T.$$

- Then

$$\mathbf{\Phi} = [\phi_1, \phi_2, ...\phi_l] \in \mathbb{R}^{n \times l}$$

are the l eigenvectors corresponding to the largest eigenvalues of $\mathcal{R}$ satisfying:

$$\frac{\sum_{j=1}^{l} \lambda_j}{\sum_{j=1}^{m} \lambda_j} \leq \alpha, \qquad 0 < \alpha \leq 1.$$

Lemma 1

Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix, and $\mathbf{x}$ is a solution of:

$$\mathcal{A}\mathbf{x} = \mathbf{b}. \tag{3}$$

Let $\mathbf{x}_i, \mathbf{b}_i \in \mathbb{R}^n$, $i = 1, ..., m$, be vectors linearly independent ($l.i.$) and

$$\mathcal{A}\mathbf{x}_i = \mathbf{b}_i. \tag{4}$$

The following equivalence holds

$$\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \qquad \Leftrightarrow \qquad \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i. \tag{5}$$

Lemma 1 (proof)

$$\text{Proof} \Rightarrow \qquad \mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \Rightarrow \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i. \tag{6}$$

Substituting $\mathbf{x}$ from (6) into $\mathcal{A}\mathbf{x} = \mathbf{b}$ and using the linearity of $\mathcal{A}$ we obtain:

$$\mathcal{A}\mathbf{x} = \sum_{i=1}^{m} c_i \mathcal{A}\mathbf{x}_i = \sum_{i=1}^{m} c_i \mathbf{b}_i = \mathbf{b}. \qquad \text{Similarly for} \Leftarrow \qquad \boxtimes$$

Lemma 2

If the the deflation matrix $\mathcal{Z}$ is constructed with a set of m vectors

$$\mathcal{Z} = \begin{bmatrix} \mathbf{x}_1 & \ldots & \ldots & \mathbf{x}_m \end{bmatrix}, \tag{7}$$

such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ *l.i.*, then the solution of system (3) is obtained with one iteration of DCG.

Lemma 2 (proof)

Proof.
The relation between $\hat{\mathbf{x}}$ and $\mathbf{x}$ is given by [3]:

$$\mathbf{x} = \mathcal{Q}\mathbf{b} + \mathcal{P}^T\hat{\mathbf{x}}.$$

For the first term $\mathcal{Q}\mathbf{b}$, taking $\mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i$ we have:

$$\begin{aligned}
\mathcal{Q}\mathbf{b} &= \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T\left(\sum_{i=1}^{m} c_i \mathbf{b}_i\right) = \mathcal{Z}(\mathcal{Z}^T\mathcal{A}\mathcal{Z})^{-1}\mathcal{Z}^T\left(\sum_{i=1}^{m} c_i \mathcal{A}\mathbf{x}_i\right) \\
&= \mathcal{Z}(\mathcal{Z}^T\mathcal{A}\mathcal{Z})^{-1}\mathcal{Z}^T\left(\mathcal{A}\mathbf{x}_1 c_1 + ... + \mathcal{A}\mathbf{x}_m c_m\right) = \mathcal{Z}(\mathcal{Z}^T\mathcal{A}\mathcal{Z})^{-1}(\mathcal{Z}^T\mathcal{A}\mathcal{Z})\mathbf{c} \\
&= \mathcal{Z}\mathbf{c} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_m\mathbf{x}_m = \sum_{i=1}^{m} c_i\mathbf{x}_i = \mathbf{x}.
\end{aligned}$$

Lemma 2 (proof)

Therefore,

$$\mathbf{x} = \mathcal{Q}\mathbf{b}, \tag{8}$$

is the solution to the original system.
For the second term of the equation, $\mathcal{P}^T\hat{\mathbf{x}}$, we compute the deflated solution $\hat{\mathbf{x}}$.

$$\begin{aligned}
\mathcal{P}\mathcal{A}\hat{\mathbf{x}} &= \mathcal{P}\mathbf{b} \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= (\mathcal{I} - \mathcal{A}\mathcal{Q})\mathbf{b} && \text{using } \mathcal{A}\mathcal{P}^T = \mathcal{P}\mathcal{A} \text{ [3] and definition of } \mathcal{P}, \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= \mathbf{b} - \mathcal{A}\mathcal{Q}\mathbf{b} \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= \mathbf{b} - \mathcal{A}\mathbf{x} = 0 && \text{taking } \mathcal{Q}\mathbf{b} = \mathbf{x} \text{ from above}, \\
\mathcal{P}^T\hat{\mathbf{x}} &= 0 && \text{as } \mathcal{A} \text{ is invertible.}
\end{aligned}$$

