Preconditioners for incompressible flows

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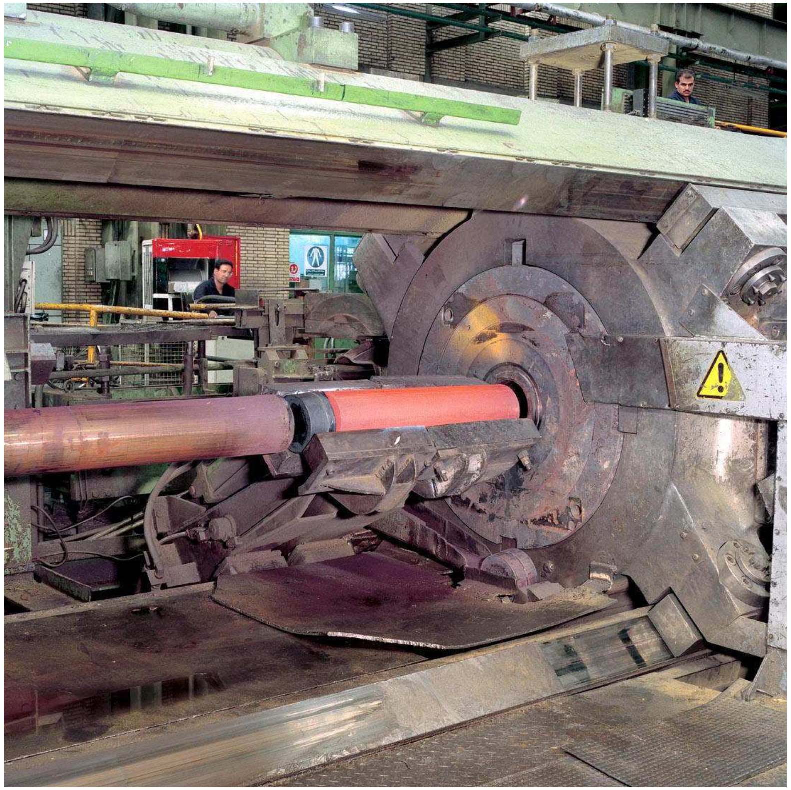
62nd Annual Meeting of the American Physical Society's Division of Fluid Dynamics (DFD)

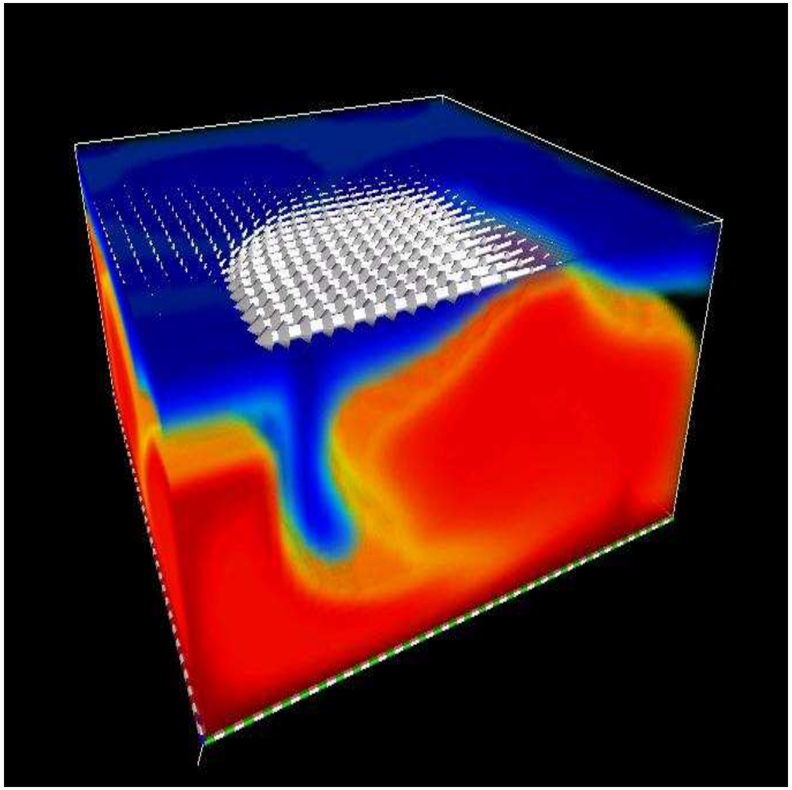
November 24, 2009

Introduction

Extrusion process

Mantle convection





Incompressible Stokes Equation

$$div\sigma = \rho f \quad \text{in} \quad \Omega$$

$$\sigma = -p\mathbf{I} + \frac{1}{2}\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

