Preconditioners for incompressible flows

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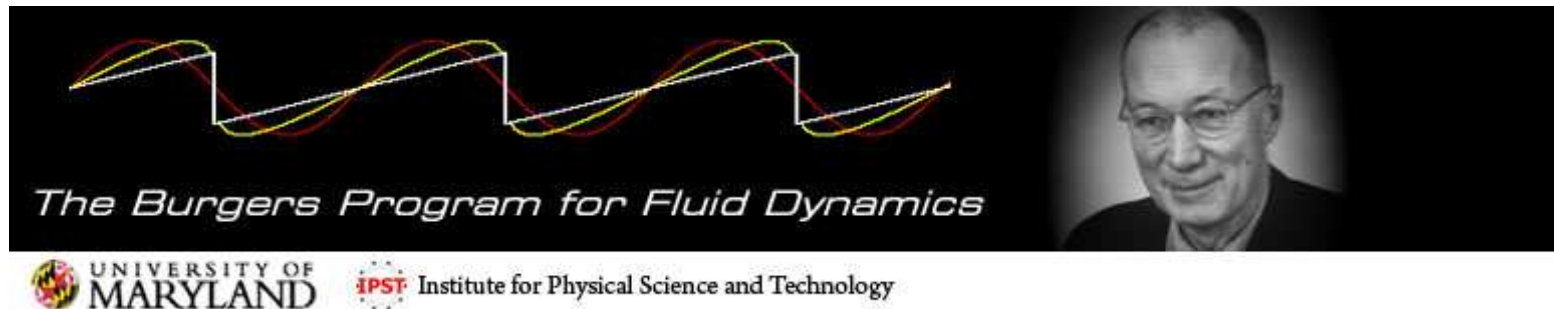
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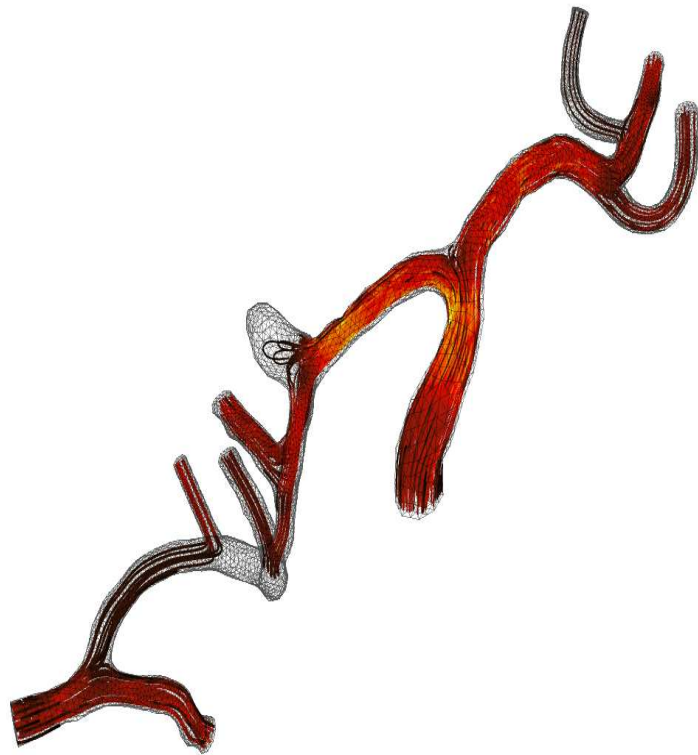
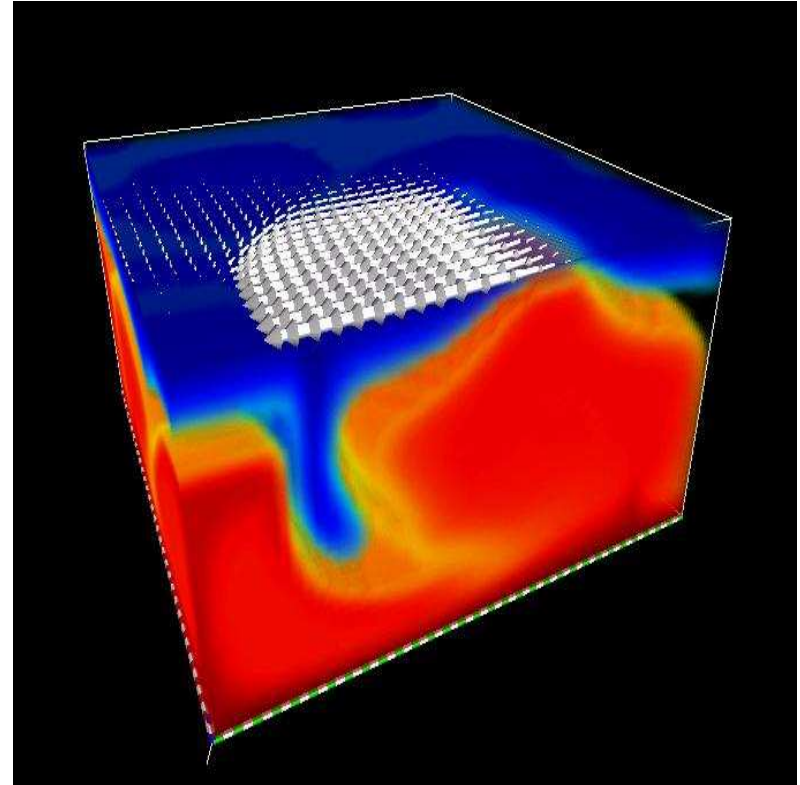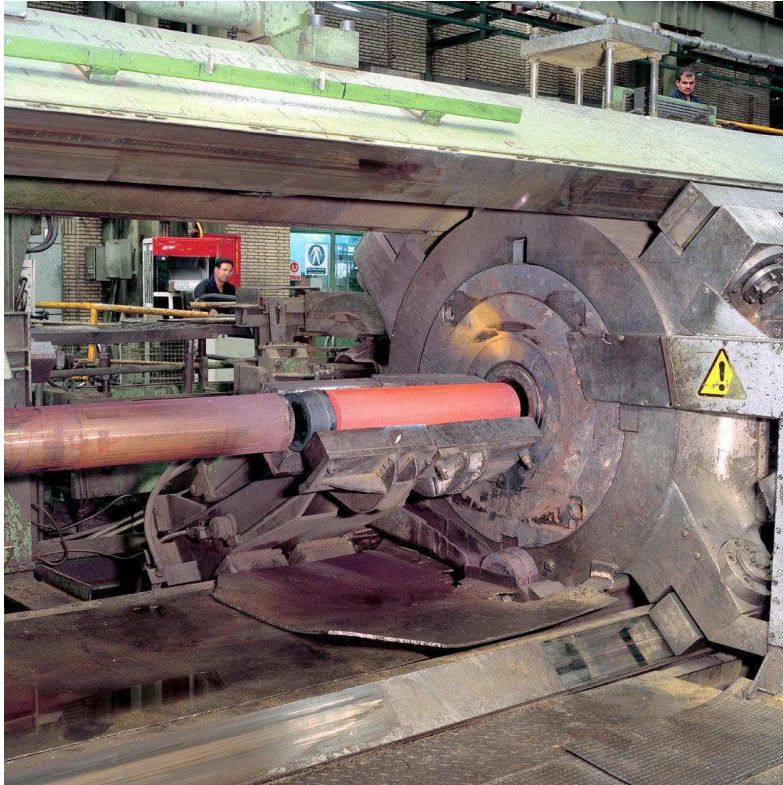**Numerical Analysis Group, TU Delft**
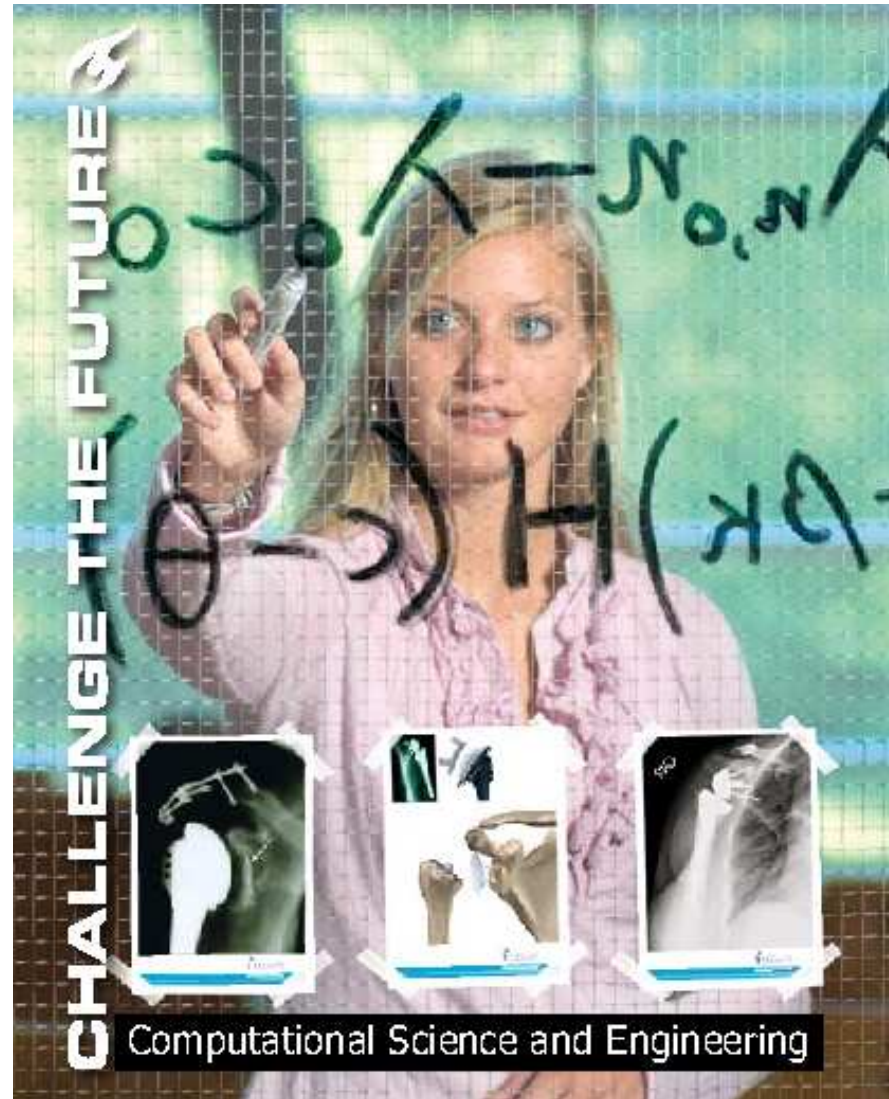
Outline

