Preconditioned Krylov methods for the incompressible Navier Stokes equations

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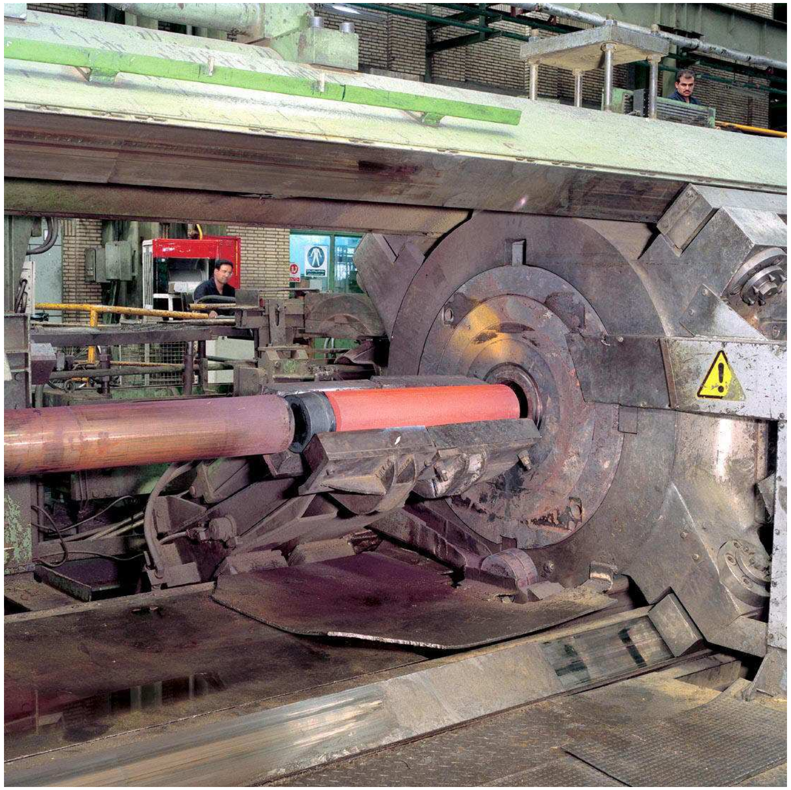
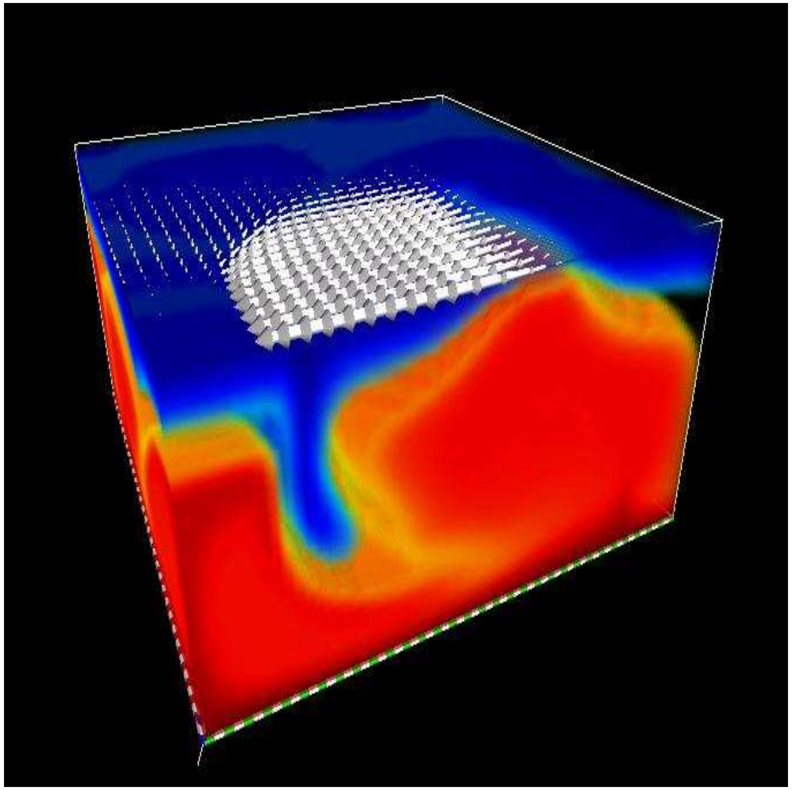
1. Introduction





2. Problem

$$-\nu\nabla^2\mathbf{u} + \mathbf{u}.\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega$$

$$\nabla.\mathbf{u} = 0 \quad \text{in} \quad \Omega.$$

$\mathbf{u}$ is the fluid velocity vector

p is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).