$\mathbf{u}$ is the fluid velocity vector, p is the pressure field

$\mu > 0$ is the variable dynamic viscosity coefficient

ρ is the variable density coefficient

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \text{ on } \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \text{ on } \partial\Omega_N.$$

Linear system

Matrix form after discretization by FEM:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, A is vector Laplacian matrix
- Sparse linear system, Symmetric indefinite (Stokes problem)
- Saddle point problem having large number of zeros on the main diagonal

Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1}B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

- Pressure Mass Matrix (PMM) [Bramble and Pasciak, 1988]
  $\hat{S} = -1/\nu Q_p$, Q_p is the pressure mass matrix
- Pressure convection diffusion (PCD) [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$
- Least squares commutator (LSC) [Elman et al, 2002]
  $\hat{S} = -(BQ_u^{-1}B^T)(BQ_u^{-1}FQ_u^{-1}B^T)^{-1}(BQ_u^{-1}B^T)$,
  Q_u is the velocity mass matrix

SIMPLE-type preconditioners

SIMPLER [Vuik et al, 2000]

$z = \mathcal{U}_{bt}^{-1}\mathcal{L}_b^{-1}r$ and then $z = z + \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$ where

$M_l = M_u = D$, $D = diag(F)$ and $\hat{S} = -BD^{-1}B^T$

MSIMPLER [Rehman etal, 2009]

$D = Q_u$, which is the velocity mass matrix.

The Schur method

- Instead of solving $\mathcal{A}x = b$, the factored system $\mathcal{L}_b\mathcal{D}_b\mathcal{U}_b x = b$ is solved.
- $M_l = M_u = F$
- Instead of approximating S in $\mathcal{D}_b$, the matrix vector product $BF^{-1}B^T p$ is computed in each step of the Krylov method.
- GCR is employed as it allows a variable preconditioner.
- The pressure mass matrix is used as preconditioner for the Schur-complement system.
- Efficient solver for the velocity subsystem is required. We use an Algebraic Multigrid Method (ML).

Scaling I

The pressure-mass matrix scaling

The standard pressure-mass matrix is defined independently of the viscosity

$$(Q_p)_{i,j} = \int_\Omega \phi_i \phi_j d\Omega, \tag{1}$$

In case of variable viscosity, we consider two alternatives:

1. Explicit scaling:

$$Q_{pe} = S_v^{-1} Q_p S_v^{-1}, \text{where} S_v = diag(\sqrt{\nu})$$

2. Implicitly scaling: This is done at the time of formation of the pressure-mass matrix. In this case, the smaller value of ν will dominate the definition of Q_{pi} (due to its inversion) at the nodes that are shared by more elements.

Scaling II

System matrix scaling

- *In high viscosity contrast problem, if we use convergence criteria based on the L_2 norm, some preconditioners e.g. PMM, lead to fewer iterations. However, an inaccurate solution is obtained with this convergence criteria.*
- *If we use a preconditioner for the Schur-complement that involves the diagonal of the velocity matrix D^{-1}, the error in the iterative method using a direct method for the subsystems becomes small. This has been verified for LSC_D, $BD^{-1}B^T$ and SIMPLE.*
- We use S_m as scaling matrix given:

$$S_m = \begin{bmatrix} \sqrt{diag(F)} & 0 \\ 0 & \sqrt{diag(BD^{-1}B^T)} \end{bmatrix}.$$

- We solve $S_m^{-1}\mathcal{A}S_m^{-1}S_m x = S_m^{-1}b$
- Convergence criteria are now based on the scaled L_2 norm

Numerical Experiments (The Stokes problem, constant ν)