1. Introduction

2. Problem

3. Krylov solvers and preconditioners

4. ILU-type preconditioners

5. Block preconditioners

6. Stokes preconditioners

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1. Introduction

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1. Introduction

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2. Problem

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

$\mathbf{u}$ is the fluid velocity vector

p is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \text{ on } \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \text{ on } \partial\Omega_N.$$

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Finite element discretization

Discrete weak formulation:

$$X_h \subset (H_0^1(\Omega))^d, \quad M_h \subset L^2(\Omega)$$

Find $\mathbf{u_h} \in X_h$ and $p_h \in M_h$

$$\nu \int_\Omega \nabla \mathbf{u_h} : \nabla \mathbf{v_h} d\Omega + \int_\Omega (\mathbf{u_h}.\nabla \mathbf{u_h}).\mathbf{v_h} d\Omega - \int_\Omega p_h(\nabla.\mathbf{v_h}) d\Omega = \int_\Omega \mathbf{f}.\mathbf{v_h} d\Omega, \quad \forall \mathbf{v_h} \in X_h,$$

$$\int_\Omega q_h(\nabla.\mathbf{u_h}) d\Omega = 0 \quad \forall q_h \in M_h.$$

Matrix notation:

$$A\mathbf{u} + N(\mathbf{u}) + B^T p = \mathbf{f}$$

$$B\mathbf{u} = 0.$$

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Finite element discretization

For a unique solution, finite elements must satisfy the LBB condition

$$\inf_{q \in M_h} \sup_{v \in X_h} \frac{(\nabla . \mathbf{v_h}, q_h)}{\|\mathbf{v_h}\|_{X_h} \|q_h\|_{M_h}} \geq \gamma \geq 0.$$

- Taylor-Hood elements [Taylor and Hood, 1973]: Bi-quadratic velocity and bi-linear pressure (Q2-Q1).

- Crouzeix Raviart elements [Crouzeix and Raviart, 1973]: Bi-quadratic velocity, discontinuous linear pressure (Q2-P1).

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Finite element discretization

Non-zero pattern of the Stokes matrix with two families of elements

Q2-Q1

Q2-P1

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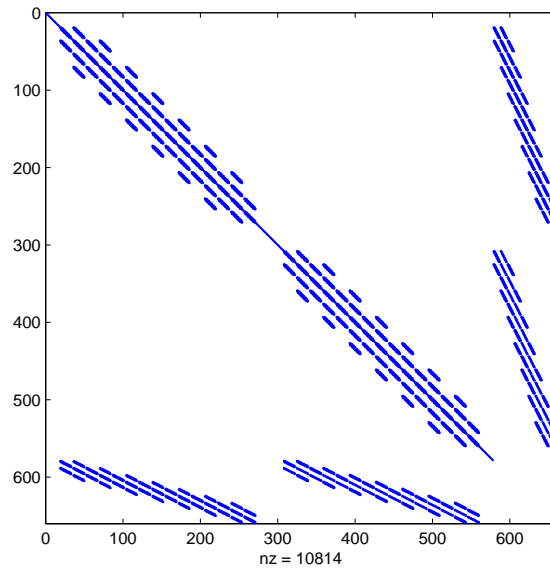
Linear system

Matrix form after linearization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

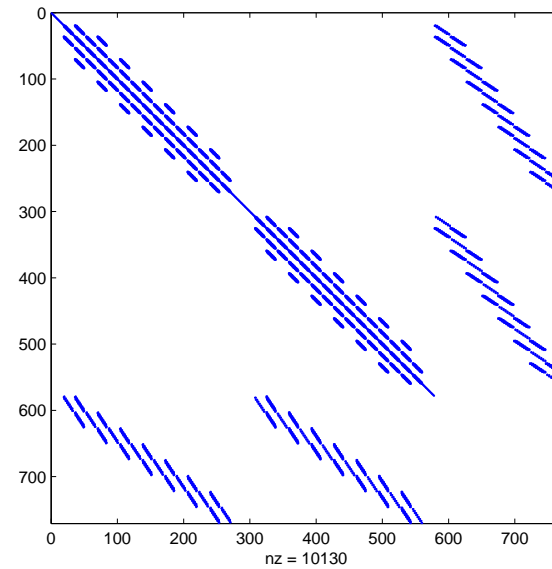where $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, A is vector Laplacian matrix
- $F = \nu A + N$ in Picard linearization, N is vector-convection matrix
- $F = \nu A + N + W$ in Newton linearization, W is the Newton derivative matrix
- B is the divergence matrix
- Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.
- Saddle point problem having large number of zeros on the main diagonal

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3. Krylov Solvers and preconditioners

- **Direct method:**

  To solve $\mathcal{A}x = b$,

  factorize $\mathcal{A}$ into upper U and lower L triangular matrices ($LUx = b$)

  First solve $Ly = b$, then $Ux = y$

- **Classical Iterative Schemes:**

  Methods based on matrix splitting, generates sequence of iterations

  $x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s$, where $\mathcal{A} = M - N$

  Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**

  $x_{k+1} = x_k + \alpha_k p_k$

  Some well known methods are

  CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986],

  GMRESR[1994], GCR[1986], IDR(s)[2007]

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IDR and IDR(s) (Induced Dimension Reduction)

- Sonneveld developed IDR in the 1970's. IDR is a finite termination (Krylov) method for solving nonsymmetric linear systems.

- Analysis showed that IDR can be viewed as Bi-CG combined with linear minimal residual steps.

- This discovery led to the development of first CGS, and later of Bi-CGSTAB (by van der Vorst).

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IDR and IDR(s) (continued)

- As a result of these developments the basic IDR-idea was abandoned for the Bi-CG-approach.

- Recently, Sonneveld and van Gijzen discovered that the IDR-approach was abandoned too soon and proposed a generalization of IDR: IDR(s).

- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
  SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

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The IDR approach for solving $Ax = b$

Generate residuals $\boldsymbol{r}_n = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_n$ that are in subspaces $\mathcal{G}_j$ of decreasing dimension.

These nested subspaces are related by

$$\mathcal{G}_j = (\boldsymbol{I} - \omega_j \boldsymbol{A})(\mathcal{G}_{j-1} \cap \mathcal{S})$$

where

- $\mathcal{S}$ is a fixed proper subspace of $\mathbb{C}^N$. $\mathcal{S}$ can be taken to be the orthogonal complement of s randomly chosen vectors $\boldsymbol{p}_i, i = 1 \cdots s$.

- The parameters $\omega_j \in \mathbb{C}$ are non-zero scalars.

It can be proved that ultimately $\boldsymbol{r}_n \in \{\boldsymbol{0}\}$ (IDR theorem).

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IDR versus Bi-CG

IDR(s) forces the residual to be in an increasingly small subspace, while Bi-CG constructs a residual in an increasingly large subspace. Yet, IDR(s) is closely related to:

- Bi-CGSTAB: IDR(1) and Bi-CGSTAB are mathematically equivalent.

- ML(k)BiCGSTAB (Yeung and Chan, 1999): This method generalizes Bi-CGSTAB using multiple 'shadow residuals'. Mathematically IDR(s) and ML(k)BiCGSTAB differ in the selection of the parameters ω_j.
  IDR(s) uses simpler recurrences, less vector operations and memory than ML(k)BiCGSTAB, and is more flexible (e.g. to avoid break down).

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Prototype IDR(s) algorithm.

while $\|\boldsymbol{r}_n\| > TOL$ **or** $n < MAXIT$ **do**

    for $k = 0$ to s **do**

        Solve $\boldsymbol{c}$ from $\boldsymbol{P}^H d\boldsymbol{R}_n \boldsymbol{c} = \boldsymbol{P}^H \boldsymbol{r}_n$

        $\boldsymbol{v} = \boldsymbol{r}_n - d\boldsymbol{R}_n \boldsymbol{c}$; $\boldsymbol{t} = \boldsymbol{A}\boldsymbol{v}$;

        if $k = 0$ **then**

            $\omega = (\boldsymbol{t}^H \boldsymbol{v})/(\boldsymbol{t}^H \boldsymbol{t})$;

        end if

        $d\boldsymbol{r}_n = -d\boldsymbol{R}_n \boldsymbol{c} - \omega \boldsymbol{t}$; $d\boldsymbol{x}_n = -d\boldsymbol{X}_n \boldsymbol{c} + \omega \boldsymbol{v}$;

        $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + d\boldsymbol{r}_n$; $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + d\boldsymbol{x}_n$;

        $n = n + 1$;

        $d\boldsymbol{R}_n = (d\boldsymbol{r}_{n-1} \cdots d\boldsymbol{r}_{n-s})$; $d\boldsymbol{X}_n = (d\boldsymbol{x}_{n-1} \cdots d\boldsymbol{x}_{n-s})$;

    end for

end while

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More information

More information: http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html

- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
  SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

- The relation of IDR(s) with Bi-CGSTAB, and how to derive generalizations of Bi-CGSTAB using IDR-ideas can be found in: Bi-CGSTAB as an induced dimension reduction method (with Sleijpen).

- A high quality IDR(s) implementation is described in: An elegant IDR(s) variant that efficiently exploits bi-orthogonality properties.