$\Omega \subset \mathbf{R}^{2 \text{ or } 3}$ is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \;\text{ on }\; \partial\Omega_D, \quad \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0 \;\text{ on }\; \partial\Omega_N.$$

Linear system

Matrix form after linearization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = \nu A$ in Stokes problem, A is vector Laplacian matrix
- $F = \nu A + N$ in Picard linearization, N is vector-convection matrix
- $F = \nu A + N + W$ in Newton linearization, W is the Newton derivative matrix
- B is the divergence matrix
- Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.
- Saddle point problem having large number of zeros on the main diagonal

3. Krylov Solvers and preconditioners

- **Direct method:**
  To solve $\mathcal{A}x = b$,
  factorize $\mathcal{A}$ into upper U and lower L triangular matrices ($LUx = b$)
  First solve $Ly = b$, then $Ux = y$

- **Classical Iterative Schemes:**
  Methods based on matrix splitting, generates sequence of iterations
  $x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s$, where $\mathcal{A} = M - N$
  Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**
  $x_{k+1} = x_k + \alpha_k p_k$
  Some well known methods are
  CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986], GMRESR[1994], GCR[1986], IDR(s)[2007]

IDR and IDR(s) (Induced Dimension Reduction)

- Sonneveld developed IDR in the 1970's. IDR is a finite termination (Krylov) method for solving nonsymmetric linear systems.

- Analysis showed that IDR can be viewed as Bi-CG combined with linear minimal residual steps.

- This discovery led to the development of first CGS, and later of Bi-CGSTAB (by van der Vorst).

IDR and IDR(s) (continued)

- As a result of these developments the basic IDR-idea was abandoned for the Bi-CG-approach.

- Recently, Sonneveld and van Gijzen discovered that the IDR-approach was abandoned too soon and proposed a generalization of IDR: IDR(s).

- P. Sonneveld and M.B. van Gijzen IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
  SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

More information: http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html

4. ILU-type Preconditioners

A linear system $\mathcal{A}x = b$ is transformed into $P^{-1}\mathcal{A}x = P^{-1}b$ such that

- $P \approx \mathcal{A}$
- Eigenvalues of $P^{-1}\mathcal{A}$ are more clustered than $\mathcal{A}$
- $Pz = r$ cheap to compute

Several approaches, we will discuss here

- ILU preconditioner
- Preconditioned IDR(s) and Bi-CGSTAB comparison
- Block preconditioners

SILU preconditioners

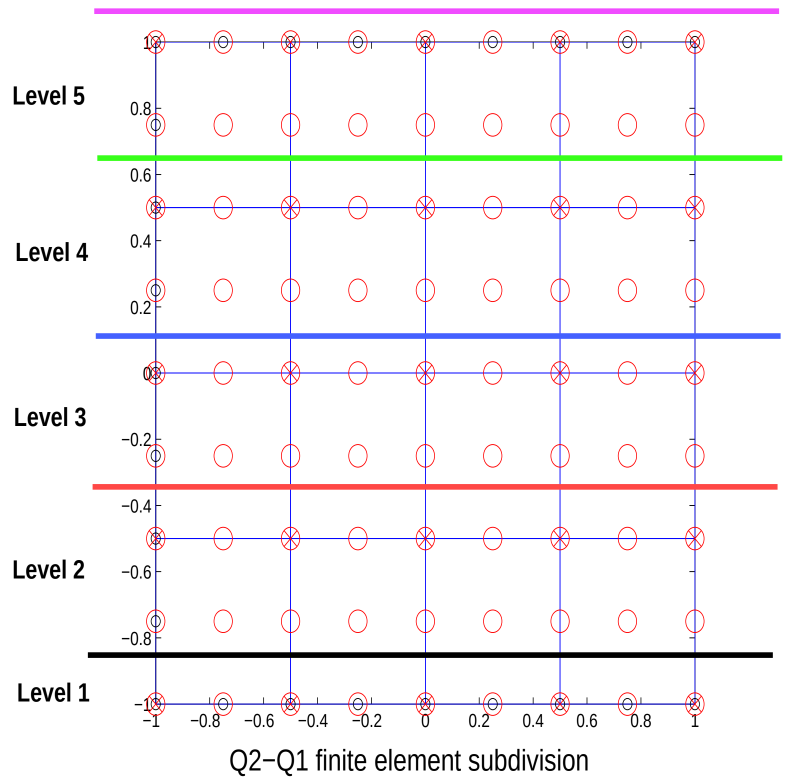
New renumbering Scheme

- Renumbering of grid points:
  - Sloan algorithm [Sloan - 1986]
  - Cuthill McKee algorithms [Cuthill McKee - 1969]
- The unknowns are reordered by p-last or p-last per level methods
  - In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually it produces a large profile but avoids breakdown of LU decomposition.
  - In **p-last per level reordering**, unknowns are reordered per level such that at each level, the velocity unknowns are followed by the pressure unknowns.

what are the levels ?