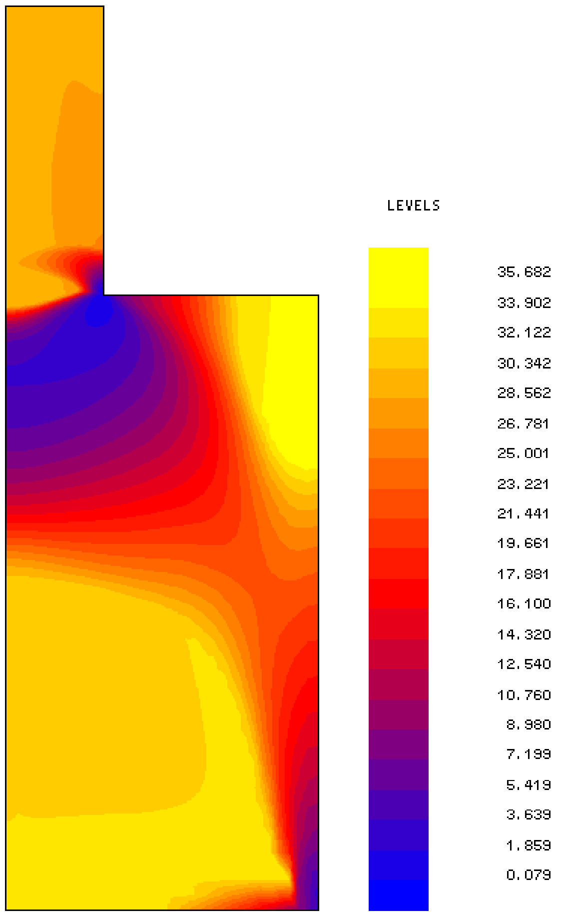
The Stokes driven cavity flow problem with Q2-Q1 discretization with AMG/CG for the velocity subsystem solves and ICCG(0) for the Schur subsystem solves. Solution accuracy is 10^{-6}.

Preconditioner	Grids			
	32×32	64×64	128×128	256×256
	iter.(time in seconds)			
PMM	11(1.4)	10(5.6)	9(23.6)	9(97)
LSC	10(1.38)	13(8.3)	17(54)	22(319)
MSIMPLER	13(1.5)	16(8)	22(50)	29(300)
Schur(1)	6(3)	5(10.2)	5(46)	6(221)
Schur(6)	1(2)	1(10.6)	1(53)	1(251)

Numerical Experiments (The Stokes problem, variable ν)

Extrusion problem
A round aluminum rod is heated and pressed through a die.

The viscosity model used describes the viscosity as function of shear stress and temperature, which are highest at the die where the aluminum is forced to flow into a much smaller region.



Numerical Experiments (The Stokes problem, variable ν)

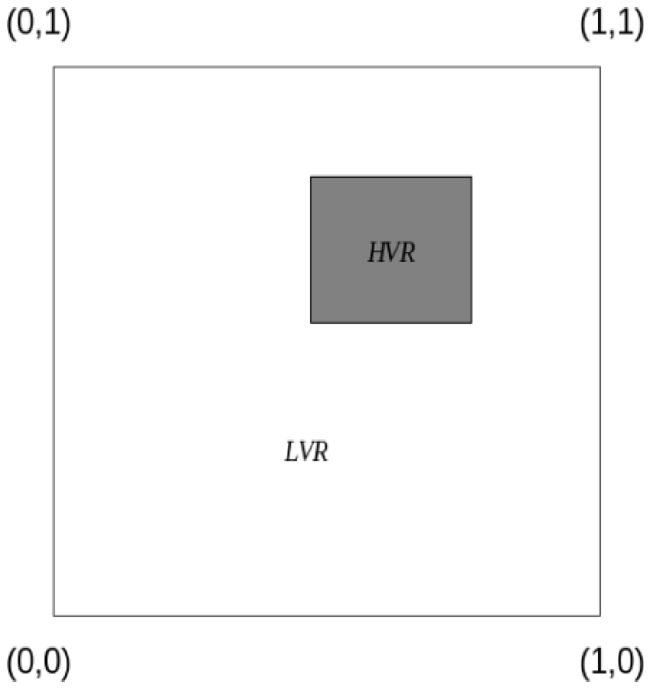
The variable-viscosity Stokes problem with Q2-Q1 discretization with AMG/CG for the velocity subsystem and ICCG(0) (PMM, Schur method) or AMG/CG (LSC, LSC_D, MSIMPLER) for the Schur subsystem. Solution accuracy is 10^{-6}.