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4. ILU-type Preconditioners

A linear system $\mathcal{A}x = b$ is transformed into $P^{-1}\mathcal{A}x = P^{-1}b$ such that

- $P \approx \mathcal{A}$

- Eigenvalues of $P^{-1}\mathcal{A}$ are more clustered than $\mathcal{A}$

- $Pz = r$ cheap to compute

Several approaches, we will discuss here

- ILU preconditioner

- Preconditioned IDR(s) and Bi-CGSTAB comparison

- Block preconditioners

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ILU preconditioners

Incomplete LU preconditioners

$\mathcal{A} = LU - R$, where R consist of dropped entries that are absent in the index set $\mathcal{S}(i,j)$.

$\mathcal{S} = \{(i,j)|\ a_{ij} \neq 0\}$ [Classical ILU by Meijerink and van der Vorst, 1977].

- cheap and easy to implement

- if $\|R\|$ is large, give poor convergence (**reordering**)

- Instability due to large $\|L^{-1}\|$ and $\|U^{-1}\|$

- Pivoting, reordering, scaling and shifted diagonal schemes are used to make ILU factors more stable and effective.

[Elman -1989, Chow and Saad -1997]

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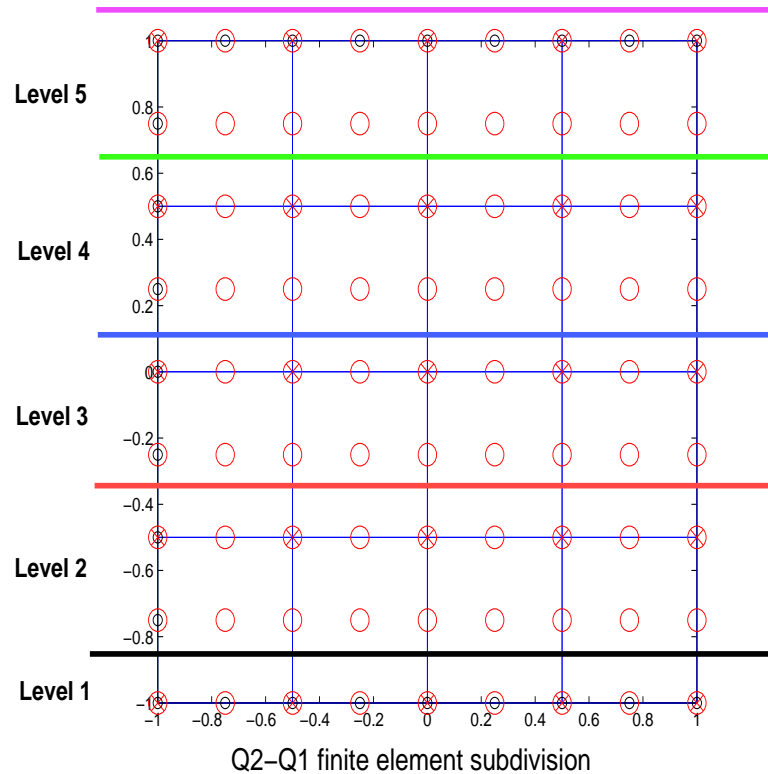**T U** Delft

SILU preconditioners

New renumbering Scheme

- Renumbering of grid points:
  - Sloan algorithm [Sloan - 1986]
  - Cuthill McKee algorithms [Cuthill McKee - 1969]

- The unknowns are reordered by p-last or p-last per level methods
  - In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually it produces a large profile but avoids breakdown of LU decomposition.
  - In **p-last per level reordering**, unknowns are reordered per level such that at each level, the velocity unknowns are followed by the pressure unknowns.

what are the levels ?

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SILU preconditioner

4×4 Q2-Q1 grid

Q2–Q1 finite element subdivision

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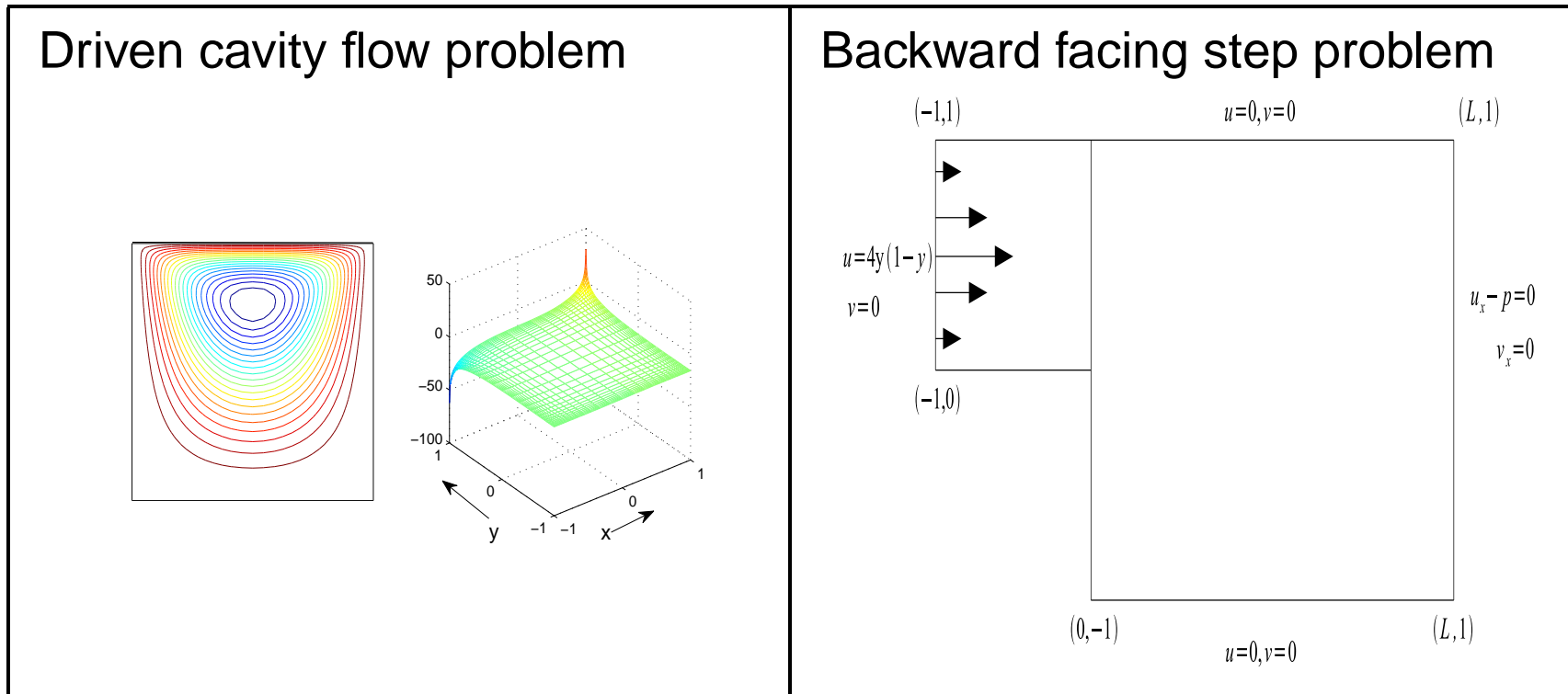**T U** Delft

SILU preconditioners

- In direct solver, reordering improves the profile and bandwidth of the matrix.

- Improve the convergence of the ILU preconditioned Krylov subspace method

- Minimizes dropped entries in ILU ($\|\mathcal{A} - \bar{L}\bar{U}\|_F$)

- May give stable factorization ($\|I - \mathcal{A}(\bar{L}\bar{U})^{-1}\|_F$)

[Dutto-1993, Benzi et al 1999, Duff and Meurant-1989, Wille-2004, Bollhöfer and Saad - 2006, Saad -2005]

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SILU preconditioners

- **Proposition 1:** If we use the p-last ordering and assume that the ILUD decomposition of F exists then the ILUD decomposition exists because every column of B^T contains a non-zero element . Note that if B^T has a zero column then $\mathcal{A}$ is singular.

- **Proposition 2:** For an arbitrary ordering we suppose that ILUD decomposition exists for all $j < i$, where the ith row is related to the continuity equation. If the ith (pressure) unknown is preceded by at least one velocity unknown with a non-zero connection to this pressure unknown (so there is one $k < i$ such that $a_{i,k} \neq 0$), then the ILUD decomposition exists and $d_i > 0$.

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Numerical experiments (SILU preconditioner)

Driven cavity flow problem	Backward facing step problem

The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol$.

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Numerical experiments (SILU preconditioner)

- Bandwidth($\mathcal{A}$)=$\max_i\{\beta_i(\mathcal{A}), 1 \leq i \leq n\}$

- Profile($\mathcal{A}$)=$\sum_{i=1}^{n} \beta_i(\mathcal{A})$

16×16 driven cavity flow with Q2-Q1 discretization

Profile = 52195, Bandwidth = 570
p−last ordering with lexicographic numbering

Profile =31222, Bandwidth = 212
p−last per level ordering with Sloan renumbering

Profile =47468, Bandwidth =160
p−last per level ordering with Cuthill McKee renumbering