SILU preconditioner

4×4 Q2-Q1 grid

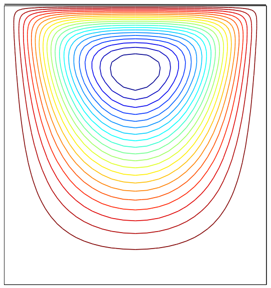
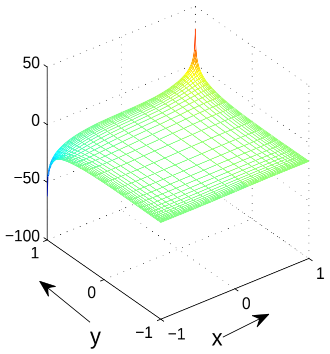
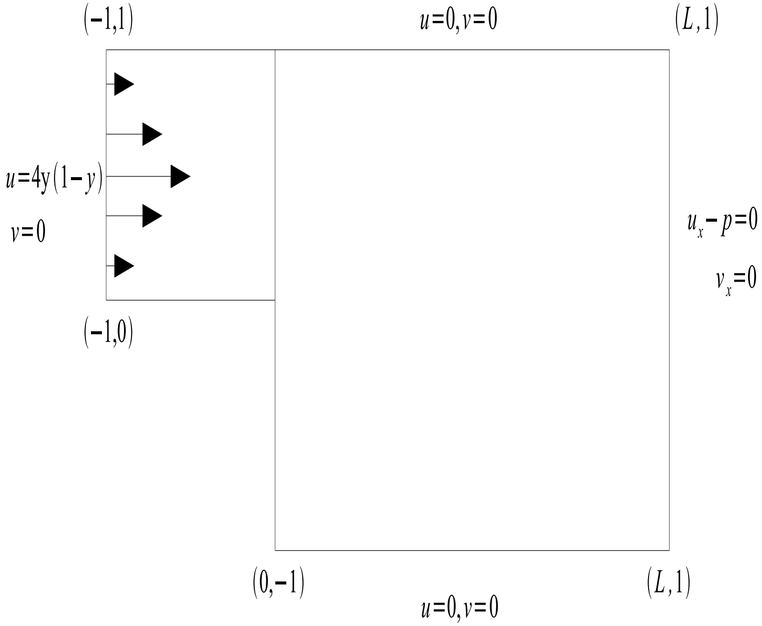


SILU preconditioners

- In direct solver, reordering improves the profile and bandwidth of the matrix.
- Improve the convergence of the ILU preconditioned Krylov subspace method
- Minimizes dropped entries in ILU ($\|\mathcal{A} - \bar{L}\bar{U}\|_F$)
- May give stable factorization ($\|I - \mathcal{A}(\bar{L}\bar{U})^{-1}\|_F$)

[Dutto-1993, Benzi et al 1999, Duff and Meurant-1989, Wille-2004, Bollhöfer and Saad - 2006, Saad -2005]

Numerical experiments (SILU preconditioner)

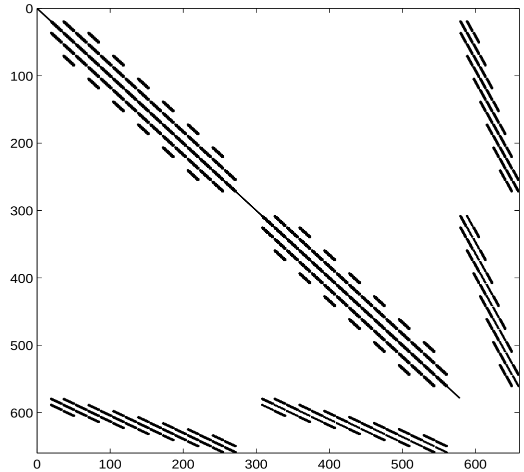
Driven cavity flow problem	Backward facing step problem
 	

The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol$.