Then we have obtained the solution

$$\mathbf{x} = \mathcal{Q}\mathbf{b} + \mathcal{P}^T\hat{\mathbf{x}} = \mathcal{Q}\mathbf{b},$$

in one step of DCG.

$\boxtimes$

Deflation vectors

Single-phase, $\mathbf{T}\mathbf{p}^n = \mathbf{q}^n$

Recycling linearly independent (l.i.) solutions

Compute l.i. solutions with ICCG	Construct **Z**	Use **Z** to solve with DICCG
$\mathbf{T}\mathbf{p}_i = \mathbf{q}_i,$	$\mathbf{Z} = \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix},$	$\mathbf{T}\mathbf{p} = \mathbf{q}.$

Deflation vectors

Two-phases, $\mathbf{T}^n\mathbf{p}^n = \mathbf{q}^n$

Training phase approach

Compute snapshots with ICCG	Construct correlation matrix	Obtain POD basis and use it to construct $\mathbf{Z}_m$
$\mathbf{T}^i\mathbf{p}^i = \mathbf{q}^i,$	$\mathbf{C} = \mathbf{X}\mathbf{X}^T$	$\mathbf{Z} = [\phi_1, ...\phi_n]$

$$\mathbf{X} = [\mathbf{p}^1 \cdots \mathbf{p}^n]$$

Use $\mathbf{Z}$ to solve $\mathbf{Tp} = \mathbf{q}$ with DICCG, different problems.

Heterogeneous permeability (Neumann and Dirichlet boundary conditions).

The experiments were performed for single-phase flow, with the following characteristics:

- $nx = ny = 64$ grid cells.
- 5 linearly independent snapshots.

System configuration								
Well pressures (bars)					Boundary conditions (bars)			
	$W1$	$W2$	$W3$	$W4$	$P(y=0)$	$P(y=Ly)$	$\frac{\partial P(x=0)}{\partial n}$	$\frac{\partial P(x=Lx)}{\partial n}$
	-5	-5	$+5$	$+5$	0	3	0	0
Snapshots								
	$W1$	$W2$	$W3$	$W4$	$P(y=0)$	$P(y=Ly)$	$\frac{\partial P(x=0)}{\partial n}$	$\frac{\partial P(x=Lx)}{\partial n}$
$\mathbf{z}_1$	-5	0	0	0	0	0	0	0
$\mathbf{z}_2$	0	-5	0	0	0	0	0	0
$\mathbf{z}_3$	0	0	-5	0	0	0	0	0
$\mathbf{z}_4$	0	0	0	-5	0	0	0	0
$\mathbf{z}_5$	0	0	0	0	0	3	0	0

Table : Table with the well configuration and boundary conditions of the system and the snapshots used for the Case 1.

Heterogeneous permeability (Neumann and Dirichlet boundary conditions).

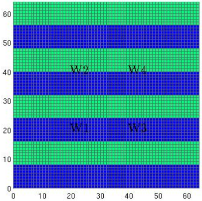


Figure : Heterogeneous permeability, 4 wells.

κ_2 (mD)	10^{-1}	10^{-2}	10^{-3}
ICCG	75	103	110
DICCG	1	1	1

Table : Number of iterations for different contrasts between the permeability of the layers for the ICCG and DICCG methods.

Numerical experiments

Heterogeneous permeability (Neumann boundary conditions).

The experiments were performed for single-phase flow, with the following characteristics:

- $nx = ny = 64$ grid cells.
- Neumann boundary conditions.
- 15 snapshots, 4 linearly independent.
- W1 = W2 = W3 = W4 = -1 bars, W5 = +4 bars.

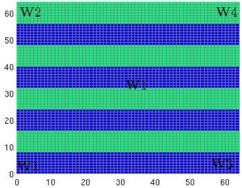


Figure : Heterogeneous permeability layers.