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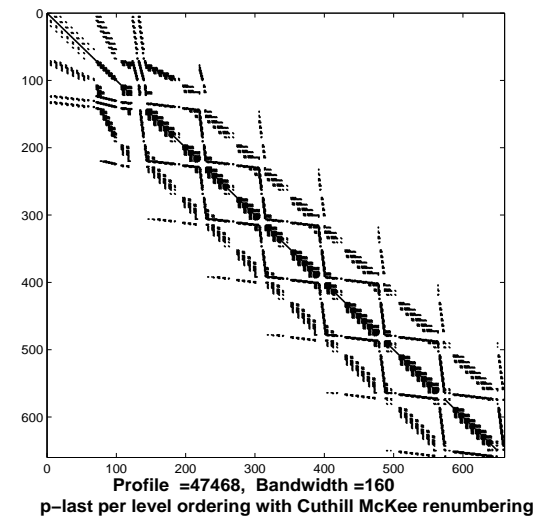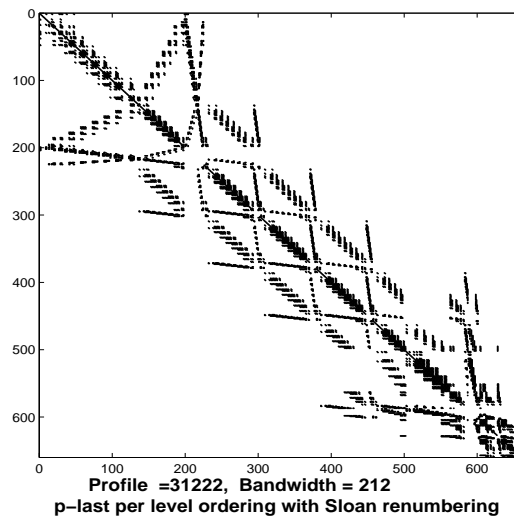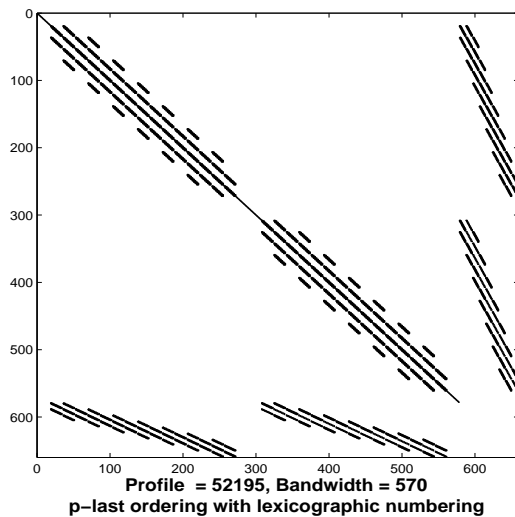
Numerical experiments (SILU preconditioners)

Stokes Problem in a square domain with Bi-CGSTAB, $accuracy = 10^{-6}$, Sloan renumbering

Grid size	$Q2 - Q1$		$Q2 - P1$	
	p-last	p-last per level	p-last	p-last per level
16×16	36(0.11)	25(0.09)	44(0.14)	34(0.13)
32×32	90(0.92)	59(0.66)	117(1.08)	75(0.80)
64×64	255(11.9)	135(6.7)	265(14)	165(9.0)
128×128	472(96)	249(52)	597(127)	407(86)

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Numerical experiments (SILU preconditioners)

Effect of grid increase(Left) and Reynolds number(Right) on inner iterations for the Navier Stokes backward facing step problem with $accuracy = 10^{-2}$ using the p-last-level reordering

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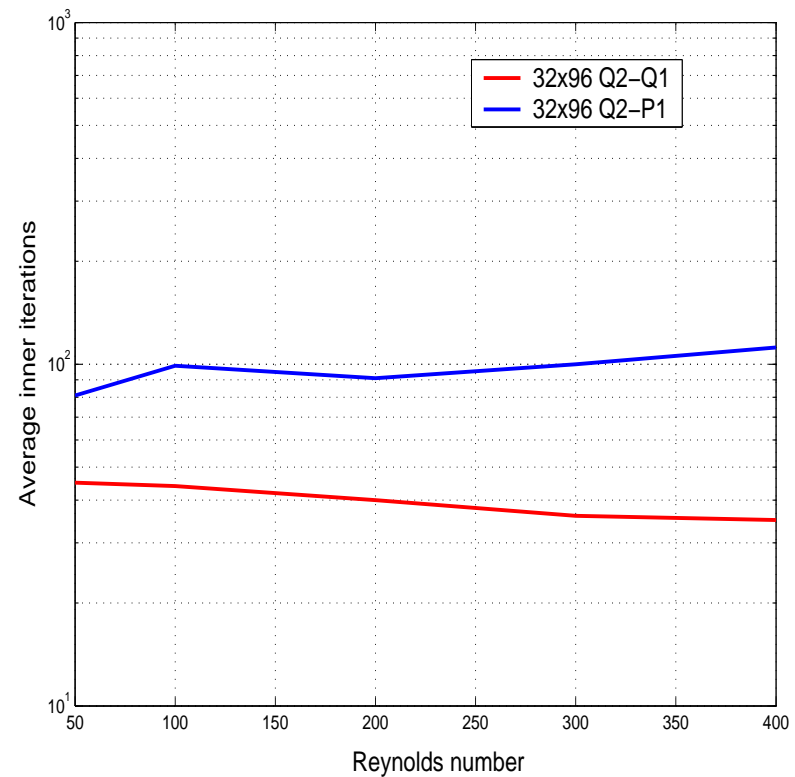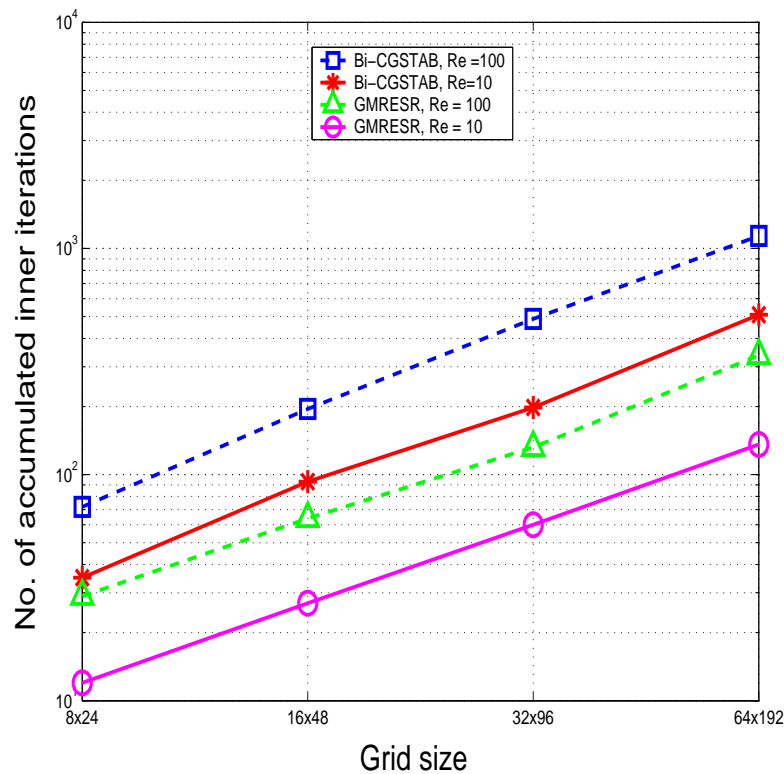
Numerical Experiments (IDR(s))

IDR(s): Top: 32×32, Bottom: 64×64 driven cavity Stokes flow problem

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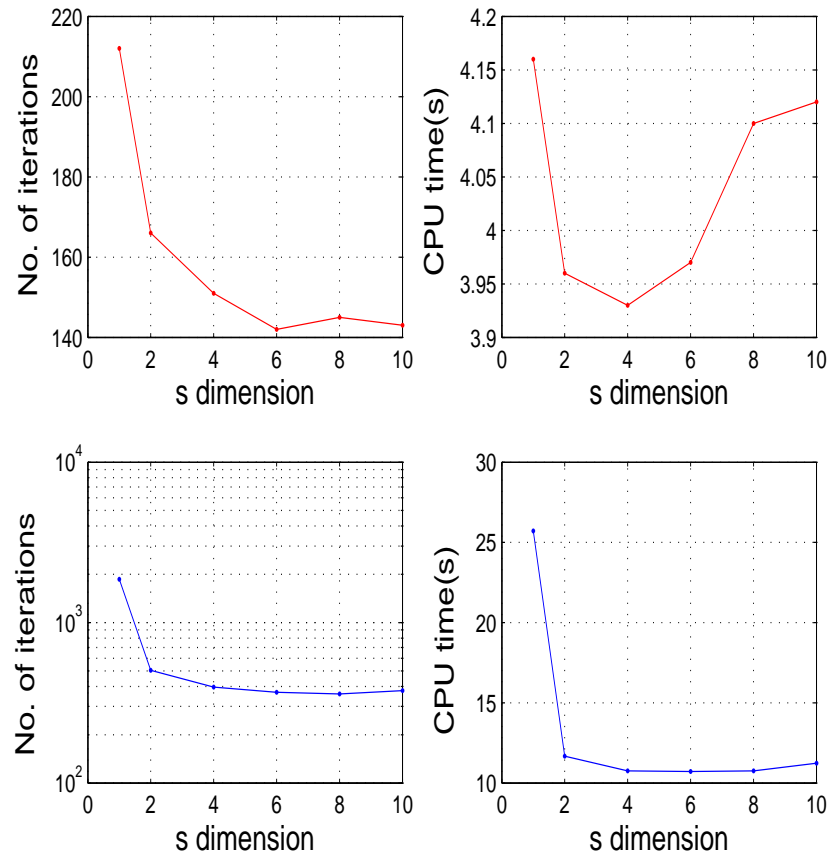
Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioner: Comparison of iterative methods for increasing grid size for the driven cavity Stokes flow problem.