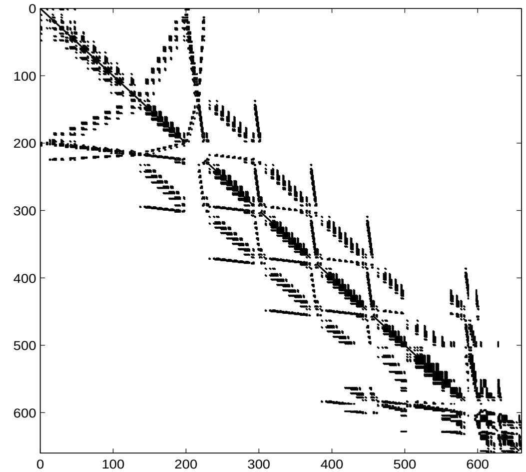
Numerical experiments (SILU preconditioner)

- Bandwidth($\mathcal{A}$)=$\max_i\{\beta_i(\mathcal{A}), 1 \leq i \leq n\}$
- Profile($\mathcal{A}$)=$\sum_{i=1}^{n} \beta_i(\mathcal{A})$

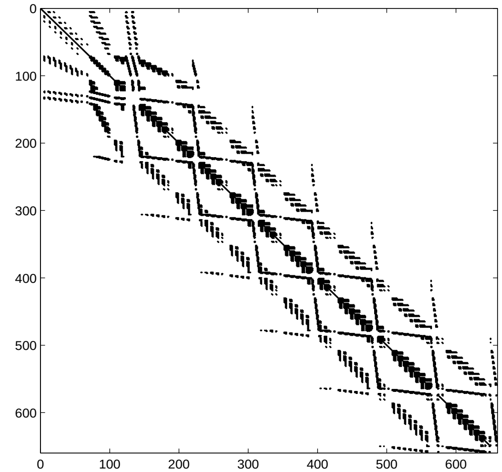
16×16 driven cavity flow with Q2-Q1 discretization



Profile = 52195, Bandwidth = 570
p−last ordering with lexicographic numbering



Profile =31222, Bandwidth = 212
p−last per level ordering with Sloan renumbering



Profile =47468, Bandwidth =160
p−last per level ordering with Cuthill McKee renumbering

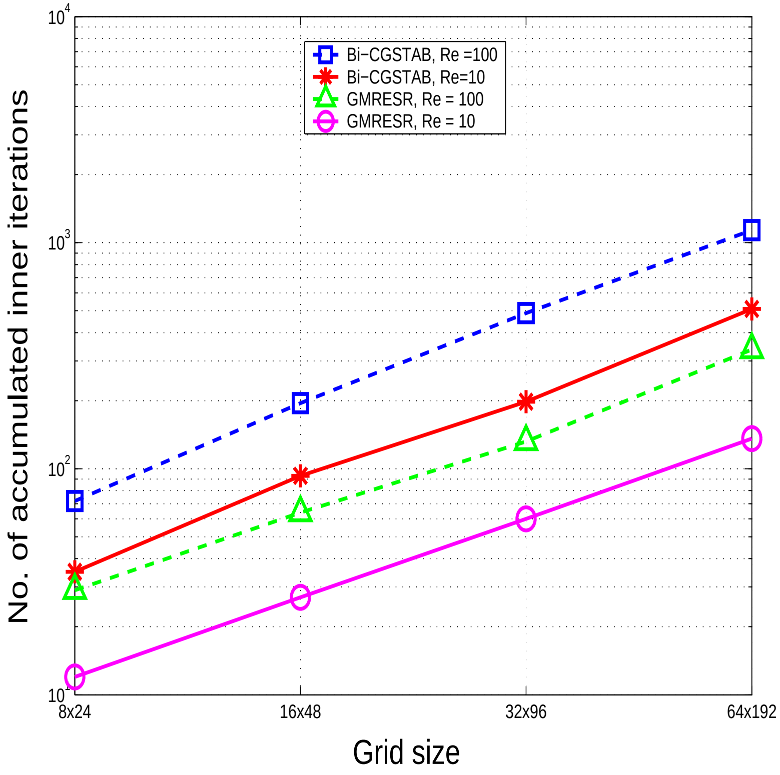
Numerical experiments (SILU preconditioners)