Grid ↓	Levels/N	PMM	LSC	MSIMPLER	Schur
tol ⟶		10^{-3}	10^{-3}	10^{-1}, 10^{-3}	10^{-6}
	iter.(time in seconds)				
$66k$	3/394	19(51)	11(35)	15(35)	1(104)
$195k$	4/152	18(183)	13(188)	19(138)	1(370)
$390k$	5/300	18(429)	14(480)	19(360)	1(869)
$595k$	5/408	19(743)	15(871)	19(693)	1(1478)
$843k$	6/112	19(1229)	15(1406)	21(989)	1(2686)

Numerical Experiments (The Stokes problem, jumping ν)

A 2D geodynamics test model called SINKER

Low Viscosity Region (LVR) has density $\rho_1 = 1$ and viscosity $\nu_1 = 1$,
High Viscosity Region (HVR) has density $\rho_2 = 2$ and viscosity $\nu_2 =$ ($1,\ 10^3,$ or 10^6).



Numerical Experiments (The Stokes problem, jumping ν)

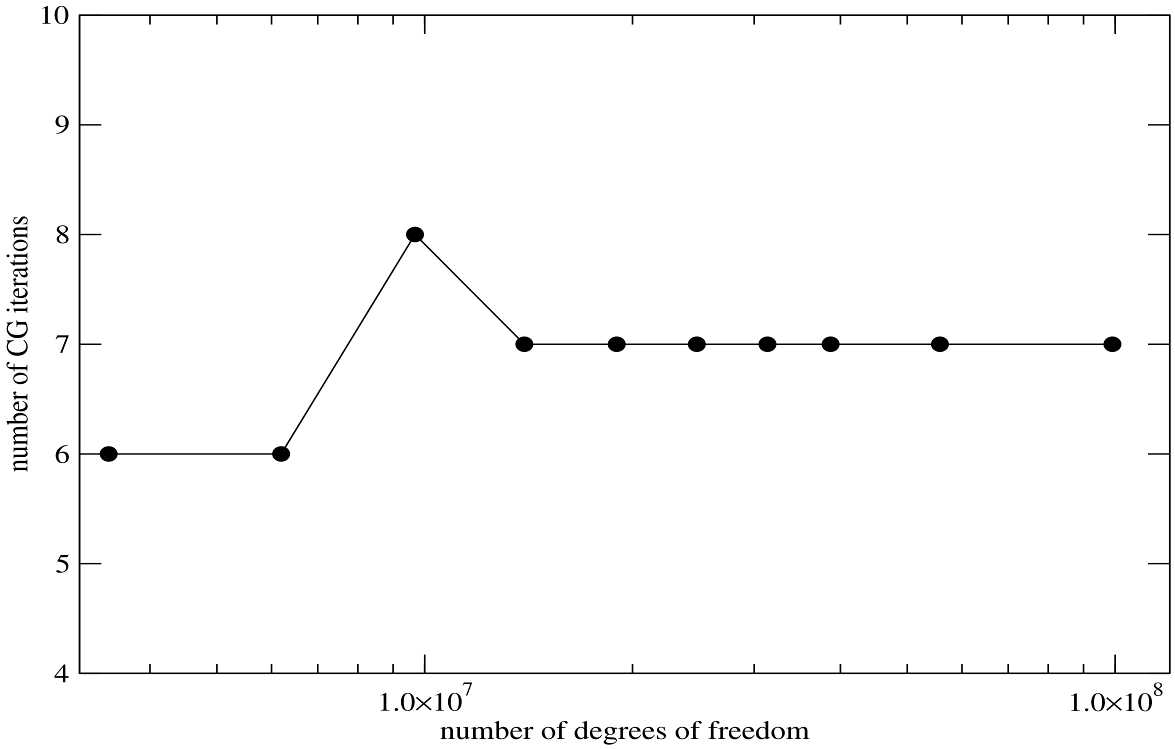
Iterative solution of the Stokes problem with SINKER configuration , accuracy = 10^{-6}.