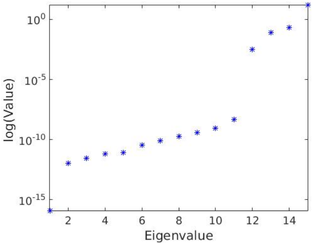


Figure : Eigenvalues of the data snapshot correlation matrix.

σ_2 (mD)	10^{-1}	10^{-2}	10^{-3}
ICCG	90	115	131
$DICCG_4$	1	1	1
$DICCG_{15}$	200*	200*	200*
$DICCG_{POD_4}$	1	1	1

Table : Number of iterations.

SPE 10 model

- 60*x*220*x*85 grid cells.
- Neumann boundary conditions.
- 15 snapshots, 4 linearly independent.
- W1 = W2 = W3 = W4 = -1 bars, W5 = +4 bars.

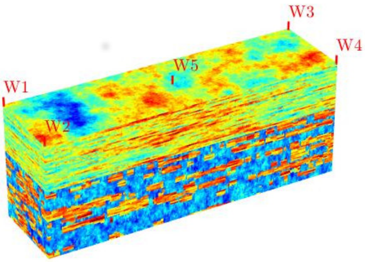


Figure : SPE 10 benchmark, permeability field.

Method	Iterations
ICCG	1011
$DICCG_{15}$	2000*
$DICCG_{4}$	1
$DICCG_{POD_4}$	1

Table : Number of iterations for ICCG and DICCG methods.

Numerical experiments

Compressible heterogeneous layered problem

35*x*35 grid cells.

Neumann boundary conditions.

W1 = W2 = W3 = W4 = 100 bars, W5 = 600 bars.

Initial pressure 200 bars.

Contrast between permeability layers of 10^1, 10^2 and 10^3.

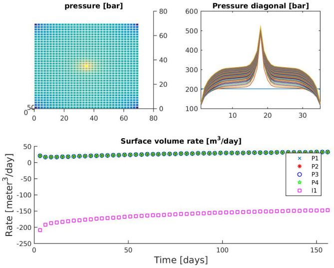


Figure : Solution, contrast between permeability layers of 10^1.

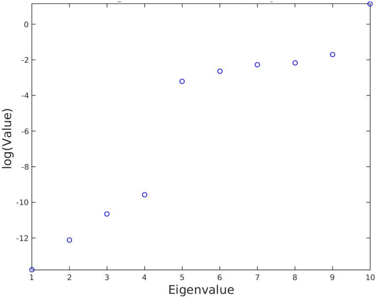


Figure : Eigenvalues of the data snapshot correlation matrix, contrast between permeability layers of 10^1.

Numerical experiments

1st NR Iteration						
$\frac{\sigma_2}{\sigma_1}$	Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10^1	780	$DICCG_{10}$	140	42	182	23
	780	$DICCG_{POD_6}$	140	84	224	29
10^2	624	$DICCG_{10}$	100	42	142	23
	624	$DICCG_{POD_7}$	100	42	142	23
10^3	364	$DICCG_{10}$	20	42	62	17
	364	$DICCG_{POD_7}$	20	42	62	17

Table : Comparison between the ICCC and DICCG methods of the average number of linear iterations for the first NR iteration for various contrast between permeability layers.

2nd NR Iteration						
$\frac{\sigma_2}{\sigma_1}$	Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10^1	988	$DICCG_{10}$	180	78	258	26
	988	$DICCG_{POD_6}$	180	198	378	38
10^2	832	$DICCG_{10}$	140	90	230	28
	832	$DICCG_{POD_7}$	140	154	294	33
10^3	884	$DICCG_{10}$	110	90	200	23
	884	$DICCG_{POD_7}$	110	150	260	29

Table : Comparison between the ICCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers.

Compressible SPE 10 problem

60x220x85 grid cells.

Neumann boundary conditions.

W1 = W2 = W3 = W4 = 100 bars, W5 = 600 bars.

Initial pressure 200 bars.

Contrast in permeability of 3x10^7.