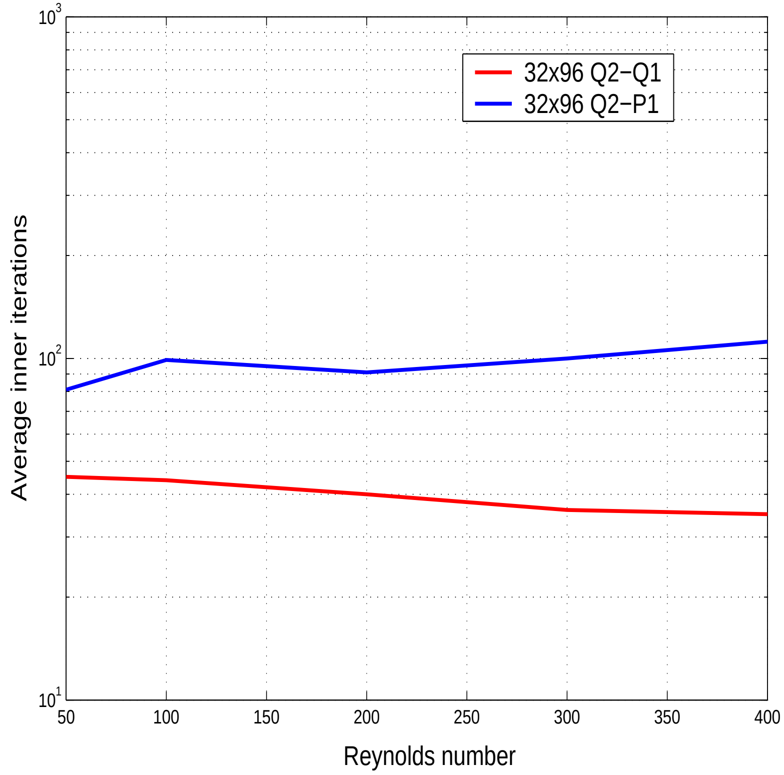
Stokes Problem in a square domain with Bi-CGSTAB,
$accuracy = 10^{-6}$, Sloan renumbering

	$Q2 - Q1$		$Q2 - P1$	
Grid size	p-last	p-last per level	p-last	p-last per level
16×16	36(0.11)	25(0.09)	44(0.14)	34(0.13)
32×32	90(0.92)	59(0.66)	117(1.08)	75(0.80)
64×64	255(11.9)	135(6.7)	265(14)	165(9.0)
128×128	472(96)	249(52)	597(127)	407(86)

Numerical experiments (SILU preconditioners)

Effect of grid increase(Left) and Reynolds number(Right) on inner iterations for the Navier Stokes backward facing step problem with $accuracy = 10^{-2}$ using the p-last-level reordering





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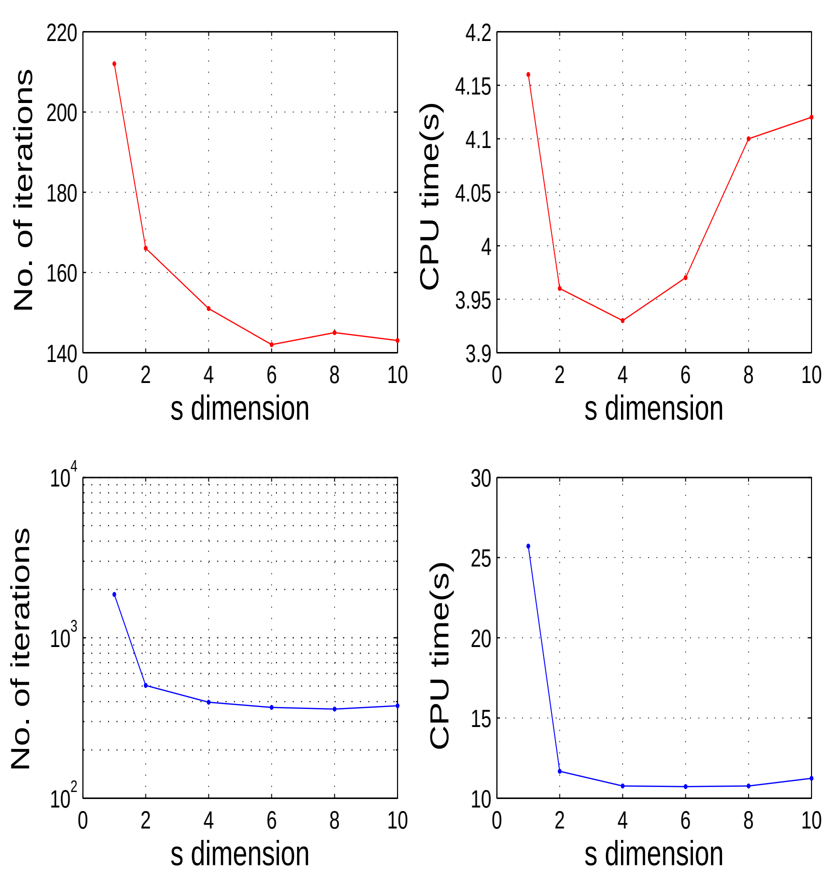
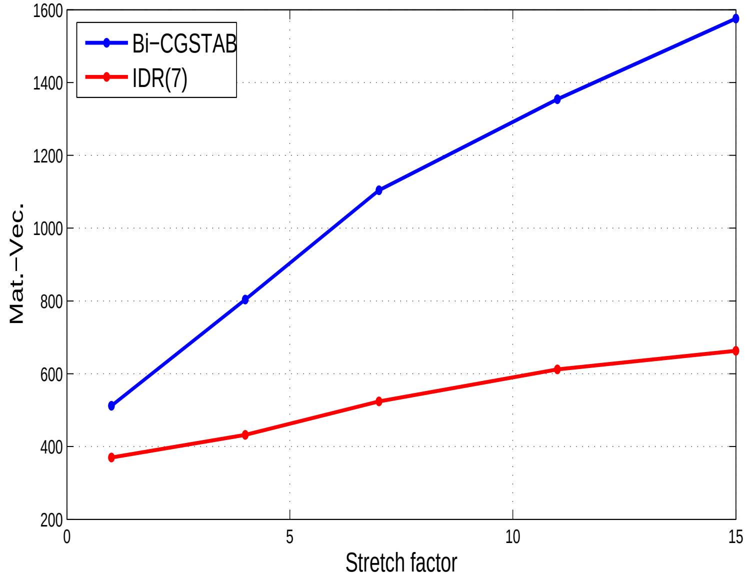
Numerical Experiments (IDR(s))