Grid	Bi-CGSTAB	IDR(4)
	Mat.-Vec. (ts)	Mat.-Vec. (ts)
16×16	38(0.01)	33(0.01)
32×32	90(0.14)	75(0.14)
64×64	214(1.6)	159(1.4)
128×128	512(16)	404(15)
256×256	1386(183)	1032(156)

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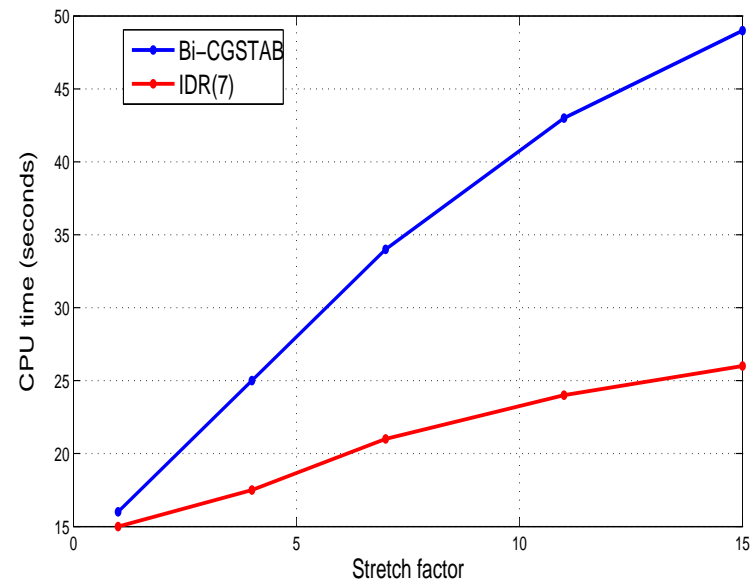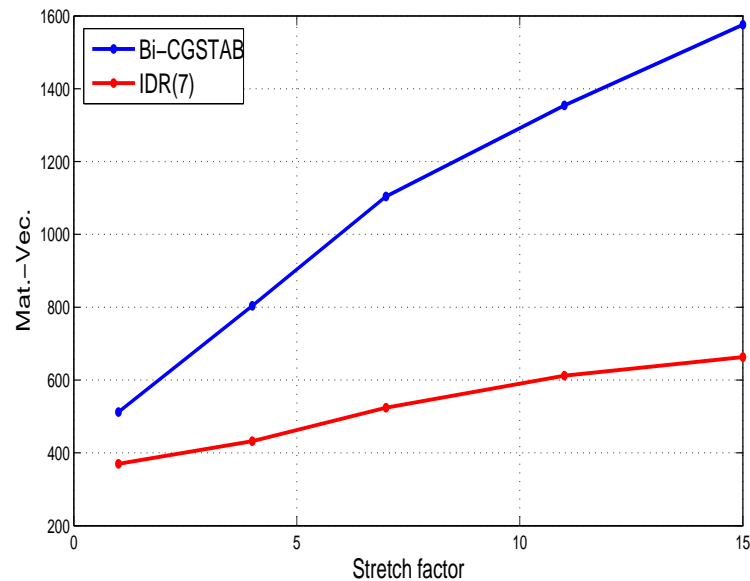
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Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for increasing stretch factor for the driven cavity Stokes problem.

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Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for the backward facing step Stokes problem.

Grid	Bi-CGSTAB	IDR(s)	
	Mat.-Vec.(ts)	Mat.-Vec.(ts)	s
32×96	214(1.3)	168(1.26)	4
64×96	NC	597(7.7)	4
96×96	NC	933(18)	4
128×96	NC	1105(31)	8

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Numerical Experiments (IDR(s) vs Bi-CGSTAB(l))

SILU preconditioned: Comparison of iterative methods

Driven Cavity Stokes problem, stretch factor 10

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
128×128	1104(36.5)	4	638(24.7)	6
256×256	5904(810)	6	1749(307)	8

Channel flow Stokes problem, length 100

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
64×64	1520(12)	4	938(8.7)	8
128×128	NC	6	8224(335)	8

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Conclusions

- SILU (Saddle point ILU) based on renumbering of grid points and reordering of unknowns is introduced

- SILU does not lead to break down (robust)

- SILU is fast for medium sized problems (efficient)

- SILU with level fill in is hard to beat for "easy" problems

- IDR(s) is a new Krylov method

- Convergence of IDR(s) is close to that of full GMRES

- IDR(s) more robust and faster than Bi-CGSTAB(l)

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5. Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1}B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{D}_b\mathcal{U}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{L}_b\mathcal{D}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

Preconditioners are based on combination of these blocks involve:

$$Fz_1 = r_1 \text{ The velocity subsystem}$$

$$S \longrightarrow \hat{S}$$

$$\hat{S}z_2 = r_2 \text{ The pressure subsystem}$$

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Block preconditioners

Block triangular preconditioners

$$P_t = \mathcal{U}_{bt} = \left[\begin{array}{cc} F & B^T \\ 0 & \hat{S} \end{array} \right]$$

- Pressure convection diffusion (PCD)   [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$, Q_p is the pressure mass matrix

- Least squares commutator (LSC)   [Elman et al, 2002]
  $\hat{S} = -(B Q_u^{-1} B^T)(B Q_u^{-1} F Q_u^{-1} B^T)^{-1}(B Q_u^{-1} B^T)$, Q_u is the velocity mass matrix

- Augmented Lagrangian approach (AL)   [Benzi and Olshanskii, 2006]
  F is replaced by $F_\gamma = F + \gamma B W^{-1} B^T$
  $\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1})$, $W = \hat{Q}_p$

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Block preconditioners

SIMPLE-type preconditioners[Vuik et al-2000]

SIMPLE	SIMPLER
$z = \mathcal{U}_b^{-1} \mathcal{L}_{bt}^{-1} r$	$z = \mathcal{U}_{bt}^{-1} \mathcal{L}_b^{-1} r$
	$z = z + \mathcal{U}_b^{-1} \mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$
$M_u = D$	$M_l = M_u = D,\ D = diag(F)$
$\hat{S} = BD^{-1}B^T$	$\hat{S} = BD^{-1}B^T$
One Poisson Solve	Two Poisson solves
One velocity solve	Two velocity solves

Lemma: In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical .

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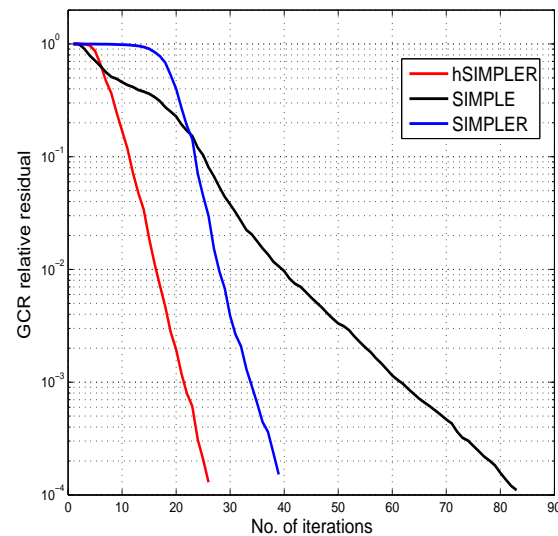# Improvements in SIMPLE-type preconditioners

We use approximate solvers for subsystems, so flexible Krylov solvers are required (GCR, FGMRES, GMRESR)

- hSIMPLER

- MSIMPLER

Improvements in SIMPLE(R) preconditioners

hSIMPLER preconditioner:

In hSIMPLER (hybrid SIMPLER), first iteration of Krylov method preconditioned with SIMPLER is done with SIMPLE and SIMPLER is employed afterwards.

- Faster convergence than SIMPLER
- Effective in the Stokes problem

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Improvements in SIMPLE(R) preconditioners

MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.

LSC: $\hat{S} \approx -(B\hat{Q_u}^{-1}B^T)(B\hat{Q_u}^{-1}\underbrace{F\hat{Q_u}^{-1}}B^T)^{-1}(B\hat{Q_u}^{-1}B^T)$

assuming $F\hat{Q_u}^{-1} \approx I$ (time dependent problems with a small time step)

$$\hat{S} = -B\hat{Q_u}^{-1}B^T$$

MSIMPLER uses this approximation for the Schur complement and updates scaled with $\hat{Q_u}^{-1}$.

-Convergence better than other variants of SIMPLE

-Cheaper than SIMPLER (in construction) and LSC (per iteration)

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Numerical Experiments (SIMPLE-type preconditioners)

Stokes backward facing step solved with preconditioned GCR(20) with $accuracy$ of 10^{-6}, PCG used as an inner solver (SEPRAN), Blue: Low inner accuracy , Red: High inner accuracy

Grid	SIMPLE	SIMPLER	hSIMPLER	MSIMPLER
	iter. (ts)	iter. (ts)	iter. (ts)	iter. (ts)
8×24	39(0.06)	26(0.05)	19(0.03)	11(0.02)
	37(0.14)	19(0.07)	17(0.06)	12(0.05)
16×46	72(0.6)	42(0.5)	31(0.34)	12(0.1)
	68(1.94)	30(0.86)	24(0.68)	15(0.44)
32×96	144(8.2)	NC	44(5.97)	16(0.9)
	117(34)	114(32)	37(10.6)	20(5.75)
64×192	256(93)	NC	89(141)	23(8.5)
	230(547)	NC	68(161)	25(60)

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Numerical Experiments (SIMPLE type preconditioners)

SIMPLE with relaxation parameter

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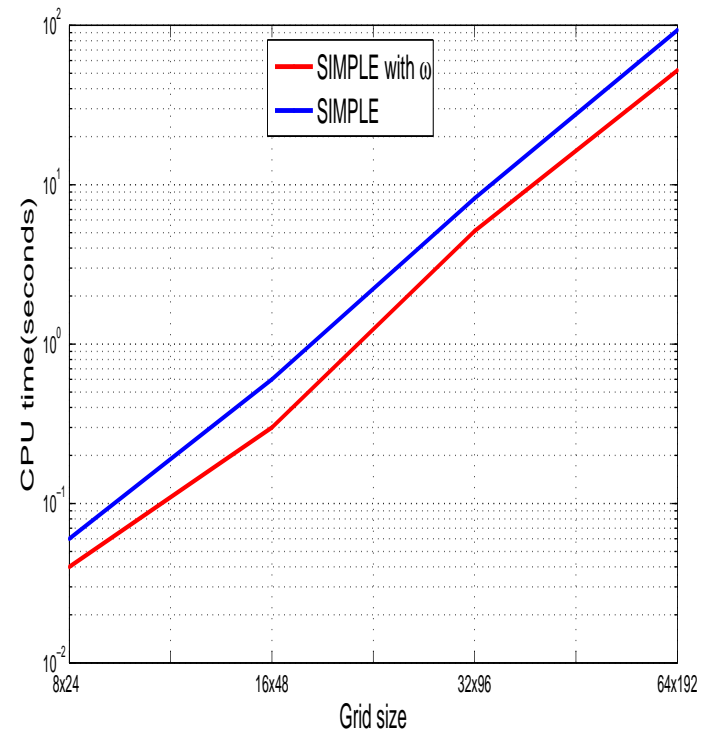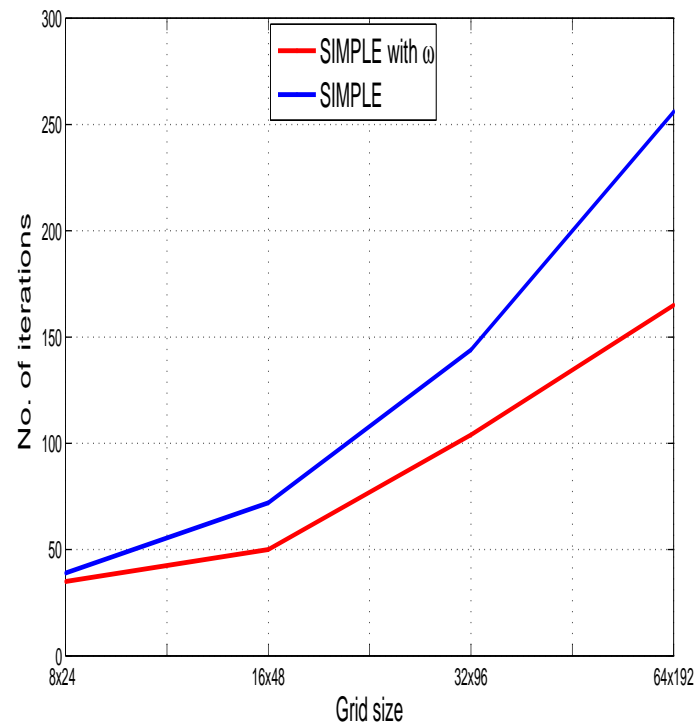
Numerical Experiments (SIMPLE type preconditioners)