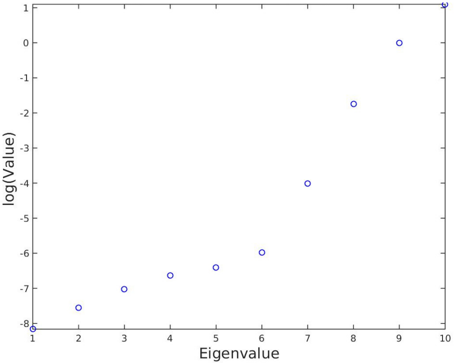


Figure : Eigenvalues of the data snapshot correlation matrix.

Numerical experiments

1^{st} NR Iteration					
Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10173	$DICCG_{10}$	1770	1134	2904	28
10173	$DICCG_{POD_4}$	1770	1554	3324	32

Table : Total number of linear iterations for the first NR iteration, full SPE 10 benchmark.

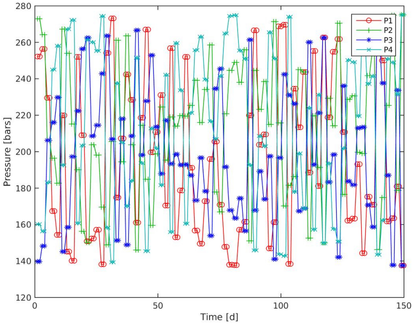
2^{nd} NR Iteration					
Total ICCG(only)	Method	ICCG Snapshots	DICCG	Total ICCG+DICCG	% of total ICCG(only)
10231	$DICCG_{10}$	1830	200	2030	20
10231	$DICCG_{POD_4}$	1830	200	2030	20

Table : Total number of linear iterations for the second NR iteration, full SPE 10 benchmark.

Numerical experiments, two-phase flow

Injection through wells, training phase approach

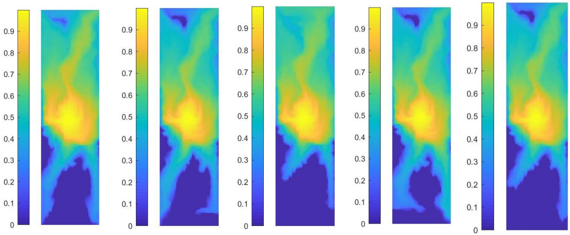
Well pressures [bars], $P_0 = 500$ [bars]				
Training phase				
$P1$	$P2$	$P3$	$P4$	I
137-275	137-275	137-275	137-275	1100
Same pressure in the producers				
200	200	200	200	800
Different pressure in the producers				
20	500	500	500	4×P



Numerical experiments, two-phase flow

Injection through wells, training phase approach

Pressure Field



Same pressure in production wells
$P_{bhp} = 200$ [bars]

dv	Total ICCG	DICCG Method	% of ICCG
10	90130	13720	15
5	90130	21522	24

Different pressure in production wells
$P_{2,3,4} = 500$ [bars], $P_1 = 20$ [bars]

dv	Total ICCG	DICCG Method	% of ICCG
10	90130	11740	13
5	90130	17855	20

Table : Number of iterations.

Injection through wells, training phase approach

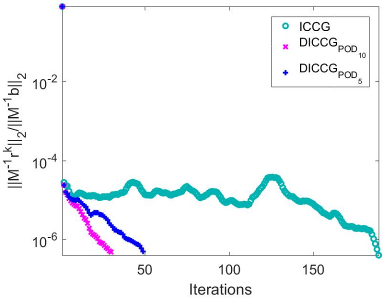


Figure : Relative residual.

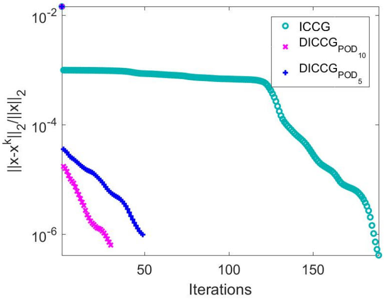


Figure : Relative true error.

Conclusions

- Solution is reached in few (1 or 2) iterations for the DICCG method in the incompressible case.
- A good choice of snapshots takes into account the boundary conditions of the problem.
- The number of iterations of the DICCG method does not depend on the contrast between the coefficients (Heterogeneous permeability example).
- The number of iterations of the ICCG method is reduced up to 80% with the DICCG method in the compressible case.
- Only a limited number of POD basis vectors is necessary to obtain a good speed-up. (for more info see [10, 11])

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