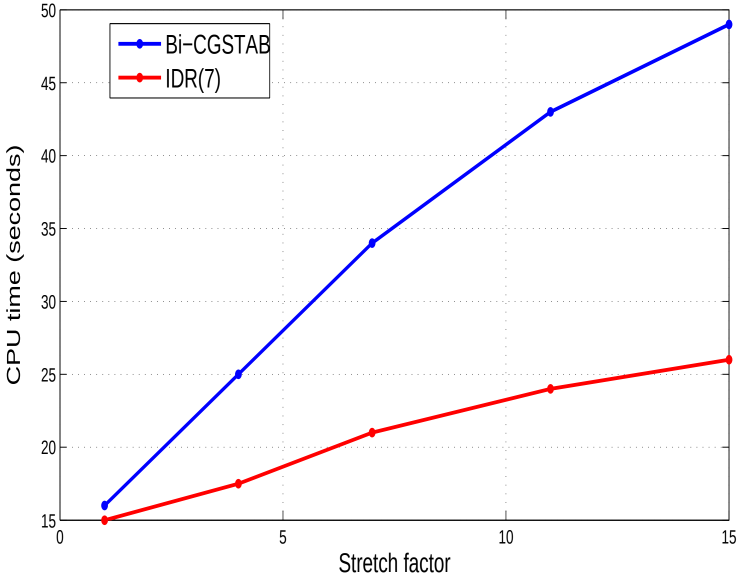
IDR(s): Top: 32×32, Bottom: 64×64 driven cavity Stokes flow problem



Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for increasing stretch factor for the driven cavity Stokes problem.





Numerical Experiments (IDR(s) vs Bi-CGSTAB(l))

SILU preconditioned: Comparison of iterative methods

Driven Cavity Stokes problem, stretch factor 10

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
128×128	1104(36.5)	4	638(24.7)	6
256×256	5904(810)	6	1749(307)	8

Channel flow Stokes problem, length 100

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
64×64	1520(12)	4	938(8.7)	8
128×128	NC	6	8224(335)	8

5. Block preconditioners

$$\mathcal{A} = \mathcal{L}_b \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BM_l^{-1} & I \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & M_u^{-1}B^T \\ 0 & I \end{bmatrix}$$

$M_l = M_u = F$ and $S = -BF^{-1}B^T$ is the Schur-complement matrix.

$$\mathcal{U}_{bt} = \mathcal{L}_b \mathcal{D}_b = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}, \quad \mathcal{L}_{bt} = \mathcal{D}_b \mathcal{U}_b = \begin{bmatrix} F & 0 \\ B & \hat{S} \end{bmatrix}.$$

Preconditioners are based on combination of these blocks involve:

$Fz_1 = r_1$ The velocity subsystem

$$S \longrightarrow \hat{S}$$

$\hat{S}z_2 = r_2$ The pressure subsystem

Block preconditioners

Block triangular preconditioners

$$P_t = \mathcal{U}_{bt} = \begin{bmatrix} F & B^T \\ 0 & \hat{S} \end{bmatrix}$$