Effect of relaxation parameter: The Stokes problem solved in Q2-Q1 discretized driven cavity problem with varying ω: 32×32 grid (Left), 64×64 grid (Right).

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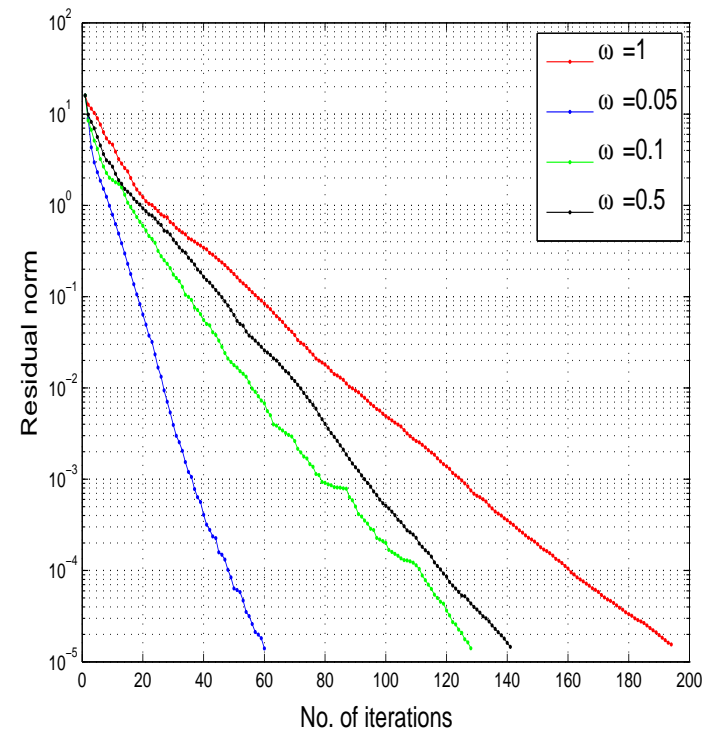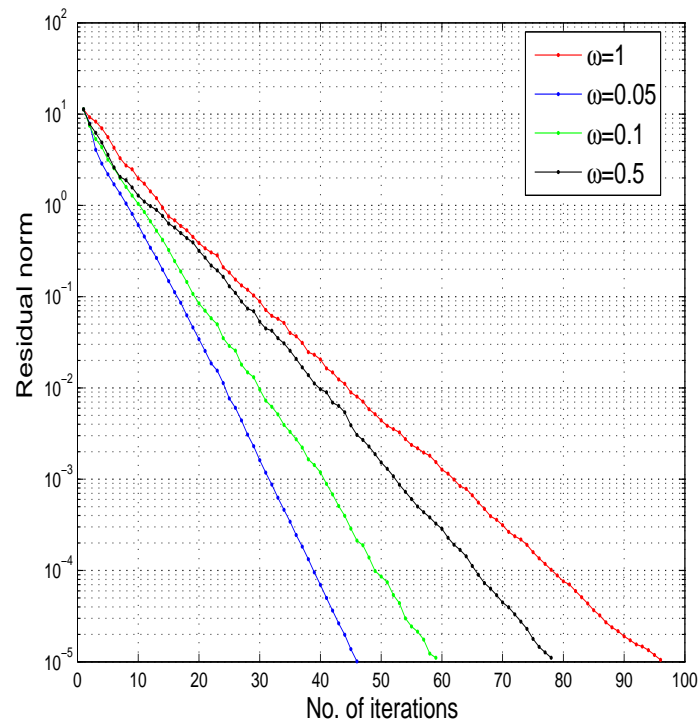
Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-6} (SEPRAN) using Q2-Q1 hexahedrons

Grid	SIMPLE	LSC	MSIMPLER
iter. $(t_s)\frac{\text{in-it-}u}{\text{in-it-}p}$			
$8 \times 8 \times 16$	$44(4)\ \frac{97}{342}$	$16(1.9)\ \frac{41}{216}$	$14(1.4)\ \frac{28}{168}$
$16 \times 16 \times 32$	$84(107)\ \frac{315}{1982}$	$29(51)\ \frac{161}{1263}$	$17(21)\ \frac{52}{766}$
$24 \times 24 \times 48$	$99(447)\ \frac{339}{3392}$	$26(233)\ \frac{193}{2297}$	$17(77)\ \frac{46}{1116}$
$32 \times 32 \times 40$	$132(972)\ \frac{574}{5559}$	$37(379)\ \frac{233}{2887}$	$20(143)\ \frac{66}{1604}$

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Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in solving the Navier-Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-2} (SEPRAN) using Q2-Q1 hexahedrons

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 32$			
100	173(462)	96(162)	321(114)
200	256(565)	145(223)	461(173)
400	399(745)	235(312)	768(267)
$32 \times 32 \times 40$			
100	240(5490)	130(1637)	1039(1516)
200	397(7040)	193(2251)	1378(2000)
400	675(11000)	295(2800)	1680(2450)

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Numerical Experiments (comparison)

3D Lid driven cavity problem (tetrahedrons):The Navier-Stokes problem is solved with accuracy 10^{-4}, a linear system at each Picard step is solved with accuracy 10^{-2} using preconditioned Krylov subspace methods. Bi-CGSTAB is used as inner solver in block preconditioners(SEPRAN)

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 16$			
20	30(20)	20(16)	144(22)
50	57(37)	37(24)	234(35)
100	120(81)	68(44)	427(62)
$32 \times 32 \times 32$			
20	38(234)	29(144)	463(353)
50	87(544)	53(300)	764(585)
100	210(1440)	104(654)	1449(1116)

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Numerical Experiments (comparison)

2D Lid driven cavity problem on 64×64 stretched grid: The Stokes problem is solved with accuracy 10^{-6}. PCG is used as inner solver in block preconditioners (SEPRAN) .

Stretch factor	LSC	MSIMPLER	SILU
	GCR iter.	GCR iter.	Bi-CGSTAB iter.
1	20	17	96
8	49	28	189
16	71	34	317
32	97	45	414
64	145	56	NC
128	NC	81	NC

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Conclusions

- MSIMPLER is introduced as a new preconditioner

- MSIMPLER is at present the fastest of all SIMPLE-type preconditioners

- In our experiments, MSIMPLER proved to be cheaper than SILU

- MSIMPLER shows better performance than LSC

- To find a robust method for a streched grid is an open problem

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6. Preconditioners for the incompressible Stokes problem

The main motivation behind this work is to develop and test preconditioners for the variable viscosity Stokes problem.

- The preconditioners that are developed for the incompressible Navier-Stokes problem or Saddle point problems are also applied to the Stokes problem.

- Block diagonal preconditioner [Rusten and Winther -1992, Silvester and Wathen-1994]

- Block triangular preconditioner [Bramble and Pasiack-1988, Elman et. al 2005]

- Recent preconditioners [May and Moresi - 2008, Burstedde et. al - 2008, Grinevich and Olshanskii -2008]

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Preconditioners for the incompressible Stokes problem

Block-triangular preconditioner

$$
P_{PMM} = \mathcal{U}_{bt} = \left[\begin{array}{cc} F & B^T \\ 0 & -1/\nu Q_p \end{array} \right]
$$

Q_p is the pressure-mass matrix.

- h-independent convergence

- Gives approximately two times faster convergence than the block diagonal approach [Elman et. al, 2005]

- One Poisson solve and one velocity solve

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Preconditioners for the incompressible Stokes problem

The Schur method

- In this method, instead of solving $\mathcal{A}x = b$, factorized system is solved by solving $\mathcal{A}x = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b x = b$.

- $M_l = M_u = F$

- Instead of approximating S in $\mathcal{D}_b$, $BF^{-1}B^T$ is treated implicitly by computing $BF^{-1}B^T p$ at each step of Krylov method.

- The pressure mass matrix is used as preconditioner for the Schur-complement system.

- Efficient solver for the velocity subsystem is required.

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Preconditioners for the incompressible Stokes problem

The Schur method

Initialize $u^{(0)}$, $p^{(0)}$ and $maxiter$ (maximum iterations)

Compute: $r_u = f - Fu^{(0)} - B^T p^{(0)}$

$r_p = g - Bu^{(0)}$

For $k = 0$ to $maxiter$

1. Solve $Fu_f = r_u$

2. Solve $-BF^{-1}B^T p_\delta = r_p - Bu_f$

3. Update $u_\delta = u_f - u_l$, where u_l is obtained by solving $Fu_l = B^T p_\delta$

4. Update $u^{(k+1)} = u^{(k)} + u_\delta$

5. Update $p^{(k+1)} = p^{(k)} + p_\delta$

6. Update $r_u = f - Fu^{(k+1)} - B^T p^{(k+1)}$

7. Update $r_p = g - Bu^{(k+1)}$

8. **If** converged **Exit**

End For

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Scaling

The pressure-mass matrix scaling

The standard pressure-mass matrix is defined independently of the viscosity

$$(Q_p)_{i,j} = \int_\Omega \phi_i \phi_j \, d\Omega, \tag{1}$$

In case of variable viscosity, we consider two alternatives:

1. Explicit scaling:

$$Q_{pe} = S_v^{-1} Q_p S_v^{-1}, \text{where} S_v = diag(\sqrt{\nu})$$

2. Implicitly scaling: This is done at the time of formation of the pressure-mass matrix. In this case, the smaller value of ν will dominate the definition of Q_{pi} (due to its inversion) at the nodes that are shared by more elements.