$Error = \|p_{exact} - p_{PMM,\ LSC_D,\ Schur}\|_2$

ν	PMM		LSC_D		Schur	
	iter.	$Error$	iter.	$Error$	iter. (inner)	$Error$
30×30						
$\nu_2 = 10^6$	12	9×10^{-4}	26	7×10^{-6}	2(18)	2×10^{-8}
$\nu_2 = 10^3$	12	2×10^{-5}	26	3×10^{-6}	2(20)	2×10^{-10}
$\nu_2 = 10^1$	11	5×10^{-6}	24	1×10^{-6}	2(16)	2×10^{-10}
60×60						
$\nu_2 = 10^6$	13	8×10^{-3}	40	6×10^{-5}	2(19)	5×10^{-8}
$\nu_2 = 10^3$	13	3×10^{-5}	40	5×10^{-6}	2(20)	3×10^{-9}
$\nu_2 = 10^1$	13	1×10^{-6}	41	3×10^{-6}	2(18)	4×10^{-10}

Scaling independent to the number grid points

Solution of the velocity system with CG preconditioned with one AMG V-cycle



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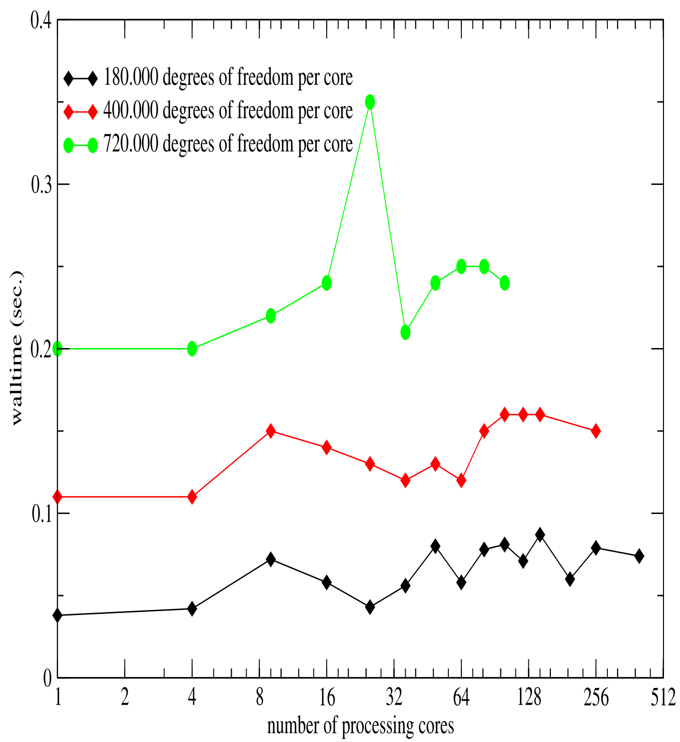
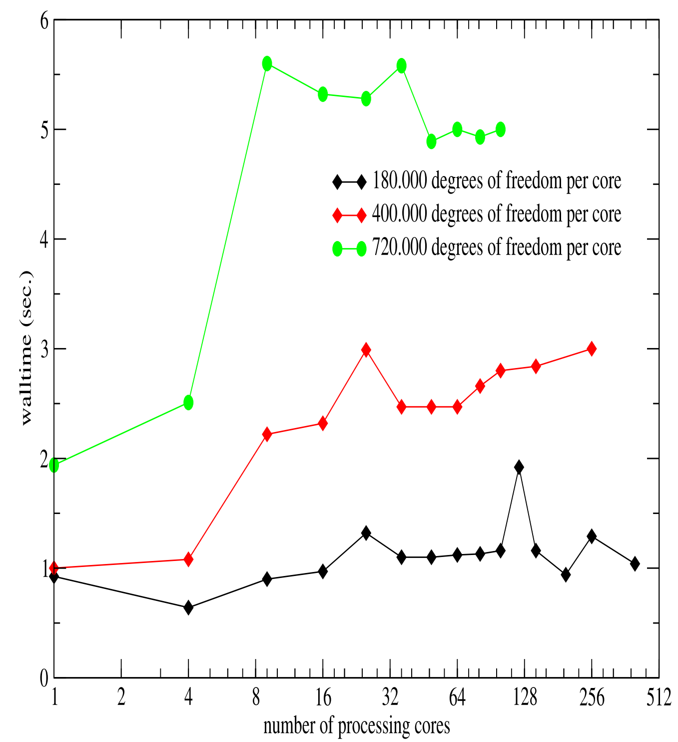
Scaling independent to the number of cores

One CG iteration



One AMG V-cycle



Conclusions

- PMM and Schur show h-independent convergence for all types of viscosity configurations and the convergence is similar.
- In high viscosity contrast problems the Schur method is the best preconditioner.
- The proposed solver is independent of the grid size, jump in the viscosity, and the number of computer cores.
- If the solver in your software (package) is older than 10 years, please update it.

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