- Pressure convection diffusion (PCD) [Kay et al, 2002]
  $\hat{S} = -A_p F_p^{-1} Q_p$, Q_p is the pressure mass matrix
- Least squares commutator (LSC) [Elman et al, 2002]
  $\hat{S} = -(BQ_u^{-1}B^T)(BQ_u^{-1}FQ_u^{-1}B^T)^{-1}(BQ_u^{-1}B^T)$, Q_u is the velocity mass matrix
- Augmented Lagrangian approach (AL) [Benzi and Olshanskii, 2006]
  F is replaced by $F_\gamma = F + \gamma B W^{-1} B^T$
  $\hat{S}^{-1} = -(\nu \hat{Q}_p^{-1} + \gamma W^{-1}), \; W = \hat{Q}_p$

Block preconditioners

SIMPLE-type preconditioners[Vuik et al-2000]

SIMPLE	SIMPLER
$z = \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1} r$	$z = \mathcal{U}_{bt}^{-1}\mathcal{L}_b^{-1} r$
	$z = z + \mathcal{U}_b^{-1}\mathcal{L}_{bt}^{-1}(r - \mathcal{A}z)$
$M_u = D$	$M_l = M_u = D$, $D = diag(F)$
$\hat{S} = BD^{-1}B^T$	$\hat{S} = BD^{-1}B^T$
One Poisson Solve	Two Poisson solves
One velocity solve	Two velocity solves

Lemma: In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical .

Improvements in SIMPLE-type preconditioners

We use approximate solvers for subsystems, so flexible Krylov solvers are required (GCR, FGMRES, GMRESR)

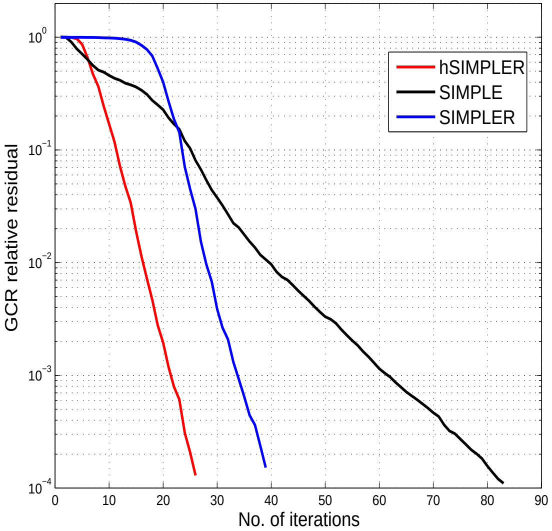
- hSIMPLER

- MSIMPLER

Improvements in SIMPLE(R) preconditioners

hSIMPLER preconditioner:

In hSIMPLER (hybrid SIMPLER), first iteration of Krylov method preconditioned with SIMPLER is done with SIMPLE and SIMPLER is employed afterwards.



- Faster convergence than SIMPLER
- Effective in the Stokes problem

Improvements in SIMPLE(R) preconditioners

MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.

LSC: $\hat{S} \approx -(B\hat{Q}_u^{-1}B^T)(B\hat{Q}_u^{-1}\underbrace{F\hat{Q}_u^{-1}}B^T)^{-1}(B\hat{Q}_u^{-1}B^T)$

assuming $F\hat{Q}_u^{-1} \approx I$ (time dependent problems with a small time step)

$\hat{S} = -B\hat{Q}_u^{-1}B^T$

MSIMPLER uses this approximation for the Schur complement and updates scaled with $\hat{Q}_u^{-1}$.

-Convergence better than other variants of SIMPLE

-Cheaper than SIMPLER (in construction) and LSC (per iteration)

Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-6} (SEPRAN) using Q2-Q1 hexahedrons

Grid	SIMPLE	LSC	MSIMPLER
iter. $(t_s)\frac{\text{in-it-}u}{\text{in-it-}p}$			
$8 \times 8 \times 16$	44(4) $\frac{97}{342}$	16(1.9) $\frac{41}{216}$	14(1.4) $\frac{28}{168}$
$16 \times 16 \times 32$	84(107) $\frac{315}{1982}$	29(51) $\frac{161}{1263}$	17(21) $\frac{52}{766}$
$24 \times 24 \times 48$	99(447) $\frac{339}{3392}$	26(233) $\frac{193}{2297}$	17(77) $\frac{46}{1116}$
$32 \times 32 \times 40$	132(972) $\frac{574}{5559}$	37(379) $\frac{233}{2887}$	20(143) $\frac{66}{1604}$