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Scaling

The velocity-mass matrix scaling

Use of AMG (ML) requires a constant number of unknowns per grid point.

- If one velocity component is prescribed, this will lead to only 1(2D) or 2(3D) degrees of freedom on those points.

- Approximate BC: $c_n u_n + \sigma^{nt} = c_n \bar{u}$, where c_n a large number, $\bar{u}$ is the prescribed value of the normal velocity and σ^{nt} is the tangential component of the stress tensor.

- To get convergence with LSC and MSIMPLER, we must update the velocity mass matrix by multiplying the entries corresponding to the boundary elements with the approximate boundary conditions by a factor of c_n.

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Scaling

The velocity-mass matrix scaling

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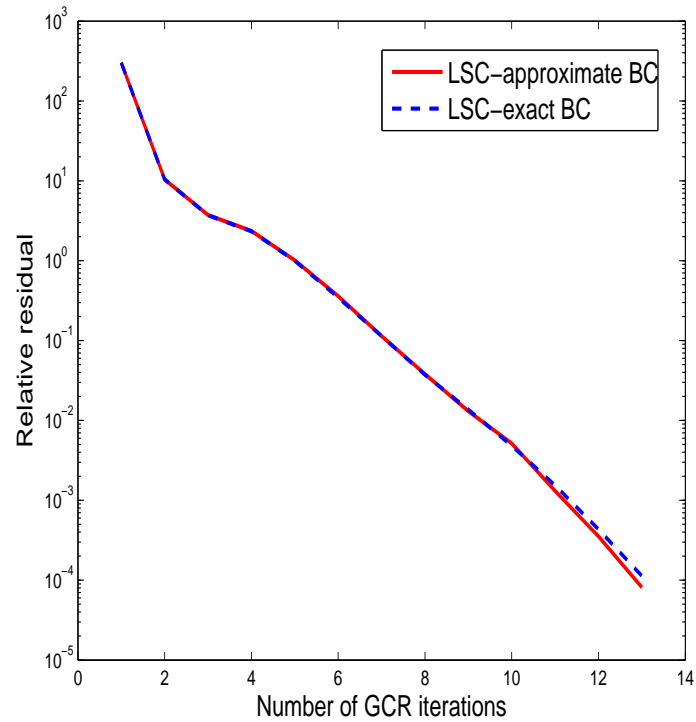
Scaling

System matrix scaling

- *In high viscosity contrast problem, if we use convergence criteria based on the L_2 norm, some preconditioners e.g. PMM, lead to fewer iterations. However, an inaccurate solution is obtained with this convergence criteria.*

- *If we use a preconditioner for the Schur-complement that involves the diagonal of the velocity matrix D^{-1}, the error in the iterative method using a direct method for the subsystems becomes small. This has been verified for LSC_D, $BD^{-1}B^T$ and SIMPLE.*

- We use S_m as scaling matrix given:

$$
S_m = \begin{bmatrix} \sqrt{diag(F)} & 0 \\ 0 & \sqrt{diag(BD^{-1}B^T)} \end{bmatrix}.
$$

- We solve $S_m^{-1} \mathcal{A} S_m^{-1} S_m x = S_m^{-1} b$

- Convergence criteria is now based on scaled L_2 norm

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Numerical Experiments (The Stokes problem, constant ν)

The Stokes driven cavity flow problem with Q2-Q1 discretization with AMG/CG for the velocity subsystem solves and ICCG(0) for the Schur subsystem solves. Solution accuracy is 10^{-6}.

Preconditioner	Grids			
	32×32	64×64	128×128	256×256
	iter.(time in seconds)			
PMM	11(1.4)	10(5.6)	9(23.6)	9(97)
LSC	10(1.38)	13(8.3)	17(54)	22(319)
LSC_D^*	22(3.2)	37(25)	80(275)	180(2880)
MSIMPLER	13(1.5)	16(8)	22(50)	29(300)
Schur(1)	6(3)	5(10.2)	5(46)	6(221)
Schur(6)	1(2)	1(10.6)	1(53)	1(251)

$*$: LSC_D uses diagonal of the velocity system in the Schur complement approximation.

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Numerical Experiments (The Stokes problem, constant ν)

Solution of **the constant-viscosity Stokes problem** with accuracy 10^{-6}: Left, the total number of iterations required for the velocity subsystem. Right, the total number of iterations required for the pressure subsystem.
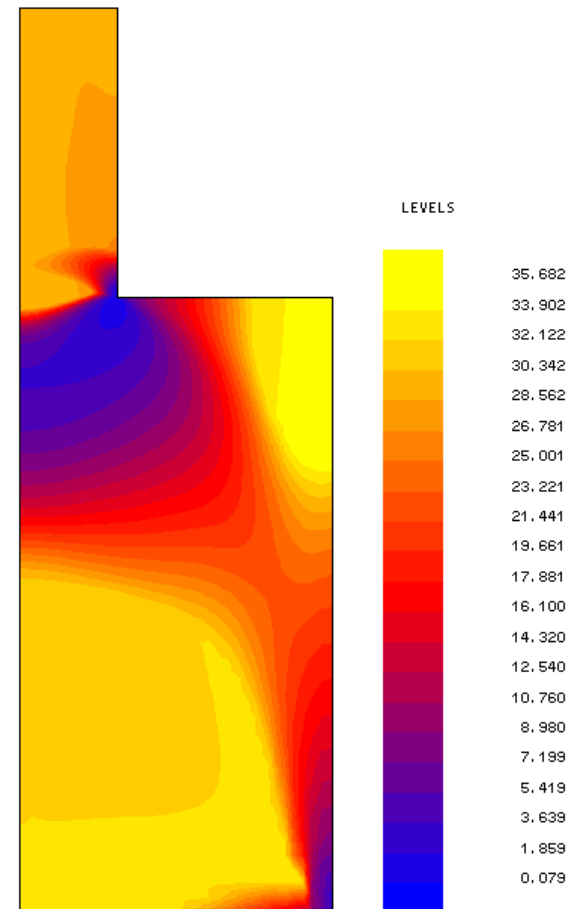
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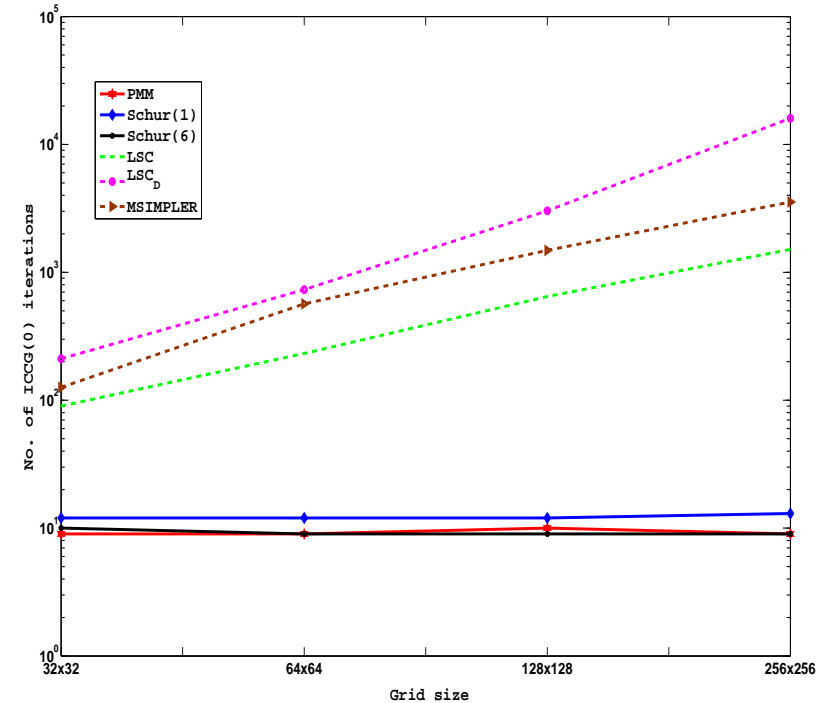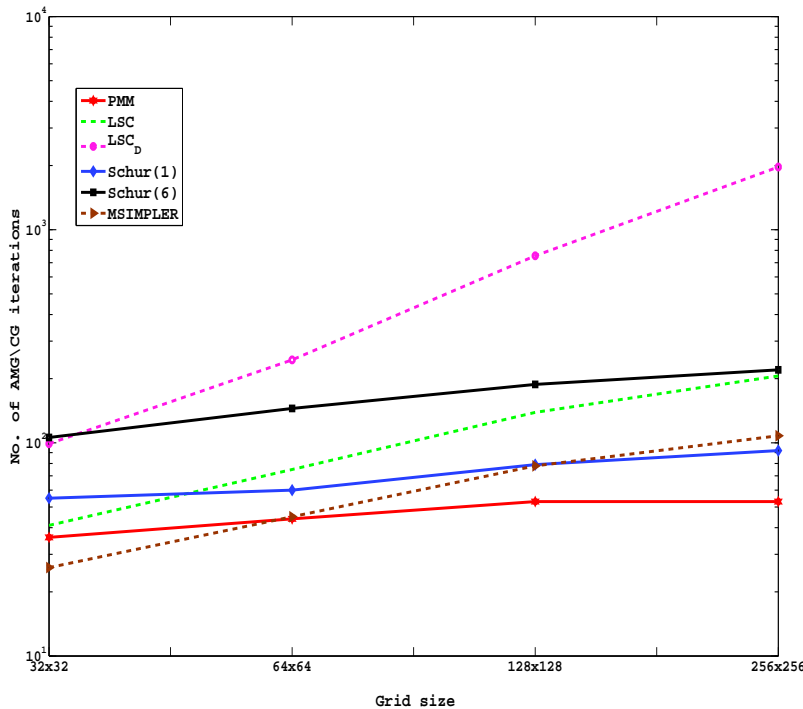
Numerical Experiments (The Stokes problem, variable ν)

Extrusion problem

A round aluminum rod is heated and pressed
through a die.
The viscosity model used describes
the viscosity as function of shear stress
and temperature,
which are highest at the die where
the aluminum is forced to flow into a
much smaller region.

LEVELS

35.682
33.902
32.122
30.342
28.562
26.781
25.001
23.221
21.441
19.661
17.881
16.100
14.320
12.540
10.760
8.980
7.199
5.419
3.639
1.859
0.079

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Numerical Experiments (The Stokes problem, variable ν)

The variable-viscosity Stokes problem with Q2-Q1 discretization with AMG/CG for the velocity subsystem and ICCG(0) (PMM, Schur method) or AMG/CG (LSC, LSC_D, MSIMPLER) for the Schur subsystem. Solution accuracy is 10^{-6}.

Grid $\downarrow$	Levels/N	PMM	LSC	LSC_D	MSIMPLER	Schur
tol $\longrightarrow$		10^{-3}	10^{-3}	10^{-6}	10^{-1}, 10^{-3}	10^{-6}
	iter.(time in seconds)					
$66k$	3/394	19(51)	11(35)	74(357)	15(35)	1(104)
$195k$	4/152	18(183)	13(188)	129(2650)	19(138)	1(370)
$390k$	5/300	18(429)	14(480)	> 1000	19(360)	1(869)
$595k$	5/408	19(743)	15(871)	> 1000	19(693)	1(1478)
$843k$	6/112	19(1229)	15(1406)	> 1000	21(989)	1(2686)

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Numerical Experiments (The Stokes problem, variable ν)

Solution of the variable viscosity Stokes problem with accuracy 10^{-6}: Left, the total number of iterations required for the velocity subsystem. Right, the total number of iterations required for the pressure subsystem.

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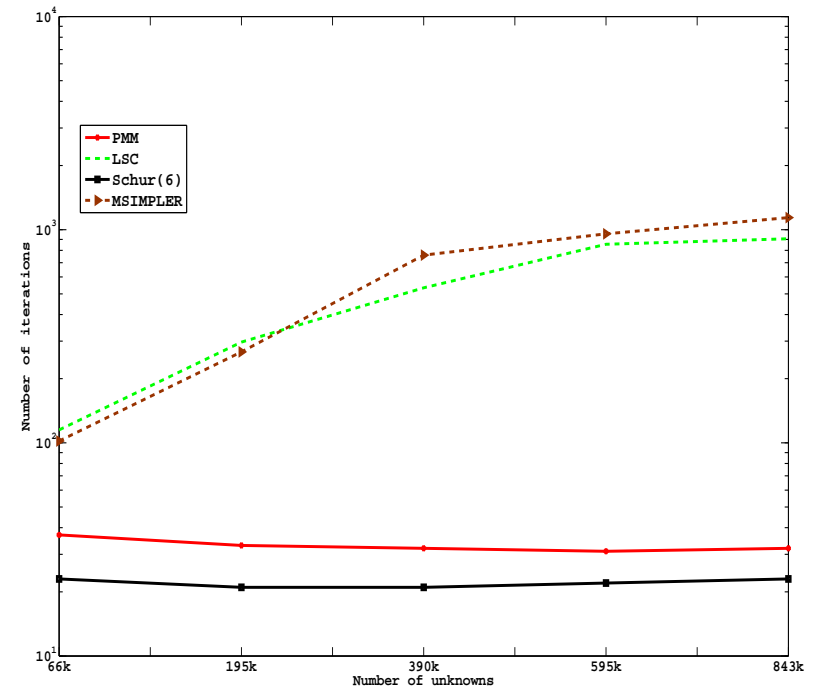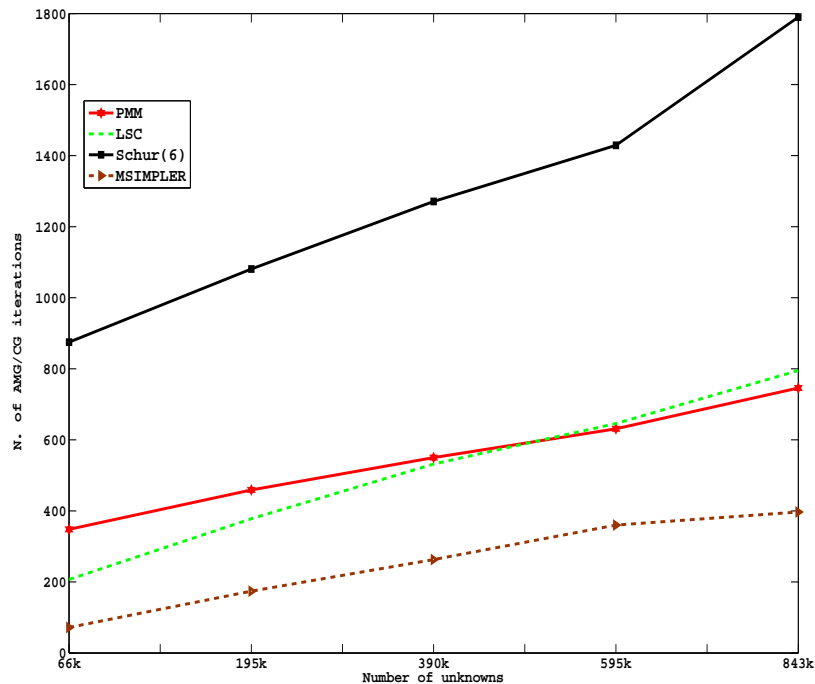
Numerical Experiments (The Stokes problem, jumping ν)

Two dimensional domain for the variable-viscosity Stokes problem (Left). At right, a 2D geodynamics test model: LVR represents the low viscosity region with density $\rho_1 = 1$ and viscosity $\nu_1 = 1$, and HVR denotes the high viscosity region with density $\rho_2 = 2$, and constant viscosity ν_2 ($1,\ 10^3$ and 10^6).

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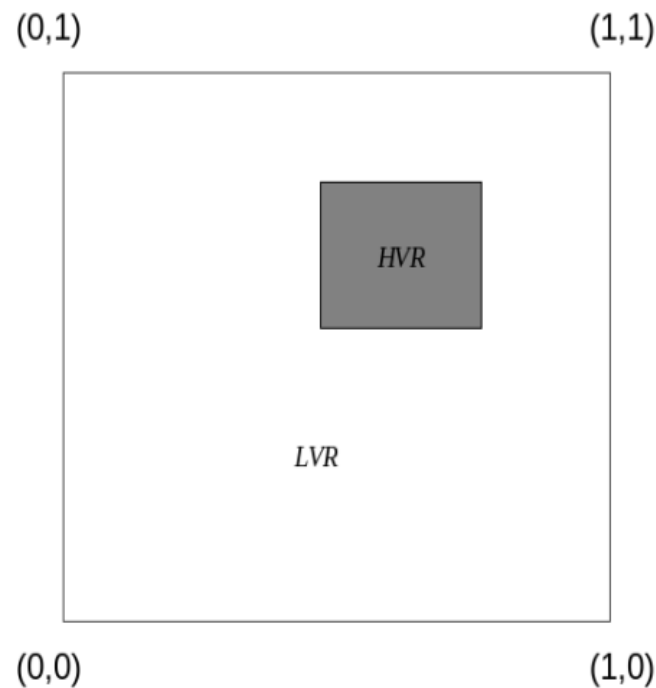
Numerical Experiments (The Stokes problem, jumping ν)

Iterative solution of the Stokes problem with SINKER configuration , accuracy = 10^{-6}.

$Error = \|p_{exact} - p_{PMM, \ LSC_D, \ Schur}\|_2$

ν	PMM		LSC_D		Schur	
	iter.	$Error$	iter.	$Error$	iter. (inner)	$Error$
30×30						
$\nu_2 = 10^6$	12	9×10^{-4}	26	7×10^{-6}	2(18)	2×10^{-8}
$\nu_2 = 10^3$	12	2×10^{-5}	26	3×10^{-6}	2(20)	2×10^{-10}
$\nu_2 = 10^1$	11	5×10^{-6}	24	1×10^{-6}	2(16)	2×10^{-10}
60×60						
$\nu_2 = 10^6$	13	8×10^{-3}	40	6×10^{-5}	2(19)	5×10^{-8}
$\nu_2 = 10^3$	13	3×10^{-5}	40	5×10^{-6}	2(20)	3×10^{-9}
$\nu_2 = 10^1$	13	1×10^{-6}	41	3×10^{-6}	2(18)	4×10^{-10}

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Numerical Experiments (The Stokes problem, jumping ν)

Remark: The Schur method gives more accurate results might be the fact that it uses subfactors of the system iteratively in a classical way. If that is the case, we may also expect better results when using Richardson type iterative improvement of the form:

$$x_{k+1} = x_k + PMM^{-1}(b - \mathcal{A}x_k).$$

We observe that Richardson also gives better accuracy than PMM and LSC.

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Conclusions

- The effect of (dis)continuous variation of the viscosity on the solver is large.

- PMM shows h-independent convergence for all types of viscosity configurations.

- For the problems that require subsystem to be accurately solved, PMM and the Schur method give similar convergence.

- In a high viscosity contrast problem, the Schur complement matrix must be approximated by an operator that contains viscosity information for the problem.

- In large viscosity contrast problems, to get results efficiently, the Schur method is a better option to use than the other preconditioners.

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Conclusions

Further research

- To reduce the effect of stretching on the convergence of the preconditioned Krylov subspace methods.

- Use of deflation-type schemes in the variable discontinuous viscosity Stokes problem.

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