Numerical Experiments (comparison)

3D Backward facing step: Preconditioners used in solving the Navier-Stokes problem with preconditioned GCR(20) with $accuracy$ of 10^{-2} (SEPRAN) using Q2-Q1 hexahedrons

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 32$			
100	173(462)	96(162)	321(114)
200	256(565)	145(223)	461(173)
400	399(745)	235(312)	768(267)
$32 \times 32 \times 40$			
100	240(5490)	130(1637)	1039(1516)
200	397(7040)	193(2251)	1378(2000)
400	675(11000)	295(2800)	1680(2450)

Numerical Experiments (comparison)

3D Lid driven cavity problem (tetrahedrons):The Navier-Stokes problem is solved with accuracy 10^{-4}, a linear system at each Picard step is solved with accuracy 10^{-2} using preconditioned Krylov subspace methods. Bi-CGSTAB is used as inner solver in block preconditioners(SEPRAN)

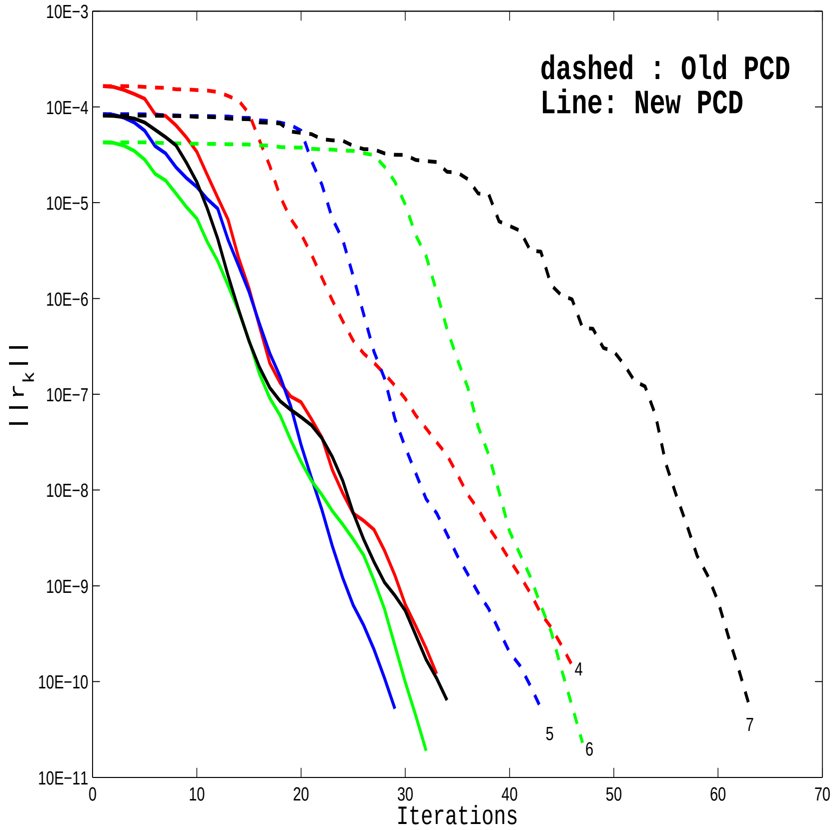
Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 16$			
20	30(20)	20(16)	144(22)
50	57(37)	37(24)	234(35)
100	120(81)	68(44)	427(62)
$32 \times 32 \times 32$			
20	38(234)	29(144)	463(353)
50	87(544)	53(300)	764(585)
100	210(1440)	104(654)	1449(1116)

Numerical Experiments (comparison)

2D Lid driven cavity problem on 64×64 stretched grid: The Stokes problem is solved with accuracy 10^{-6}. PCG is used as inner solver in block preconditioners (SEPRAN) .

Stretch factor	LSC	MSIMPLER	SILU
	GCR iter.	GCR iter.	Bi-CGSTAB iter.
1	20	17	96
8	49	28	189
16	71	34	317
32	97	45	414
64	145	56	NC
128	NC	81	NC

New PCD, better boundary conditions



6. Conclusions

- *In ILU, A new scheme for the renumbering of grid points and reordering of unknowns is introduced that prevents the break down of the resulting SILU preconditioner and leads to faster convergence of Krylov subspace methods.*
- *MSIMPLER is at present the fastest of all SIMPLE-type preconditioners.*
- *In our experiments, MSIMPLER proved to be cheaper than SILU, especially when the problem is solved with high accuracy.*
- MSIMPLER shows better performance than LSC. Both have similar convergence characteristics.
- New PCD behaves much better than classical PCD.

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